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Some Explanations of Dobinski's Formula

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The geometric, algebraic, and combinatorial explanations of Dobinski's formula are presented by mixed volumes of compact convex sets, Möbius inversion, difference operator, and species. The employed method may be useful in proving some other combinatorial identities.

1. Introduction

For a positive integer m and an integer n, define the number

$$C(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} k^n.$$

If n is nonnegative, then it can be proved by induction that

$$(-1)^m C(n,m) = \sum_{\substack{l_1+\cdots+l_m=n \\ l_1,\ldots,l_m \geq 1}} \frac{n!}{l_1!\cdots l_m!} = \begin{cases} m!S(n,m), & m \leq n \\ 0, & m > n, \end{cases} \quad (1)$$

where $S(n,m)$ is the Stirling number of the second kind.

The purpose in writing this note is to provide geometric, algebraic, and combinatorial methods to prove the identity (1). Let B_n be the nth Bell

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number. Dobinski's formula

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

can be easily obtained by identity (1) because

$$\begin{aligned} B_n &= \sum_{m=0}^{\infty} S(n,m) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} k^n \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(-1)^{m-k}}{(m-k)!} \cdot \frac{k^n}{k!} \\ &= \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \frac{(-1)^{m-k}}{(m-k)!} \cdot \frac{k^n}{k!} \\ &= e^{-1} \sum_{m=0}^{\infty} \frac{k^n}{k!}. \end{aligned}$$

Moreover, it is interesting that the numbers $C(n,m)$ $(n > 0)$ have various combinatorial explanations and $C(n,m)$ $(n < 0)$ are the harmonic numbers that are related to formal power series of logarithmic type [11]. The employed method here may be a useful technique in proving some other combinatorial identities [2, 7, 16].

In Section 2, a geometric identity is presented by the study of mixed volumes of compact convex sets. In Section 3, we use the Möbius function and difference operator to prove the same identity. In Section 4, a bijective proof of identity (1) is given by a species model.

Definition 1.1. Let $X = \{x_1, x_2, \ldots, x_s\}$ be a finite set. A multiset M on X is a function M: $X \to \mathbb{P}$, where $\mathbb{P}$ is the set of all positive integers. If $M(x_1) + M(x_2) + \cdots + M(x_s) = n$, we write $|M| = n$ for short. Let $\binom{n}{M}$ denote the multinomial coefficient $\binom{n}{M(x_1), \ldots, M(x)}$.

2. A geometric identity

We present a geometric identity that was first derived from the study of mixed volumes of compact convex sets [5]. When the identity is considered in view of combinatorics, it admits an interesting generalization that is closely related to harmonic numbers and formal power series of logarithmic type [11].

Let $\mathbb{R}^n$ be the standard Euclidean space. A subset E of $\mathbb{R}^n$ is called *convex* if for any two points p,q of E, the segment $[p,q]=\{tp+(1-t)q \mid 0\le t\le 1\}$ is contained in E. The class of all compact convex sets of $\mathbb{R}^n$ is denoted $\mathscr{K}$. Let χ be the function defined on $\mathscr{K}$ such that $\chi(K)=1$ for each nonempty $K\in\mathscr{K}$ and $\chi(\emptyset)=0$. It is well known that χ can be extended to a linear functional on the vector space $V(\mathscr{K})$ generated by indicator functions of elements of $\mathscr{K}$, and $\chi(f)$ is called the *Euler characteristic* of $f\in V(\mathscr{K})$ [6, 9, 10, 12, 14]. It can be shown that $V(\mathscr{K})$ is the valuation ring [4, 6, 8, 12, 14] of the lattice of compact convex sets, i.e., the lattice generated by finite unions of compact convex sets.

For $f,g\in V(\mathscr{K})$, let

$$(f*g)(x) = \chi(f\cdot g_x), \qquad \forall x\in\mathbb{R}^n,$$

where g_x is given by $g_x(y)=g(x-y)$ for $y\in\mathbb{R}^n$. It had been shown by Groemer [9] that $f*g$ is again an element of $V(\mathscr{K})$, and $(V(\mathscr{K}),+,*)$ is a commutative ring with identity $1_{\{o\}}$, the indicator function of the origin; the linear functional χ is a ring homomorphism from $V(\mathscr{K})$ to $\mathbb{R}$. For compact convex sets K_1 and K_2, $1_{K_1}*1_{K_2}=1_{K_1+K_2}$, so $*$ is called *Minkowski multiplication*. A more general treatment of Minkowski algebra of closed convex sets and relatively open convex sets can be found in [4].

Let A be an algebra over a commutative ring R, and let ϵ be an augmentation, i.e., an algebra homomorphism from A to R. The *Geissinger multiplication* $\circ$ on A is defined by

$$A\circ b = \epsilon(a)b+\epsilon(b)a-a\cdot b, \qquad \forall a,b\in A.$$

It can be shown that $(A,+,\circ)$ is an algebra over R, and ϵ is an algebra homomorphism. Moreover, $(A,+,\circ)$ is commutative if and only if $(A,+,\cdot)$ is commutative. The two multiplications of elements of A are related by the generalized inclusion-exclusion formula of Rota et al. (cf. [1, 8, 12, 14]), i.e., for $a_1,a_2,\ldots,a_m\in A$,

$$a_1\circ\cdots\circ a_m = \sum_{k=1}^{m}(-1)^{k-1}\sum_{i_1<\cdots<i_k} a_{i_1}\cdots a_{i_k}\prod_{i\ne i_1,\ldots,i_k}\epsilon(a_i). \qquad (2)$$

Let us consider the special augmented algebra $V(\mathscr{K})$ in which the addition is as usual, the multiplication is the Minkowski multiplication, and the augmentation is the Euler characteristic. Then for $f_1,\ldots,f_m\in V(\mathscr{K})$, the identity (2) becomes

$$f_1\circ\cdots\circ f_m = \sum_{k=1}^{m}(-1)^{k-1}\sum_{i_1<\cdots<i_k} f_{i_1}*\cdots*f_{i_k}\prod_{i\ne i_1,\ldots,i_k}\chi(f_i).$$

Hence, we have the following theorem.

THEOREM 2.1 [5]. *If each f_i, $1 \le i \le m$, is the indicator function of a compact convex set K_i, then*

$$1_{K_1} \circ \cdots \circ 1_{K_m} = \sum_{k=1}^{m} (-1)^{k-1} \sum_{i_1 < \cdots < i_k} 1_{K_{i_1} + \cdots + K_{i_k}}. \tag{3}$$

If $m = n$, we have

$$V(K_1, \ldots, K_n) = \frac{(-1)^{n-1}}{n!} V(1_{K_1} \circ \cdots \circ 1_{K_n}), \tag{4}$$

where $V(K_1, \ldots, K_n)$ is the mixed volume of $K_1, \ldots, K_n$.

The mixed volume can be defined by Minkowski's formula

$$V(\lambda_1 K_1 + \cdots + \lambda_s K_s) = \sum_{i_1=1}^{s} \cdots \sum_{i_n=1}^{s} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}$$

for compact convex sets $K_1, \ldots, K_s$ and nonnegative numbers $\lambda_1, \ldots, \lambda_s$, where $V(K_{i_1}, \ldots, K_{i_n})$ depends only on $K_{i_1}, \ldots, K_{i_n}$ and is invariant under permutations of $\{1, \ldots, n\}$ [3]. The formulas (3) and (4) are useful in deriving some combinatorial identities by choosing various compact convex sets. Let K_i be the closed unit ball B of $\mathbb{R}^n$ centered at the origin o. We have

$$V(B) = V(\underbrace{B, \ldots, B}_{n}).$$

Observe that

$$V(B) = V(\underbrace{B, \ldots, B}_{n}) = \frac{(-1)^{n-1}}{n!} \sum_{k=1}^{n} (-1)^{k-1} \sum_{i_1 < \cdots < i_k} V(1_{kB})$$

$$= \frac{(-1)^n}{n!} \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^n V(B).$$

Since $V(B) > 0$, this implies that

$$n! = (-1)^n \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^n.$$

According to [5], if $m > n$, then

$$V(1_{K_1} \circ \cdots \circ 1_{K_m}) = 0.$$

It turns out that

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} k^n = 0$$

for nonnegative n. More generally, we have the following geometric identity:

COROLLARY 2.1. *Let $V(K_1, \ldots, K_n)$ be the mixed volume of $K_1, \ldots, K_n$:*

$$\sum_{k=1}^{m} (-1)^k \sum_{i_1 < \cdots < i_k} V(K_{i_1} + \cdots + K_{i_k})$$
$$= (-1)^m \sum_{\substack{l_1 + \cdots + l_m = n \\ l_1, \ldots, l_m \geq 1}} \frac{n!}{l_1! \cdots l_m!} V\Big(\underbrace{K_1, \ldots, K_1}_{l_1}, \ldots, \underbrace{K_m, \ldots, K_m}_{l_m}\Big). \quad (5)$$

By setting $K_i = x_i B$, $1 \leq i \leq m$, the identity (5) induces

$$\sum_{k=1}^{m} (-1)^k \sum_{i_1 < \cdots < i_k} (x_{i_1} + \cdots + x_{i_k})^n$$
$$= (-1)^m \sum_{\substack{l_1 + \cdots + l_m = n \\ l_1, \ldots, l_m \geq 1}} \frac{n!}{l_1! \cdots l_m!} x_1^{l_1} \cdots x_m^{l_m}. \quad (6)$$

Identity (1) is a special case of the above polynomial identity (6) by the substitution $x_i = 1$, $1 \leq i \leq m$.

3. An algebraic identity

It is interesting to notice that $C(n,m)$ is related to harmonic numbers $C_m^{(-n)}$ [11]. In fact, if n is negative and m is positive, the harmonic number $C_m^{(n)}$ can be represented by

$$C_m^{(n)} = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} k^{-n}. \quad (7)$$

Then $C(n,m) = -C_m^{(-n)}$ whenever $n < 0$, $m \geq 1$.

Let $f(x)$ be a function defined on the set $\mathbb{Z}$ of all integers. The *difference* of f (Δf) is defined by

$$(\Delta f)(x) = f(x+1) - f(x), \qquad \forall x \in \mathbb{Z}.$$

The *kth difference* $\Delta^k f$ is

$$\Delta^k f = \Delta(\Delta^{k-1} f).$$

It is well known (Newton's formula) that

$$\Delta^m f(0) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(k), \tag{8}$$

and for nonnegative n,

$$\Delta^m 0^n = \begin{cases} m!\,S(n,m), & m \le n \\ 0, & m > n. \end{cases} \tag{9}$$

Hence, identity (1) is obtained again by rewriting the Stirling number $S(n,m)$ of the second kind [15].

The following theorem describes an algebraic identity that is obtained by the Möbius inversion on a Boolean algebra. The identities (1) and (6) are its immediate consequences.

THEOREM 3.1. *Let X be an m element set, $m \ge 1$, and w be a weight function on X with values in a commutative ring R. Then for any positive integer n,*

$$\sum_{s \subset X} (-1)^{|S|} \Big(\sum_{x \in S} w(x) \Big)^n = (-1)^{|X|} \sum_{\substack{m:\, X \to \mathbb{P} \\ |M| = n}} \binom{n}{M} \prod_{x \in M} w(x). \tag{10}$$

Proof: Let the power set $\mathscr{P}(X)$ of X be partially ordered by inclusion. Consider the incidence algebra of $\mathscr{P}(X)$ over the commutative ring R. For $n \ge 1$, $S \subset X$, define

$$f(S) = \Big(\sum_{x \in S} w(x) \Big)^n, \qquad g(S) = \sum_{\substack{m:\, S \to \mathbb{P} \\ |M| = n}} \binom{n}{M} \prod_{x \in M} w(x). \tag{11}$$

Then $f(X)$ can be written as

$$f(X) = \sum_{S\subset X} \sum_{\substack{M:S\to \mathbb{P}\\ |M|=n}} \binom{n}{M} \prod_{x\in M} w(x) = \sum_{S\subset X} g(S). \qquad (12)$$

By Möbius function $\mu(S,X)=(-1)^{|X-S|}$ [13, 15], we have

$$g(X) = \sum_{S\subset X} (-1)^{|X-S|} f(S). \qquad (13)$$

By the definition of $g(X)$ formula (10) follows immediately. ■

4. A combinatorial explanation

A *linear partition* ρ is a nonincreasing infinite sequence $(\rho_i)_{i\geq 1}$ of nonnegative integers that is eventually zero. Each nonzero ρ_i is called a *part* of ρ. The number of parts of ρ is denoted $l(\rho)$, and the product of the parts of ρ is denoted $\pi(\rho)$. Let $\mathscr{P}$ be the set of all linear partitions.

Let n and m be two positive integers. The harmonic number $C_m^{(n)} = -C(-n,m)$ can be represented combinatorially by [11]

$$C_m^{(n)} = \sum_{\substack{\rho\in\mathscr{P}\\ l(\rho)=n+1\\ \rho_1=m}} \pi(\rho)^{-1};$$

i.e., the harmonic number $C_m^{(n)}$ is equal to the sum of reciprocal product of parts of linear partitions ρ with $n+1$ parts and with no part greater than m. If we adopt the convention $0^{-n}=0$ for positive integer n, it is easily seen by the formula (8) that $C_m^n = -\Delta^m 0^n$.

Using the concept of species, the number $C(n,m)$ $(n>0,\ m>0)$ has a combinatorial explanation that can be viewed as a bijective proof of identity (1).

Let M and N be finite sets with $|M|=m$ and $|N|=n$, respectively, and let M be arbitrarily labeled, say $M=\{v_1,\ldots,v_m\}$. We define a species S_N by

$$S_N[M] = \{(A,f)\,|\,A\subset M, f\colon N\to A \text{ is a map}\}.$$

Given a weight function φ on $S_N[M]$ by

$$\varphi(A,f) = (-1)^{|A|}, \qquad \forall (A,f)\in S_N[M],$$

we have

$$\sum_{(A,f)\in S_N[M]} \varphi(A,f) = \sum_{k=1}^{m} (-1)^k \binom{m}{k} k^n. \tag{14}$$

The elements of A can be viewed as black vertices and the elements of $M - A$ can be viewed as white vertices since the subset A is distinguished. An element v of M is called a *nonimage* element of (A,f) if $v \notin f(N)$. Let $G(A,f)$ denote the set of all such nonimage elements. Now we define an operator I from $S_N[M]$ to itself by

$$I(A,f) = \begin{cases} (A,f), & G(A,f) = \varnothing \\ (A^*,f^*), & G(A,f) \neq \varnothing, \end{cases} \tag{15}$$

where (A^*,f^*) is the structure obtained from (A,f) by changing the color of the element of $G(A,f)$ whose labeling is minimal.

It is obvious to see that the operator I is an involution, and if (A,f) is not a fixed point of I, then

$$\varphi(I(A,f)) = -\varphi(A,f). \tag{16}$$

Let $F(I)$ be the set of all fixed points of I. Then we have

$$\sum_{(A,f)\in S_N[M]} \varphi(A,f) = \sum_{(A,f)\in F(I)} \varphi(A,f). \tag{17}$$

Note that (A,f) is a fixed point of I if and only if $M \subset f(N)$. But $A \subset M$ and $f(N) \subset A$, so $A = M = f(N)$. Thus, $F(I)$ can be identified with the set of all surjections from N to M.

THEOREM 4.1.

$$\sum_{(A,f)\in S_N[M]} \varphi(A,f) = (-1)^{|M|} \#\{f\colon N \to M \textit{ is surjective}\}. \tag{18}$$

One can consider a slightly more general weight function ψ on $S_N[M]$, which is defined by

$$\psi(A,f) = (-1)^{|A|}\alpha(f),$$

where α is a function on the set of all maps from N to M, and each map f: $N \to A$ is viewed as a map from N to M. Then we still have

$$\sum_{(A,f) \in S_N[M]} \varphi(A,f) = \sum_{(A,f) \in F(I)} \varphi(A,f).$$

Some numerical identities may be obtained by specifying some restrictions on α. For general treatment of the involution method, one can consult [16].

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