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The generating polynomial and Euler characteristic of intersection graphs

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Abstract

Let E^n be n-dimensional Euclidean space. A molecular space is a family of unit cubes in E^n. Any molecular space can be represented by its intersection graph. Conversely, it is known that any graph G can be represented by molecular space $M(G)$ in E^n for some n. Suppose that S_1 and S_2 are topologically equivalent surfaces in E^n and molecular spaces M_1 and M_2 are the two families of unit cubes intersecting S_1 and S_2, respectively. It was revealed that M_1 and M_2 could be transferred from one to the other with four kinds of contractible transformations if a division was small enough.

In this paper, we will introduce the generating polynomial $E_G(x)$ and the Euler characteristic $e(G)$ of a graph G. We will study several various operations performing on two graphs (surfaces). The generating polynomial of the new graph, which is obtained by performing various operations on well-studied graphs, can be expressed in terms of those of the old graphs. An immediate consequence is that the four contractible transformations do not change the Euler characteristic of a graph. Furthermore, we prove that all chordal graphs are contractible.

Key words: Intersection graph; Molecular spaces; Contractible transformations; Generating polynomial; Euler characteristic

1. Introduction

Let E^n be n-dimensional Euclidean space. The coordinates of point $x \in E^n$ are a sequence of real numbers $(x_1, x_2, \ldots, x_n)$. A unit cube U is a set $\{x=(x_1,x_2,\ldots,x_n) \mid n_i \leqslant x_i \leqslant n_i+1,\ 1 \leqslant i \leqslant n\}$ for some integer n_i. Two unit cubes are adjacent if they have common points. A molecular space is a family of unit cubes in Euclidean space E^n. Similar spaces are used in digital topology for the study of the topological properties of image arrays. Given a molecular space M, an intersection graph $G(M)$ (not unique) of M is a graph whose set of vertices is the set of unit cubes M and two vertices are adjacent if they are two adjacent cubes (see Fig. 1).

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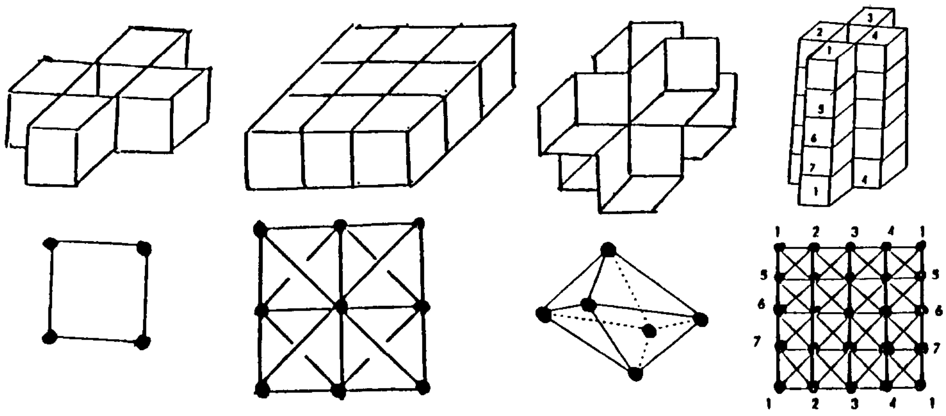

Fig. 1.

Conversely, it is known that any graph G can be represented by molecular space $M(G)$ in E^n for some n (imbedding problem). For example, the cycle graph C_p, $p \geqslant 3$, can be imbedded in E^n for $n \geqslant 3$ and the cycle graph C_5 cannot be imbedded in E^2. Contractible transformations of molecular space, based on four operations, were introduced in [7, 8]. We transfer these operations to graphs by using the connection between molecular spaces and graphs. These transformations did not change the homology group and the Euler characteristic of a graph was proved in [6–8].

Let H be a subgraph of G. $N(H)$ is then denoted by the subgraph of G which contains all neighbors of vertices of H and is called the rim of H. Let $B(H)$ be the subgraph of G which contains all vertices in $N(H)$ and H is called the ball of the graph H. Apparently, $B(H) = H \cup N(H)$. If H is a vertex a, then $N(a)$ and $B(a)$ are called the rim and the ball of the vertex a. Let vertices $a_1, a_2, \ldots, a_n \in G$; the subgraph $N(a_1, a_2, \ldots, a_n)$, $= N(a_1) \cap N(a_2) \cap \cdots \cap N(a_n)$, is called the joint rim of the vertices $a_1, a_2, \ldots, a_n$.

Definition 1.1. The following transformations are said to be contractible.

(1) *Deleting of a vertex a.* A vertex $a \in G$ can be deleted if the rim $N(a)$ is contractible.

(2) *Gluing of a vertex a.* If a subgraph A of the graph G is contractible then the vertex a can be glued to the graph G in such a manner that $N(a) = A$, i.e., the new graph is $(G; A) \oplus a$.

(3) *Deleting of an edge* (a_1a_2). The edge (a_1a_2) of a graph G can be deleted if the joint rim $N(a_1, a_2)$ is contractible.

(4) *Gluing of an edge* (a_1a_2). Let two vertices a_1 and a_2 of a graph G be nonadjacent. The edge (a_1a_2) can be set if the joint rim $N(a_1, a_2)$ is contractible.

Suppose that S_1 and S_2 are topologically equivalent surfaces in E^n and let molecular spaces M_1 and M_2 be the two families of unit cubes intersecting S_1 and S, respectively.

It was revealed that M_1 and M_2 could be transferred from one to the other with the above four kinds of contractible transformations if a division was small enough. This allows us to assume that the molecular space contains topological and, perhaps, geometrical characteristics of a continuous surface, i.e., the molecular space is the discrete counterpart of a continuous space.

The minimal representative of a homotopy class of graphs was introduced in [8]. Computing the Euler characteristic of a graph is very useful in constructing the minimal representative of a homotopy class of graphs.

In this paper, we will introduce the generating polynomial $E_G(x)$ of a graph G. We will study several different operations performed on two graphs (surfaces). A very complicated graph can be obtained by performing several operations on some well-studied graphs. The Euler characteristic, $e(G)$, of a graph G is $1-E_G(-1)$. The generating polynomial of the new graph, which is obtained by performing various operations on well-studied graphs, can be expressed in terms of those of the old graphs. Hence, the Euler characteristic of a graph can be easily derived by its generating polynomial. Hence, an immediate consequence is that four contractible transformations do not change the Euler characteristic of a graph. Furthermore, we prove that all chordal graphs are contractible.

Notation. $C(n)$ denotes the complete graph on n vertices, $K(n_1, n_2, \ldots, n_s)$ the s-partite complete graph, C_n the cycle graph on n vertices, L_n the graph of path on n vertices, $n_i(G)$ the number of i-complete subgraphs of graph G, $E_G(x)$ the generating polynomial of graph G and $e(G)$ the Euler characteristic of graph G.

2. Generating polynomial

In this paper, we treat a graph as finite and undirected with at most one edge joining a pair of vertices and no edge joining a vertex to itself. We will follow Harary [6] for any theoretical graph terminologies which were not defined here.

Let G be a graph and $n_i(G)$ the number of i-complete subgraphs of G, $i \geqslant 1$. For convenience, we always denote that $n_0(G)=1$ for all graphs G. We say that graph G is a graph of rank d if

$$d = \max\{i \mid n_i(G) > 0\}.$$

Definition 2.1. Let G be a graph of rank d. We denoted by by $E_G(x)$ the generating polynomial of the graph G, i.e., $E_G(x) = \sum_{i \geqslant 0}^{d} n_i(G) x^i$. The Euler characteristic of graph G, $e(G)$, is defined, as usual, by $e(G) = \sum_{k=1}^{s} (-1)^{k+1} n_k(G) = 1 - E_G(-1)$.

Remark. Two graphs in Fig. 2 are not isomorphic but have the same generating polynomial $1+4x+3x^2$ and Euler characteristic 1.

We would now like to discuss several operations performed on graphs and express the generating polynomial and the Euler characteristic of the new graph in terms of those of the old graphs.

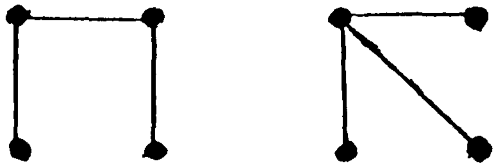

Fig. 2.

Lemma 2.2. *Let C_n be the graph of cycle and L_n be the graph of path on n vertices. Then*
(1) $E_{C_n}(x)=1+nx+nx^2$ *and* $e(C_n)=0$,
(2) $E_{L_n}(x)=1+nx+(n-1)x^2$ *and* $e(L_n)=1$.

Definition 2.3. Let two graphs G, H have disjointed vertex sets V_G, V_H and edge sets E_G, E_H, respectively. Then the following hold:
(1) *The union of G and H*, $G\cup H$, has a vertex set $V_G\cup V_H$ and edge set $E_G\cup E_H$.
(2) *The Kronecker product of G and H*, $G\otimes H$, has vertex set $\{(g,h)\mid g\in V_G, h\in V_H\}$ and edge set $\{(g,h),(g',h')\mid (g,g')\in E_G, (h,h')\in E_H\}$.
(3) *The sum of G and H*, $G\oplus H$, has vertex set $V_G\cup V_H$ and the edge set $E_G\cup E_H\cup\{(g,h)\mid g\in G, h\in H\}$.
(4) *The cartesian product of G and H*, $G\times H$, has a vertex set $\{(g,h)\mid g\in V_G, h\in V_H\}$ and edge set $\{((g,h),(g',h'))\mid (g=g', (h,h')\in E_H)$ or $(h=h', (g,g')\in E_G)\}$.

If a graph is not connected then its generating polynomials and Euler characteristic can be obtained in terms of those of its components.

Theorem 2.4. *Let F, G be two graphs. Then*
(1) $E_{G\cup H}(x)=E_G(x)+E_H(x)$,
(2) $e(G\cup H)=e(G)+e(H)$.

Let G and H be two graphs with known generating polynomials. First, we will find the number of k-complete subgraphs of the new graph, which are obtained by performing certain operations on graphs G and H, in terms of $n_i(G)$ and $n_j(H)$.

Lemma 2.5. *Let G and H be two graphs. Then.*

$$n_k(G\otimes H)=k!\cdot n_k(G)\cdot n_k(H) \quad \textit{for } k\geqslant 1.$$

Proof. Let U, V be i-complete subgraphs of G and H and their vertices sets are $\{u_1,u_2,\ldots,u_k\}$ and $\{v_1,v_2,\ldots,v_k\}$, respectively. Let σ be any permutation on the set $\{1,2,\ldots,k\}$. Then vertices $(u_j, v_{\sigma(j)})$ in $G\otimes H$ are denoted by $z_{\sigma(j)}$. Hence, these vertices $z_{\sigma(1)}, z_{\sigma(2)},\ldots,z_{\sigma(k)}$ form an i-complete subgraph of $G\otimes H$. On the other hand, if $w_1,w_2,\ldots,w_k, w_j=(s_j,t_j)$, where $s_j\in G$ and $t_j\in H$, then $s_1,s_2,\ldots,s_k$ and $t_1,t_2,\ldots,t_k$

form the k-complete subgraph of G and H, respectively. Thus we have

$$n_k(G \otimes H) = k!\, n_k(G)\, n_k(H). \qquad \square$$

We need the following definition to express the generating polynomial of the Kronecker product of graphs G and H.

Definition 2.6. Let $f(x)=\sum_{n\geqslant 0} a_n x^n$, $g(x)=\sum_{n\geqslant 0} b_n x^n$ be two polynomials. We then denote by $(f \otimes g)(x)$ the polynomial

$$\sum_{n\geqslant 0} n! \cdot a_n \cdot b_n \cdot x^n.$$

Theorem 2.7. *Let F, G be two graphs. Then*

(1) $E_{G\otimes H}(x) = E_G(x) \otimes E_H(x)$,

(2) $e(G \otimes H) = 1 - E_G(x) \otimes E_H(x)\,|_{x=-1}$.

Corollary 2.8. *Let C_n and L_n be defined as above. Then*

(1) $E_{C_n \otimes C_m}(x) = 1 + nmx + 2nmx^2$ *and* $e(C_n \otimes C_m) = -nm$,

(2) $E_{L_n \otimes L_m}(x) = 1 + nmx + 2(n-1)(m-1)x^2$ *and* $e(L_n \otimes L_m) = 2n + 2m - mn - 2$,

(3) $E_{C_n \otimes L_m}(x) = 1 + nmx + 2n(m-1)x^2$ *and* $e(C_n \otimes L_m) = 2n - mn$.

Let A and B be two graphs. We denote $A \cap B$ by the graph with the vertex set $V_A \cap V_B$ and edge set $E_A \cap E_B$. We will now study the operion "sum".

Lemma 2.9. *Let G and H be two graphs. Then*

$$n_k(G \oplus H) = \sum_{i=0}^{n} n_i(G) \cdot n_{k-i}(H).$$

Proof. Let U be an i-complete subgraph of G, and V be a j-complete subgraph of H. Then $U \oplus V$ is an $(i+j)$-complete subgraph of $G \oplus H$. On the other hand, let W be a k-complete subgraph of $G \oplus H$. Then $W \cap G$ and $W \cap H$ are complete subgraphs of G and H, respectively. Hence

$$\mathrm{n}_k(G \oplus H) = \sum_{i=0}^{n} n_i(G) \cdot n_{k-i}(H). \qquad \square$$

Theorem 2.10. *Let G, H be two graphs. Then*

(1) $E_{G\oplus H}(x) = E_G(x) \cdot E_H(x)$,

(2) $e(G \oplus H) = e(G) + e(H) - e(G)e(H) = 1 - (1 - e(G))(1 - e(H))$.

Let $C(n)$ be the complete graph on n vertices. The s-partite complete graph $K(n_1, n_2, \ldots, n_s)$ is the graph $C(n_1) \oplus C(n_2) \oplus \cdots \oplus C(n_s)$ and $C(n) = C(1) \oplus C(1) \oplus \cdots \oplus C(1)$. Hence, we have the following corollary.

Corollary 2.11. *Let* $C(n)$ *be the complete graph and* $K(n_1, n_2, \ldots, n_s)$ *be the s-partite complete graph. Then*

(1) $E_{C(n)}(x) = (1+x)^n$ *and* $e(C(n)) = 1$,

(2) $E_{K(n_1, n_2, \ldots, n_s)}(x) = \prod_{i=1}^{s}(1+n_i x)$ *and* $e(K(n_1, n_2, \ldots, n_s)) = 1 - \prod_{i=1}^{s}(1-n_i)$.

Let S_p be the p-dimension sphere. Then $K(2, 2, \ldots, 2)$ (the p-partite complete graph) is an intersection graph of S_p. Hence, the Euler characteristic of S_p is $1-(-1)^p$.

Corollary 2.12. *If* G *is a graph with* $e(G) = 1$, *then* $e(G \oplus H) = 1$ *for any graph* H.

We will now study the operation "cartesian product" performed on two graphs.

Lemma 2.13. *Let* G, H *be two graphs. Then*

$$n_k(G \times H) = n_1(G) \cdot n_k(H) + n_1(H) \cdot n_k(G)$$

for $k \geqslant 2$.

Proof. Let U and V be k-complete subgraphs of G and H, respectively. For any vertices $g \in G$, $h \in H$, let $\{g\} \times V$ and $U \times \{h\}$ be k-complete subgraphs of $G \times H$. On the other hand, we can let W be a k-complete subgraph of $G \times H$ and $\{w_i \mid w_i = (g_i, h_i)$ where $g_i \in G$, $h_i \in H$, $i = 1, 2, \ldots, k\}$ be the set of vertices of W. We claim that $g_1 = g_2 = \cdots = g_k$ or $h_1 = h_2 = \cdots = h_k$. This claim is obvious if $k = 2$. Let $k > 2$. Without loss of generality, assuming $g_1 \neq g_2$ and $h_2 \neq h_3$ then $h_1 = h_2$ and $g_2 = g_3$. Thus $w_1 w_3$ cannot be an edge in $G \times H$ since $g_1 \neq g_3$ and $h_1 \neq h_3$. This is a contradiction. Thus, our claim is true. Hence

$$n_k(G \otimes H) = n_1(G) n_k(H) + n_1(H) \cdot n_k(G). \qquad \square$$

Theorem 2.14. *Let* G, H *be two graphs. Then*

(1) $E_{G \times H}(x) = 1 + |V_G| \cdot E_H(x) + |V_H| \cdot E_G(x) - |V_G| \cdot |V_H| \cdot x - |V_G| - |V_H|$,

(2) $e(G \times H) = |V_G| \cdot e(H) + |V_H| \cdot e(G) - |V_G| \cdot |V_H|$.

Several classes of graphs can be generated by the cartesian product of cycle graph C_n and path graph L_n. For example, grid$(n, m) = L_n \times L_m$, cylinder$(n, m) = C_n \times L_m$ and torus$(n, m) = C_m \times C_n$.

Remark. The graph torus(n, m) (grid, cylinder(n, m) resp.) is different from the intersection graph of the molecular space of surface of torus (grid, cylinder resp.)

Corollary 2.15. *Let these graphs* grid(n, m), cylinder(n, m) *and* torus(n, m) *be defined as above.*

(1) $E_{\text{grid}(n, m)}(x) = 1 + nmx + (2mn - n - m)x^2$ *and* $e(\text{grid}(n, m)) = -(nm + n + m)$.

(2) $E_{\text{cylinder}(n, m)}(x) = 1 + mnx + (2mn - n)x^2$ *and* $e(\text{cylinder}(n, m)) = -n - nm$.

(3) $E_{\text{torus}(n, m)}(x) = 1 + nmx + 2nmx^2$ *and* $e(\text{torus}(n, m)) = -nm$.

We would like to generalize the concept of the operation "sum" in the following definition.

Definition 2.16. Let A and B be subgraphs of graphs G and H, respectively. Then $(G; A) \oplus (H; B)$ denotes the graph which has vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H \cup \{(a, b) \mid a \in A \text{ and } b \in B\}$. If graphs B and H are single vertex a, then we write $(G; A) \oplus a$ instead of $(G; A) \oplus (H; B)$.

Lemma 2.17. *Let G, H be two graphs. Then*

$$n_k((G; A) \oplus (H; B)) = n_k(G) + n_k(H) + \sum_{i=1}^{k-1} n_i(A) \cdot n_{k-i}(B).$$

Proof. Let W be a k-complete subgraph of $(G; A) \oplus (H; B)$. Only the following three cases occur:

(1) If $V_W \in V_G$, then W is a k-complete subgraph of G.

(2) If $V_W \in V_H$, then W is a k-complete subgraph of H.

(3) If $V_W \not\subset V_H$ and $V_W \not\subset V_G$, then $W \cap G$ is an i-complete subgraph of G and $W \cap H$ is a $(k-i)$-complete subgraph of H for some i, $0 < i < k$.

This is because there is no edge in graph $(G; A) \oplus (H; B)$ which connects the vertex in $V_{G \setminus A}$ and the vertex in $V_{H \setminus B}$. Hence

$$n_k((G; A) \oplus (H; B)) = n_k(G) + n_k(H) + \sum_{i=1}^{k-1} n_i(A) \cdot n_{k-i}(B). \qquad \square$$

Theorem 2.18. *Let A and B be subgraphs of graphs G and H, respectively. Then*

(1) $E_{(G; A) \oplus (H; B)}(x) = (E_G(x) - E_A(x)) + (E_H(x) - E_B(x)) + E_A(x) \cdot E_B(x)$,

(2) $e((G; A) \oplus (H; B)) = e(G) + e(H) - e(A)e(B)$.

In particular.

(3) $E_{(G; A) \oplus a}(x) = E_G(x) + x \cdot E_A(x)$,

(4) $e((G; A) \oplus a) = e(G) - e(A) + 1$.

Definition 2.19. Let A and B be subgraphs of graphs G and H, respectively, and graph A isomorphic to graph B. Let $(G; A) * (H; B)$ denote the graph which is obtained by identifying A and B.

Lemma 2.20. *Let A and B be subgraphs of graphs G and H, respectively, and graph A isomorphic to graph B. Then*

$$n_k((G; A) * (H; B)) = n_k(G) + n_k(H) - n_k(A).$$

Proof. There is no edge in graph $(G; A) * (H; B)$ which connects a vertex in $G \setminus A$ and a vertex in $H \setminus B$, and these complete subgraphs in A or B have to count only once since A and B are identified. Hence, this lemma is proved. $\square$

Theorem 2.21. *Let A and B be subgraphs of G and H, respectively, and graph A isomorphic to graph B. Then*

(1) $E_{(G;A)*(H;B)}\cdot(x)=E_G(x)+E_H(x)-E_A(x)$,

(2) $e((G;A)*(H;B))=e(G)+e(H)-e(A)$.

Definition 2.22. Let G be a graph and let a_1, a_2 be two nonadjacent vertices. We then denote by $G+(a_1a_2)$ the graph which is obtained from graph G plus an edge (a_1a_2).

Theorem 2.23. *Let G be a graph and let a_1, a_2 be two nonadjacent vertices. Let $N(a_1, a_2)$ be the subgraph on the joint rim of the vertices a_1 and a_2. Then*

(1) $E_{G+(a_1,a_2)}(x)=E_G(x)+x^2(E_{N(a_1,a_2)}(x))$,

(2) $e(G+(a_1a_2))=e(G)+1-e(N(a_1,a_2))$.

In particular, $e(G+(a_1a_2))=e(G)$ *if* $e(N(a_1,a_2))=1$.

3. Contractible graphs

Definition 3.1. The family T of all contractible graphs can be determined below:

- The trivial graph $K(1)$ belongs to T.
- Any graph of T can be obtained from the trivial graph by inductive application of contractible transformations (1)–(4) as above.

From the above definition, the complete graph $C(n)\in T$ for all $n\geqslant 1$. Using Theorem 2.23 and Corollary 2.12, the following theorem is easy to prove by induction.

Theorem 3.2. *Four contractible transformations do not change the Euler characteristic of a graph G.*

Corollary 3.3. *If a graph G is contractible then* $e(G)=1$.

The perfect graph was introduced by Berge [1] in the early 1960s. Since then many classes of graphs, interesting in their own right, have been shown to be perfect. One of the first classes of graphs to be recognized as being perfect was the class of *chordal graphs* (or triangulated graphs). A graph G is called a chordal graph if every cycle of length strictly greater than 3 possesses a chord, i.e., an edge joining two nonconsecutive vertices of the cycle. Equivalently, G does not contain an induced subgraph isomorphic to C_n for $n>3$. Let v be a vertex of a graph G. If $N(v)$ is a complete subgraph (clique) then v is called a *simplical vertex.*

An elimination process will produce an ordering α of the vertices $v_1, v_2, \ldots, v_n$ so that each v_i is a simplicial vertex in the subgraph induced by $v_1, v_2, \ldots, v_i$. In other words, each set $\{v_j \mid v_j\in N(v_i): j>i\}$ forms a clique. The ordering α is called a *perfect elimination ordering* (PEO).

Theorem 3.4 (Fulkerson and Gross [4]). *A graph G is chordal if and only if it has a PEO. Moreover, any simplicial vertex can start a PEO.*

According to the above theorem, all chordal graphs have a PEO. Using Definition 3.1, the following theorem can be proved by induction.

Theorem 3.5. *All chordal graphs are contractible.*

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