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A BIJECTIVE PROOF ON CIRCULAR COMPOSITIONS

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Abstract. The study of the preimage enumeration problem of an endofunction on circular compositions is motivated by the study of coloring circular-arc graphs. In this paper we establish a 1-1 correspondence between preimages of a given circular composition S and a proper S-sequences, and also provide a necessary and sufficient condition for a sequence of subsets of the natural numbers to be a proper S-sequence of some circular composition S.

1. Introduction. A graph G is an interval graph (also known as a circular-arc graph) if there exists a family $\mathcal{F}$ of arcs of the unit circle and a one-to-one correspondence between vertices of G and arcs of $\mathcal{F}$ such that two vertices are connected if and only if their corresponding arcs overlap.

A proper c-coloring of a graph G is a mapping from the vertices of G to the set $\{1, 2, 3, \ldots, c\}$ such that no two adjacent vertices are mapped to the same number. The chromatic number $\chi(G)$ is the smallest value of c for which there exists a proper c-coloring of G. It is known that chromatic number of an interval graph G is equal to the size of its maximum clique.

Given an angular position θ, let $S(\theta)$ denote the set of arcs which pass through θ; $|S(\theta)|$ is known as the density of θ. Let $\Delta(G)$ and $\delta(G)$ denote the maximum density and the minimum density of G.

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Let θ_1 be angular position such that $|S(\theta_1)|$ is maximum. Since any two arcs in $S(\theta_1)$ overlap each other, no two arcs in $S(\theta_1)$ can be assigned the same color. Hence $\chi(G) \geq \Delta(G)$.

Let θ_2 be an angular position such that $|S(\theta_2)|$ is minimum. We assign the colors $1, 2, 3, \ldots, \delta(G)$ to arcs in $S(\theta_2)$ and assign other colors to other arcs. Let $F = G \backslash S(\theta_2)$. F is an interval graph and $\chi(F) = \Delta(F)$. Therefore, there exists a $\Delta(G) + \delta(G)$-coloring of G. Since $\chi(G) \geq \Delta(G)$, we have $\Delta(G) + \delta(G) < 2\Delta(G) \leq 2\chi(G)$.

K. Tsai [9] has observed that an attempt to calculate the expected value of $(\Delta(G) + \delta(G))/\chi(G)$ leads one to the study of the preimage of the endofunction f (defined below) on circular compositions. Readers may also note that the study of circular compositions is similar to the game of Bulgarian Solitaire which was discussed in a programming and problem-solving seminar [4] at the Department of Computer Science at Standford University.

A circular composition $S = (s_1, \ldots, s_m)$ is an arbitrary composition of a non-negative integer n on m circularly labeled positions around a disk. For the sake of brevity, we henceforth refer to a circular composition simply as a state. The set of all states with m positions whose values add up to n is denoted $\mathbf{T}(n, m)$. A move f is performed on a state in the following way: for each i, $1 \leq i \leq m$, the value at position i (a non-negative integer s_i) is distributed clockwise, one unit at a time, to itself and the following $(s_i - 1)$ positions. The preimage of a state S is $\mathbf{B}(S) = \{T \in \mathbf{T}(m, n) : f(T) = S\}$. [11] contains the following result: (a) The necessary and sufficient conditions for cycle-states, root-states, and leaf-states. (b) The sharp upper and lower bounds for the length of a path from a given non-trivial state to its nearest LS in $\mathbf{T}(n, m)$. (c) Regardless of the initial state, one is sure to reach a cycle-state, which has only the values $[n/m]$ and $[(n + m - 1)/m]$ at all positions, in at most $m - 1$ moves. But [11] did not answer Dr. K. Tsai's original problem of finding the number of preimages for a given circular composition S.

In Section 2 we present definitions and preliminary material relating to circular compositions. In Section 3 we demonstrate bijection between preimages of a circular composition S and proper S-sequences, thus obtaining a formula for finding the number of preimages for a given circular composition S. In Section 4, we provide a necessary and sufficient condition for a sequence of subsets of the natural numbers to be a proper S-sequence for some circular composition and give some examples.

2. Preliminaries. We require some definitions from [11].

Definition 1. A *cycle-state* is a state S such that there exists some $k > 0$ for which $f^k(S) = S$.

Definition 2. A (n, m)-*configuration* is a matrix C with n rows and m columns, with entries either 0 or 1, having a total of n entries equal to 1. Let $\mathcal{C}(n, m)$ be the set of all (n, m)-configurations.

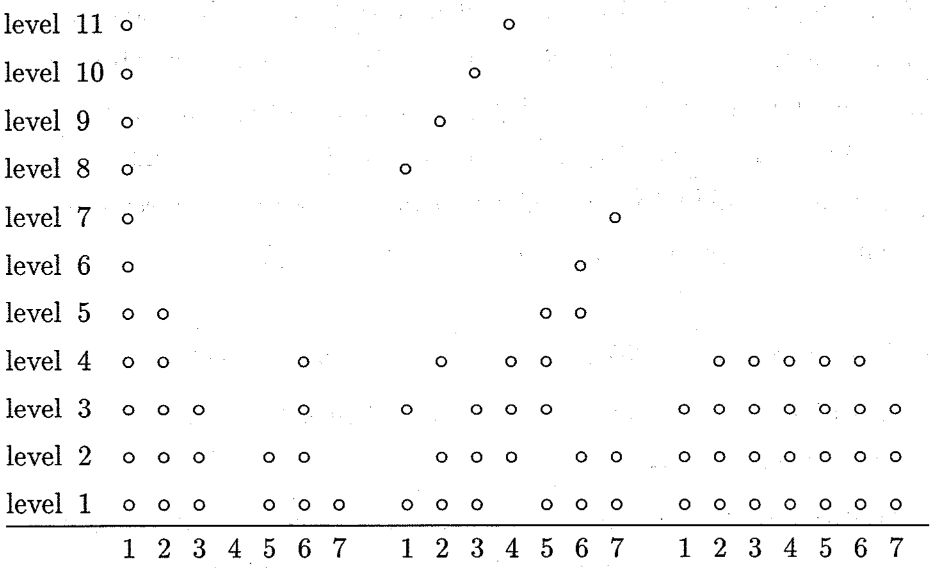
According to our usage row 1 is at the bottom, and we will use the term "level i" to refer to row i, and "position j" to refer to column j (positions are always added modulo m). We will frequently refer to any entry of 1 in C as a coin. A state $S = (s_1, s_2, \dots, s_m) \in \mathbf{T}(n, m)$ is viewed as a configuration which has ones in levels $1, \dots, s_j$ of position j and zeros everywhere else. We thus have $\mathbf{T}(n, m) = \{C \in \mathcal{C}(n, m) :$ no 1 in the matrix C has a 0 beneath it$\}$.

Here is an equivalent definition of the move f.

Definition 3. Let $f_1 : \mathbf{T}(n, m) \to \mathcal{C}(n, m)$ (also referred to as the *first step of a move*) be the function which moves each k-level coin in a given position to level k of the $(k-1)$-st subsequent position. (figure $1(a) \to 1(b)$).

Let $f_2 : \mathcal{C}(n, m) \to \mathbf{T}(n, m)$ (also referred to as the *second step of a move*) be the function which "compresses" each position by eliminating the vertical gaps (i.e., zeros) between coins and letting the coins fall to the bottom of each column. (figure $1(b) \to 1(c)$)

A *move* $f : \mathbf{T}(n, m) \to \mathbf{T}(n, m)$ consists of successively performing the first and second steps; in other words, $f = f_2 \circ f_1$.



(a) S (b) $f_1(S)$ (c) $f_2 \circ f_1(S) = f(S)$

Figure 1. a move: $f((e530241)) = (3444443)$ where $e = 11$.

Definition 4. Let $S \in \mathcal{C}(n,m)$. A k-level coin x in position j is called a *slanted coin* if a coin exists at every $(k-i)$ level in the ith previous position to j for $1 \le i < k$, i.e., there are no gaps along the diagonal line L which passes through level k of position j, level $k-1$ of position $j-1, \ldots,$ level 1 of position $j-k+1$. Infinite lines which are parallel to L will be referred to as *right-diagonal lines*.

The following lemma is evident.

Lemma 5. Given a configuration $C \in \mathcal{C}(n,m)$, every coin is slanted if and only if the following condition holds: for all $j = 1, \ldots, m$ and $i > 1$, if there is a coin in every level i of position j, then there is a coin in level $i-1$ of position $j-1$.

An element $T \in \mathbf{B}(S)$ is obtained from S by performing the backward move f^{-1}.

Definition 6. A *backward move* consists of the following two steps:

1. In the *first backward step*, f_2^{-1} (not unique), coins in each position may or may not be lifted some levels so that all coins are slanted coins.

2. In the *second backward step*, f_1^{-1}, each k-level coin is moved to level k of the $(k-1)$-st previous position.

A *backward move* $f^{-1}: \mathbf{T}(n,m) \to \mathbf{T}(n,m)$ consists of successively performing the first backward step and the second backward step; in other words, $f^{-1} = f_1^{-1} \circ f_2^{-1}$.

Since f_1^{-1} is unique, there is a 1-1 correspondence between preimages $T \in \mathbf{B}(S)$ and configurations $f_2^{-1}(S)$. In Section 3 we will count the elements of $\mathbf{B}(S)$ by counting configurations $C \in f_2^{-1}(S)$; these are configurations with s_i coins in position i, lifted in such a way that every coin is a slanted coin. We will call such configurations slanted configurations of state S.

3. Preimage of state S and proper S-sequence. Now for our main result. We establish a bijection between the set of slanted configurations of a state S and a certain collection of finite sequences, thus obtaining a formula for $|\mathbf{B}(S)|$.

Definition 7. Given a particular arrangement of a slanted coins in position j of a configuration C, a *slot* in position $j+1$ of C is a level at which a slanted coin could be placed.

If there are s_j slanted coins at levels $h_1, \ldots, h_{s_j}$ in position j, then there are s_j+1 slots in position $j+1$ at levels $1, h_1+1, \ldots, h_{s_j}+1$. We will always refer to slots $1, 2, \ldots, s_j+1$ going from the lower slot (which is always at level 1) on up.

For a given state $S = (s_1, \ldots, s_m)$, let A_j be the set $\{1, 2, \ldots, s_{j-1}+1\}$. Let $\mathcal{H}(S)$ be the set of m-element sequences $a = (a_1, a_2, \ldots, a_m)$ in which $a_j \subset A_j$ and $|a_j| = s_{j-1}+1-s_j$. There are $\binom{s_{j-1}+1}{s_{j-1}+1-s_j} = \binom{s_{j-1}+1}{s_j}$ choices for each a_j, so there are

$$\prod_{j=1}^{m} \binom{s_{j-1}+1}{s_j}$$

elements in $\mathcal{H}(S)$. Since there are $s_{j-1}+1$ slots in position j and s_j coins in position j, we can view a letter a_j of a given sequence a as a set of slots in position j which are to be left blank.

Suppose we now add the condition that at each position j, the top slot $s_{j-1}+1$ is not to be left blank (i.e., $s_{j-1}+1 \notin a_j$ for each j). In this case there are $\binom{s_{j-1}}{s_j-1}$ choices for each a_j, so the number of elements of $\mathcal{H}(S)$ which satisfy this requirement is $\prod_{j=1}^{m}\binom{s_{j-1}}{s_j-1}$.

Let $\mathcal{W}(S)=\{a\in\mathcal{H}(S):$ there exists some j such that $s_{j-1}+1\in a_j\}$. It follows that

$$|\mathcal{W}(S)|=\prod_{j=1}^{m}\binom{s_{j-1}+1}{s_j}-\prod_{j=1}^{m}\binom{s_{j-1}}{s_j-1}.$$

We henceforth refer to a sequences $a\in\mathcal{W}(S)$ as *proper S-sequences.*

There is a simple algorithm for constructing a slanted configuration of state S from a proper S-sequence $a=(a_1,\ldots,a_m)$ of $\mathcal{W}(S)$. In the following algorithm, a space is *unmarked* if it contains neither a coin nor an X. At the beginning of the algorithm all spaces are unmarked.

Step 0. Let $i=0$.

Step 1. Let $i=i+1$. If there are any unmarked spaces in level i, go to step 2. If there are no unmarked spaces in level i, then stop.

Step 2. Consider all unmarked spaces in level i, one at a time, going from left to right. If an unmarked space in position j of level i is the q-th slot in position j and $q\notin a_j$, then place a coin at position j in level i. If an unmarked space is the q-th slot in position j and $q\in a_j$, then mark an X at position j in level i, and mark with an X every unmarked space which the right-diagonal line from position j of row i passes through. (Each of the infinitely many unmarked spaces on the right-diagonal line is at a level greater than i.) Go to step 1.

Example 1. $S=(3,4,4,4,4,4,3)$, $a=(\{2\},\emptyset,\{4\},\{1\},\{5\},\{3\},\{3,4\})$.

We have $A_1=A_2=\{1,2,3,4\}$, $A_3=A_4=A_5=A_6=A_7=\{1,2,3,4,5\}$. For each i we have $a_i\subset A_i$ and $|a_i|=s_{i-1}+1-s_i$, and $5=\max a_5=\max A_5$, so a is a proper S-sequence. The reader may verify that the configuration of coins which results from performing the algorithm on a is shown in Figure 1(b).

We can immediately state some facts about the algorithm. Each element of a given a_j corresponds to a right-diagonal line of X's which is marked down. The number of elements in all of the a_j, for $j = 1, 2, \ldots, m$, is $\sum_{j=1}^{m}(s_j + 1 - s_{j-1}) = m$. The algorithm will stop only when the m-th right-diagonal line of X's is marked down, since at that point there will be no unmarked spaces left. In level 1 every space is a slot, and if the algorithm has been performed on level $1, \ldots, i$, then any space in level $i+1$ that is not a slot must already be marked with an X. Consequently, after performing step 1, the unmarked spaces in level i are precisely the spaces in that level which are slots in their respective positions.

Let g be the function which acts on an element $a \in \mathcal{W}(S)$ by performing the algorithm described above.

Lemma 8. *The function g is well-defined from $\mathcal{W}(S)$ to $f_2^{-1}(S)$.*

Proof. $f_2^{-1}(S)$ is the set of all slanted configurations of state S, i.e. configurations with s_j coins in position j and with every coin slanted. Since coins can only be placed into slots, all coins in $g(a)$ are slanted, so we need only prove that $g(a)$ has precisely s_j coins in every position j.

Case 1: $g(a)$ has some position j which contains more than s_j coins. Let k be the position which is the first one in the course of the algorithm to receive $s_k + 1$ coins. We "interrupt" the algorithm and consider the configuration C which exists immediately after the (s_k+1)-st coin is placed into position k. Let $c_j =$ the number of coins which are in position j of configuration C, so $c_k = s_k + 1$ and for $j \neq k$, $c_j \leq s_j$. By step 2 of the algorithm and the definition of the sequence a, position k of C must have $s_{k-1} + 1 - s_k$ slots that have been marked with an X in addition to its $s_k + 1$ coins, so the uppermost coin in position k of C must be occuping slot $s_{k-1} + 2$. But if slot $s_{k-1} + 2$ exists in position k of C, then $c_{k-1} \geq s_{k-1} + 1$ coins, which contradicts our choice of k.

Case 2: $g(a)$ has some position j which contains fewer than s_j coins. Let $t_i =$ the number of coins which are in position i of configuration $g(a)$, so $t_j < s_j$. If $t_{j-1} = s_{j-1}$, then there are $s_{j-1}+1$ slots in position j, and since the algorithm permits us to leave at most $s_{j-1}+1-s_j$ slots blank in position j, it follows that $t_j = s_j$. But this contradicts our assumption, so we have $t_{j-1} < s_{j-1}$. Iteratively, we have $t_i < s_i$ for all i. Let k be a position at which the top slot is to be left blank, i.e. $s_{k-1}+1 \in a_k$ (such a position must exist by the definition of a). The algorithm cannot stop before the right-diagonal line of X's corresponding to $s_{k-1}+1 \in a_k$ has been marked down, but that line must originate at slot $s_{k-1}+1$ of position k, and if slot $s_{k-1}+1$ of position k is to exist then we must have $t_{k-1} \geq s_{k-1}$. This contradiction implies that the algorithm can never stop; but if it never stops then clearly for every j we have $t_j > s_j$.

Theorem 9. $|\mathbf{B}(S)| = \prod_{j=1}^{m} \binom{s_{j-1}+1}{s_j} - \prod_{j=1}^{m} \binom{s_{j-1}}{s_j - 1}$.

Proof. We need only show that g is a bijection. Injectivity is simple; if $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$ are distinct elements of $\mathcal{W}(S)$, then for some k we have $a_k \neq b_k$, so the coins in position k of the configuration $g(a)$ occupy different slots than the coins in position k of $g(b)$, and $g(a)$ and $g(b)$ must be distinct configurations.

Choose a configuration $C \in f_2^{-1}(S)$. For each j, $1 \leq j \leq m$, let $a_j =$ {the slots in position $j+1$ which are blank}. To prove surjectivity, we will show that the sequence $a = (a_1, \ldots, a_m)$ is in $\mathcal{W}(S)$ and that $g(a) = C$. Clearly $a \in \mathcal{H}(S)$.

Suppose that there is no position j such that the top slot is left blank, i.e., we have $s_j + 1 \notin a_j$ for all j. Let x be a coin of maximal height in C, and let us say the level of x is k and it is at position j. Level $k+1$ of position $j+1$ must be the top slot in position $j+1$, but since the top slot is never left blank, there must be a coin at level $k+1$ of position $j+1$, which is impossible. It follows that there must exist some j such that $s_{j-1}+1 \in a_j$, so $a \in \mathcal{W}(S)$.

We prove $g(a) = C$ by induction on levels. In level 1, it has coins in precisely those positions j such that $1 \notin a_j$; this condition characterizes the placement of coins in level 1 in C as well. Assume that $g(a)$ and C are identical in levels $1, \ldots, h$. In level $h+1$ of C, there are coins in precisely those positions j such that level h of position j is a slot q which is not in a_j. This condition characterizes the coins in level $h+1$ of $g(a)$ as well, so C and $g(a)$ are identical up to level $h+1$. By induction, we have $g(a) = C$ and the map g is surjective.

We single out some special cases as corollaries:

Corollary 10. If S is such that for some j we have $s_j = 0$, then $\mathbf{B}(S) = \prod_{j=1}^{m} \binom{s_{j-1}+1}{s_j}$.

Corollary 11. If a state $S = (k, k, \ldots, k)$, then $\mathbf{B}(S) = (k+1)^m - k^m$.

Corollary 12. If a state S contains a coin which is not slanted, then $\mathbf{B}(S) = 0$.

4. Characterization of proper S-sequences. Given a state S, we have shown how to obtain the set of proper S-sequences $\mathcal{W}(S)$ which corresponds to $\mathbf{B}(S)$. It is natural to ask the following questions: Given some finite sequence a of finite sets of natural numbers, under what conditions does there exist a state S such that a is a proper S-sequence?

In this section we will provide a necessary and sufficient condition on a. Furthermore, we show that given a, we can determine S without performing $g(a)$.

It is easy to derive a necessary condition on a: if $a = (a_1, \ldots, a_m)$ is a proper S-sequence, then $|a_i| = s_{i-1} + 1 - s_i$ for each i, so

$$\sum_{i=1}^{m} |a_i| = \sum_{i=1}^{m} (s_{i-1} + 1 - s_i) = m. \tag{1}$$

We will show that this condition is sufficient as well.

Let $a = (a_1, \ldots, a_m)$ be a sequence of sets of natural number which satisfies condition (1). If S is a state such that a is a proper S-sequence, then $s_2 = s_1 + 1 - |a_2|$, $s_3 = s_2 + 1 - a_3 = s_1 + 2 - |a_2| - |a_3|$, and for each $i = 1, \dots, m$, we have $s_i = s_1 + c_i$, with

$$c_i = i - 1 - \sum_{j=2}^{i} |a_j| \tag{2}$$

Note that $c_1 = 0$. If $a_j \neq \emptyset$, then let $\bar{a}_j$ denote $\max\{v | v \in a_j\}$. If a is a proper S-sequence, then there must exist an i such that $s_i + 1 = \bar{a}_{i+1}$, i.e. $s_1 + c_i + 1 = \bar{a}_{i+1}$.

Let s'_1 be the number which satisfies the following condition: there exists some k such that $s'_1 + c_k + 1 = \bar{a}_{k+1}$, and if q is such that there exists a j for which $q + c_j + 1 = \bar{a}_{j+1}$, then $q \leq s_1$. In other words, $s'_1 = \max\{x_i | x_i + c_i + 1 = \bar{a}_{i+1},\ i = 1, 2, \dots, m\}$. We also let $s'_i = s'_1 + c_i$ for $i = 2, 3, \dots, m$. Then we have the following characterization theorem.

Theorem 13. *Let $a = (a_1, \dots, a_m)$ be a sequence of sets of natural numbers. If a satisfies condition* (1) *above, then a is a proper S-sequence for $S = (s'_1, \dots, s'_m)$ with s'_i as defined above.*

Proof. We must show that for all j, a_j is a $(s'_{j-1} + 1 - s'_j)$-element subset of $\{1, \dots, s'_{j-1} + 1\}$ and that for some k we have $s'_k \in a_{k+1}$. By the definition of s'_j and equation (2), we have

$$s'_{j-1} - s'_j + 1 = c_{j-1} - c_j + 1 = j - 2 - \sum_{i=2}^{j-1} |a_i| - \Big(j - 1 - \sum_{i=2}^{j} |a_i|\Big) + 1 = |a_j|.$$

The definition of s'_1 implies that $s'_{i-1} + 1 = s'_1 + c_{i-1} + 1 \geq \bar{a}_i$ for every i, so we have $a_i \subset \{1, \dots, s'_{i-1} + 1\}$ for every i. Let k be such that $s'_1 + c_k + 1 = \bar{a}_{k+1}$. Then since $s'_k = s'_1 + c_k$, we have $s'_k + 1 = \bar{a}_{k+1}$.

We close this paper by giving some examples.

Example 2. Let $a = (\{h_1\}, \{h_2\}, \dots, \{h_m\})$ for some natural numbers $h_1, \dots, h_m$. Condition (1) is clearly satisfied since $|a_i| = 1$. For $i = 1, \dots, m$, we have $c_i = i - 1 - (i - 1) = 0$, so the equations expressing s_i in terms of s_1 are all simply $s_i = s_1$ for all $i = 1, \dots, m$. For $k = 1, \dots, m$ we have

$\bar{a}_{k+1} = h_{k+1}$, so let $h = \max\{h_k\}$; we have $s'_1 = h-1$ and the desired state S is $(h-1, h-1, \ldots, h-1)$. Note that in the case $h_1 = h_2 = \cdots = 1$ we obtain the trivial circular composition $S = (0, \ldots, 0)$.

Example 3. Let a be the m-element sequence $(\{h_1, h_2, \ldots, h_m\}, \emptyset, \ldots, \emptyset)$ with $h_1 < \cdots < h_m$. By formula (2) we have $c_i = i-1$ so $s_i = s_1 + i - 1$ for $i = 1, \ldots, m$. Clearly $k = m$ is the only value at which $\bar{a}_{k+1}$ is defined, so we have $\bar{a}_1 = h_m$ and $s_1 = h_m - 1$. This gives us $S = (h_m - 1, h_m, h_m + 1, \ldots, h_m + m - 2)$.

Example 4. Let $a = (\{2\}, \emptyset, \{4\}, \{1\}, \{5\}, \{3\}, \{3, 4\})$. We have $c_1 = c_7 = 0$, $c_2 = c_3 = c_4 = c_5 = c_6 = 1$. The values for $\bar{a}_i$ are 2, undefined, 4, 1, 5, 3, 4 for $i = 1, \ldots, 7$ respectively, so $s'_1 = 3$ is the maximum value such that for some k we have $s'_1 + c_k + 1 = \bar{a}_{k+1}$. Consequently we obtain $S = (3, 4, 4, 4, 4, 4, 3)$, which agrees with Example 1.

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