A Remarkable Endofunction Involving Compositions

By *Yeong-Nan Yeh*

We define a state as an arbitrary composition of a non-negative integer n on m circularly labeled positions around a disk. A move is defined as the following endofunction: for each i, $1 \leq i \leq m$; the value at position i (a non-negative integer s_i) is distributed clockwise, one unit at a time, to itself and the following $(s_i - 1)$ positions. The structures of states keep changing in irregular ways as we perform a series of moves. Definitions and necessary and sufficient conditions for cyclic states, root states, and leaf states are given in this paper. We provide the sharp upper bounds for the length of a path from a given nontrivial state to its nearest leaf state and for the length of a path from a given nontrivial state to its farthest leaf state in $\mathbf{T}(n, m)$. Surprisingly, it turns out that regardless of the initial state, one is sure to reach a cyclic state, which has only the values $[n/m]$ and $[(n + m - 1)/m]$ at all positions, in at most $m - 1$ moves.

1. Introduction

In December 1983, the problem of Bulgarian solitaire was discussed in a programming and problem-solving seminar at the Department of Computer Science at Stanford University [1]. Bulgarian solitaire [2,3,4] is played with a fixed number of coins. Starting with an arbitrary partition S of these coins, a move is

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performed on S by the following rule: each part of S is reduced by one coin, and the set of these coins is added as a new part. Bulgarian solitaire concerns the behavior of the endofunction, defined earlier, on the set of all partitions of these coins. In this paper we study a similar game.

Let us fix two positive integers n and m, and let $\mathbf{T}(n, m)$ be the set of all compositions $S = (s_1, s_2, \ldots, s_m)$ of n in at most m nonzero parts, where all s_i are non-negative integers. We think of an element $S \in \mathbf{T}(n, m)$ as defining a column S_i of s_i identical coins in position i around a table where an orientation has been defined: position $i+1$ is, say, to the left of position i. We call the elements of $\mathbf{T}(n, m)$ "states" and the numbers $\{1, 2, \ldots, m\}$ "positions," to which we always add modulo m. Given a position p, we use the terms "i-th subsequent position" and "i-th previous position" to refer to positions $p+i$ and $p-i$, respectively.

DEFINITION 1. *An (n, m) configuration is a matrix C with n rows and m columns, with entries either 0 or 1, having a total of n entries equal to 1. Let $C(n, m)$ be the set of all (n, m) configurations.*

We think of row 1 as being at the bottom, and we use the term "level i" to refer to row i, and "position j" to refer to column j.

DEFINITION 2. *A standard (n, m) configuration is an (n, m) configuration with the property that, for each column j, the entries are in decreasing order, i.e., $\{i \mid C_{ij} = 1\}$ is an integer interval $\{1, 2, \ldots, s_j\}$.*

Viewing column j of a configuration C as a position that contains coins placed at levels $\{i \mid C_{ij} = 1\}$, it is clear there is a bijection between the set $T(n, m)$ of all states and the set of all standard (n, m) configurations. In this paper we study the following endofunction f of $\mathbf{T}(n, m)$:

DEFINITION 3. *A move is the endofunction f which operates by having each position i distribute simultaneously its s_i coins, one at a time, to itself and to the subsequent $(s_i - 1)$ positions*

We call $f(S)$ the state obtained from S after a move. For example, if $S = (2, 7, 0, 3) \in \mathbf{T}(n, m)$, then $f(S) = (1, 1, 0, 0) + (1, 2, 2, 2) + (0, 0, 0, 0) + (1, 1, 0, 1) = (3, 4, 2, 3)$.

DEFINITION 4. *A state $S \in \mathbf{T}(n, m)$ is called a cycle state (abbreviated CS) if S appears more than once in the sequence $\{S, f(S), f^2(S), f^3(S), \ldots\}$. $S \in \mathbf{T}(n, m)$ is a root state (abbreviated RS) if $f(S) = S$. $S \in \mathbf{T}(n, m)$ is a leaf state (abbreviated LS) if there exists no state T for which $f(T) = S$.*

These types of states can also be defined in terms of the digraph G_f of the endofunction f: S is CS if it is part of a cycle in G_f, RS if it is the node of a one-cycle (loop) in G_f, and LS if it is a leaf of G_f. A sharp upper bound is an upper bound that can be reached.

Section 2 contains some definitions and auxiliary lemmas. In Section 3, we give necessary and sufficient conditions for cycle states, root states, and leaf

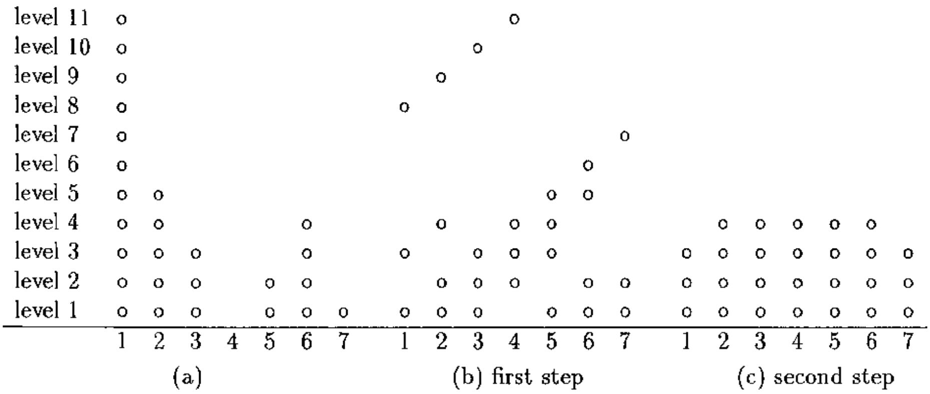


Figure 1. Move: $f((e530241)) = (3444443)$ where $e = 11$.

states, and we discuss the connectedness of the digraph G_f of f on $\mathbf{T}(n, m)$. In Section 4, we provide the sharp upper bounds for the length of a path from a given nontrivial state to its farthest LS and for the length of a path from a given nontrivial state to its farthest LS in $\mathbf{T}(n, m)$. We also prove that regardless of the initial state, one is sure to reach a cyclic state, which has only the values $[n/m]$ and $[(n + m - 1/m]$ at all positions, in at most $m - 1$ moves.

2. Auxiliary lemmas

We frequently use an equivalent definition of a move:

DEFINITION 5. *Let $f_1 : \mathcal{C}(n, m) \to \mathcal{C}(n, m)$ (also referred to as the first step of a move) be the endofunction that moves the k-level coin in a given position to level k of the $(k - 1)$-st subsequent position (see Figure 1, (a) $\to$ (b)).*

Let $f_2 : \mathcal{C}(n, m) \to \mathbf{T}(n, m)$ (also referred to as the second step of a move) be the function that "compresses" each position by eliminating the vertical gaps between coins and letting the coins fall to the bottom of each column (Figure 1, (b) $\to$ (c)).

A move $f : \mathbf{T}(n, m) \to \mathbf{T}(n, m)$ consists of successfully performing the first and second steps; in other words, $f = f_2 \circ f_1$.

For a state $S \in \mathbf{T}(n, m)$, we let $\mathrm{Max}(S)$ ($\mathrm{Min}(S)$ respectively) $= \{i \mid$ Position i has the maximal (minimal respectively) height among all positions of state $S\}$, and $\max(S)$ ($\min(S)$ respectively) be the maximal (minimal respectively) size among all positions of S. A slightly different description of the first step of a move proves useful.

DEFINITION 6. *Let $S \in \mathbf{T}(n, m)$ and $\min(S) = k$. The first step of a move, f_1, is the function that fixes all coins in levels 1 to k and slides the entire i-th row over by $i - 1$ positions for $i > k$.*

DEFINITION 7. *Let $S \in \mathbf{T}(n,m)$. A k-level coin $x \in S_j$ is called a slanted coin if a coin exists at every $(k-i)$ level in the i-th previous position to j for $1 \le i \le k$, i.e., there are no gaps along the diagonal line that passes through level k of position j, level $k-1$ of position $j-1, \ldots$, level 1 of position $j-k+1$.*

There follows a characterization of a state in which all coins are slanted.

LEMMA 1. *Let $S \in \mathbf{T}(n,m)$. Each coin in S is slanted if and only if $s_j - s_{j-1} \le 1$ for all j.*

Proof: ($\Rightarrow$) For $k > 1$, if a k-level coin is slanted, then there must be a $(k-1)$–level coin in the previous position "supporting" it. This implies that $s_j - s_{j-1} \le 1$ for all j.

($\Leftarrow$) For any k-level coin $x \in S_i$, $1 \le i \le m$, there exists a $(k-1)$–level coin in its previous position for all $k > 1$. Iteratively, there exists a $(k-i)$–level coin in its i-th previous position, so x is slanted. ■

DEFINITION 8. *Let $S \in \mathbf{T}(n,m)$. An i-level coin $x \in S_j$ is called a compressed coin if at position j there is a coin at level k for every k satisfying $1 \le k \le i$, i.e., there are no gaps from level 1 to level i along the vertical line in position j.*

We also have occasion to study backward moves.

DEFINITION 9. *A backward move consists of the following two steps:*

1. *In the first backward step, f_2^{-1} (not unique), coins in each position may or may not be lifted some levels so that all coins are slanted coins (Figure 2, $(a) \to (b)$).*
2. *In the second backward step, f_1^{-1}, each k-level coin is moved to level k of the $(k-1)$-st previous position (Figure 2, $(b) \to (c)$).*

A backward move f^{-1}: $\mathbf{T}(n,m) \to \mathbf{T}(n,m)$ consists of successively performing the first backward step and the second backward step; in other words, $f^{-1} = f_1^{-1} \circ f_2^{-1}$.

When we refer to a state $S \in \mathbf{T}(n,m)$, we are always referring to the standard configuration. Nonstandard configurations (which can occur when we consider only the first step of a move or backmove) are explicitly denoted as elements of $C(n,m)$.

Given an arbitrary configuration $C \in C(n,m)$, some coins may be neither compressed nor slanted. If $S \in \mathbf{T}(n,m)$, then all coins of S are compressed but not necessarily slanted. All coins of $f_1(S) \in C(n,m)$ are slanted but not necessarily compressed, and each coin occupies the same level in $f_1(S)$ that it did in S. f_2 acts by compressing each column of coins, so all coins in $f(S)$ are compressed and slanted for all $S \in \mathbf{T}(n,m)$. Some coins may be at a lower level after performing f, but no coins can be at a higher level. Correspondingly, after

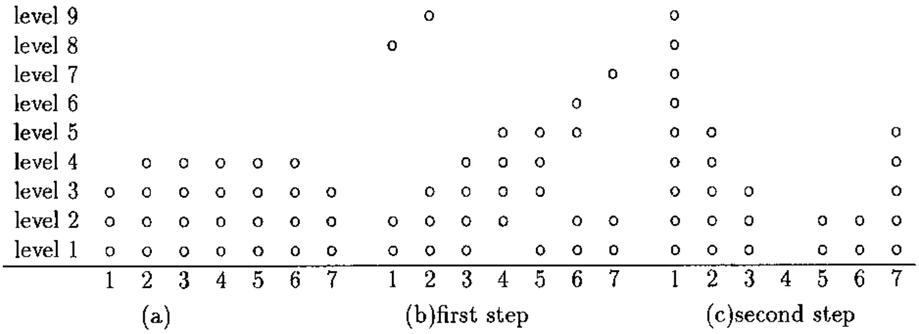


Figure 2. Backward move: (9530225) $\in \mathbf{B}((3444443))$.

performing a backward move on S, some coins may have their levels lifted but no coin can be dropped from its level. After the backward move f^{-1} has been completed on S, all coins are compressed but not necessarily slanted.

The following lemma characterizes the movement of coins under f.

LEMMA 2. *Let $x \in S_i$ be a k-level coin. Then x will be a j-level coin in $(f(S))_{i+k-1}$, with $j = |\{t \mid 0 \le t < k, k - s_{i+t} \le t\}|$.*

Proof: For any $t \in \{0, 1, \ldots, k\}$, the occupant of level $k - t$ in position $i + t$ is a coin if $k - s_{i+t} \le t$, otherwise there is an empty space. By Definition 4, whatever is at level $k - t$ in position $i + t$ ends up in $f_1(S)$ at level $k - t$ of position $k + i - 1 (= k - t + i + t - 1)$; therefore, the height of x in $f(S)$ is equal to $|\{t \mid 0 \le t \le k, k - s_{i+t} \le t\}|$, since f_2 only eliminates the spaces between coins. ■

From this lemma it follows easily that the top-level coin in a column does not go down any levels after a move if and only if $t \ge s_j - s_{j+t}$ for all $t \in \{1, 2, \ldots, s_j\}$.

DEFINITION 10. *Let $S \in \mathbf{T}(n, m)$ and $j \in \mathrm{Max}(S)$. A top-level coin $x \in S_j$ is called an unstable coin if there exists a positive integer k such that $s_i > s_{i+1} = \cdots = s_{i+k-1} > s_{i+k}$ and $s_i - s_{i+k} \ge 2$.*

According to Definition 10, there exists at least one unstable coin in state S if $\max(S) - \min(S) > 1$.

LEMMA 3. *Let $S \in \mathbf{T}(n, m)$ and let $x \in S_i$ be an unstable coin. The coin x drops down in at most $m - 1$ moves.*

Proof: The coin $x \in S_i$ is at the s_i level where $s_i > 1$. Let k, $1 \le k \le m - 1$ be the least positive integer such that $s_i > s_{i+1} = \cdots = s_{i+k-1} > s_{i+k}$ and $s_i - s_{i+k} \ge 2$, so there is an empty space at level $s_i - 1$ in position $i + k$ of state S. We prove this lemma by showing that x drops in at most k moves.

If $k = 1$, then $s_i - s_{i+1} \ge 2$, and hence $1 \notin \{t \mid 0 \le t \le s_i, s_i - s_{i+t} \le t\}$. So by Lemma 2, coin x drops after one move. Now suppose the result holds for $k < d$ for some $m - 1 > d > 1$.

Let $k = d$. We perform f on S. If x drops, then the lemma holds; if x does not drop, then by Lemma 2 x is at level s_i in position $i + s_i - 1$. Lemma 2 also tells us that a coin is in position $i + s_i + d - 2$ in $f(S)$ iff its level in S plus its position in S equal $i + s_i + d - 1$. So the coins in position $i + s_i + d - 2$ of state $f(S)$ are those that satisfy this condition, namely, the 1-level coin in position $i + s_i + d - 2$ of state S, the 2-level coin in position $i + s_i + d - 3$ of state $S, \ldots,$ and the $(s_i - 2)$–level coin in position $i + d + 1$ of state S. We know from the definition of an unstable coin that there can be neither an $(s_i - 1)$–level coin in position $i + d$ of state S, an s_i-level coin in position $i + d - 1$ of state S (since $d > 1$), nor a coin in any position of S whose level is greater than s_i since $s_i = \max(S)$. So position $i + s_i + d - 2$ of $f(S)$ can have at most $s_i - 2$ coins. The induction hypothesis, applied to coin x in state $f(S)$, tells us that x drops in at most $m - 1$ moves, and the lemma is proved. ∎

3. Cycle states, root states, and leaf states

In this section we give necessary and sufficient conditions for cycle states (CS), root states (RS), and leaf states (LS), and we discuss the connectedness of the digraph G_f of f on $\mathbf{T}(n, m)$.

THEOREM 1. *A state $S \in \mathbf{T}(n, m)$ is a CS if and only if* $\max(S) - \min(S) \le 1$.

Proof: ($\Rightarrow$) Let state S be a CS. Suppose that $\max(S) - \min(S) > 1$. Then there exists an unstable coin x in state S. By Lemma 3, the coin x will have fallen from its level after $(m - 1)$ moves are performed. Thus the state S can never appear again, no matter how many moves are performed, since a coin is never lifted from its level after a move.

($\Leftarrow$) Let $\max(S) - \min(S) \le 1$. If $\max(S) = \min(S)$, then it is clear from Definition 5 that state S is a CS (even an RS). If the state S has two different sizes, t and $t + 1$, in all positions, then by Definition 5, all coins from level 1 to level t stay fixed during a move, while the $(t + 1)$–level coins move t positions clockwise. Thus the state S is a CS since it must reappear after m or fewer moves are performed. ∎

The following corollary is now obvious.

COROLLARY 1. *The state $S \in \mathbf{T}(n, m)$ is a CS if and only if*

$$\min(S) = [n/m] \quad \textit{and} \quad \max(S) = [(n + m - 1)/m].$$

Suppose S is a CS of $\mathbf{T}(n, m)$ with m not a divisor of n. Let t and $t + 1$ be the two sizes that occur in S. Viewing the positions of size t and $t + 1$ as the digits 0 and 1 respectively, we have a binary circular word w_S which corresponds to the state S. Suppose the circular word w_S has period p; then state S appears again after $p/(t, p)$ moves.

Example 1. Let states S and T be $(3, 3, 3, 4)$ and $(4, 5, 4, 5)$. Then w_S and w_T are $(0, 0, 0, 1)$ and $(0, 1, 0, 1)$, and S and T repeat after, respectively, $4/(3, 4) = 4$ and $2/(4, 2) = 1$ moves.

We know that $p/(t, p) = 1$ if and only if $p \mid t$. Then we have the following characterization of RS.

THEOREM 2. *Let the state S be a CS with* $\min(S) = t$, *and let the circular word that corresponds to S have period p. Then the state S is an RS if and only if $p \mid t$.*

In Theorem 3, we give conditions for the connectedness of the digraph of f on $\mathbf{T}(n, m)$.

THEOREM 3. *The digraph of the endomorphism f on $\mathbf{T}(n, m)$ is connected iff the numbers n and m satisfy one of the following three conditions:*

(a) $n = km$ *for some positive integer k;*
(b) $n = km + 1$ *for some positive integer k with $(k, m) = 1$; and*
(c) $n = km - 1$ *for some positive integer k with $(k - 1, m) = 1$.*

Let $S, T \in \mathbf{T}(n, m)$. We define an equivalence relation $\sim$: $S \sim T$ if state S results from rotating state T through a finite number of positions around the disk. The endofunction f on the set $\mathbf{T}(n, m)$ induces an endofunction $f^\#$ on the set $\mathbf{T}(n, m)/\sim$. Let $[(s_1, s_2, \ldots, s_m)] \in \mathbf{T}(n, m)/\sim$ be the equivalence class that contains $(s_1, s_2, \ldots, s_m)$. The digraphs of $f(\mathrm{T}(6, 4)/\sim)$ and $f^\#(\mathbf{T}(6, 4)/\sim)$ are given in Figures 3 and 4.

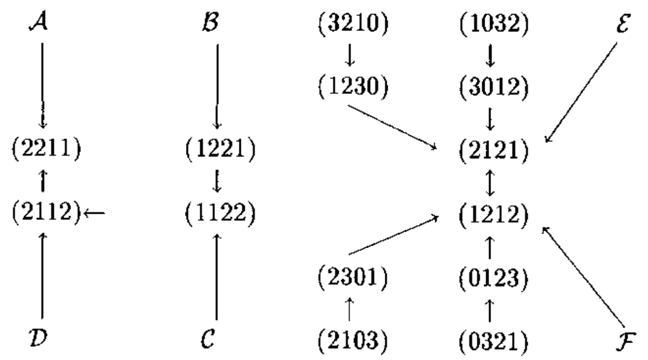
The relation $\sim$ equates members of the same cycle; hence, we have the following.

COROLLARY 2. *Every component of the digraph of $f^\#(\mathbf{T}(n, m))/\sim)$ is a rooted tree.*

THEOREM 4. *The digraph of the endomorphism $f^\#$ on $\mathbf{T}(n, m)/\sim$ is connected if and only if the numbers n and m satisfy one of the following three conditions:*

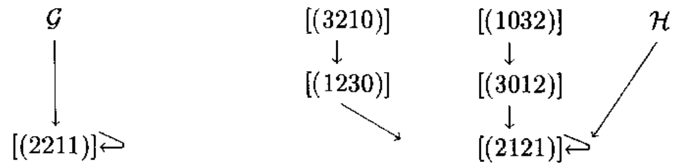
(a) $n = km$ *for some positive integer k;*
(b) $n = km + 1$ *for some positive integer k; and*
(c) $n = km - 1$ *for some positive integer k.*

To study leaf states we must again consider backward moves. A backward move from a state S is not always unqiuely determined, since there may be several ways to lift coins so that they are all slanted (see Figures 1 and 2). For $S \in \mathbf{T}(n, m)$, let $\mathbf{B}(S)$ be the set of preimages of S under the endofunction f, i.e., $\mathbf{B}(S) = \{T \in \mathbf{T}(n, m) \mid f(T) = S\}$. Each state in $\mathbf{B}(S)$ can be found by performing the appropriate first step of the backward move on $S \in \mathbf{B}(n, m)$.



$\mathcal{A}$={(6000),(5100),(1500),(1140),(1104),(2400),(2040),(2004),(3003), (3102),(2130),(2013),(1203),(2202),(1113)}
$\mathcal{B}$= {(0600),(0510),(0150),(0114),(4110),(0240),(0204),(4200),(3300), (2310),(0213),(3201),(3120),(2220),(3111)}
$\mathcal{C}$= {(0060),(0051),(0015),(4011),(0411),(0024),(4020),(0420),(0330), (0231),(3021),(1320),(0312),(0222),(1311)}
$\mathcal{D}$={(0006),(1005),(5001),(1401),(1041),(4002),(0402),(0042),(0033), (1023),(1302),(0132),(2031),(2022),(1131)}
$\mathcal{E}$= {(5010),(1050),(1410),(1014),(3030)}
$\mathcal{F}$={(0105),(0501),(4101),(0141),(0303)}

Figure 3. The digraph of f on $\mathbf{T}(6, 4)$.



$\mathcal{G}$={[(6000)],[(5100)], [(1500)],[(1140)], [(1104)],[(2400)], [(2040)],[(2004)], [(3003)], [(3102)],[(2130)],[(2013)],[(1203)],[(2202)],[(1113)]}
$\mathcal{H}$={[(5010)],[(1050)],[(1410)],[(1014)],[(3030)]}

Figure 4. The digraph of $f^{\#}$ on $\mathbf{T}(6, 4)/\sim$.

A state S is an LS if $\mathbf{B}(S) = \emptyset$. A necessary and sufficient condition for LS is given in the following theorem.

THEOREM 5. *A state S is an LS if and only if $s_j - s_{j-1} > 1$ for some j.*

Proof: ($\Rightarrow$) Suppose that for all j, $s_j - s_{j-1} \leq 1$; by Lemma 1, all coins in state S are slanted. We can choose f_2^{-1} to be the identity and perform f_1^{-1} on S

to obtain a state T such that $f(T) = S$. Thus, for S to be a leaf state, there must exist some j such that $s_j - s_{j-1} > 1$.

($\Leftarrow$) Suppose $s_j - s_{j-1} > 1$. Each slanted coin in position j except for the 1-level coin requires a coin in position $j-1$ to support it; therefore, there exists at least one coin in position j which cannot be slanted, so f_2^{-1} cannot be performed on S. ■

4. Extremal path lengths in G_f

It would be interesting to know the minimal number of backward moves that can be performed on a given state $S \in \mathbf{T}(n, m)$ (i.e., the shortest backward distance from a state S to a LS). Before we proceed further, we must define the trivial state.

DEFINITION 11. *A state S is called a trivial state if* $\max(S) = 1$ *and no two consecutive positions of state S have size one.*

Clearly $S \in \mathbf{T}(n, m)$ can be a trivial state only if $n \leq m/2$. A necessary and sufficient requirement for a trivial state is given in the following lemma.

LEMMA 4. *A state S is a trivial state if and only if* $\mathbf{B}(S) = \{S\}$.

A trivial state is an isolated loop in the digraph of f. It is a CS but not an LS. Now, we give the sharp upper bound for the length of a path from a given nontrivial state to its nearest LS in $\mathbf{T}(n, m)$.

THEOREM 6. $\mathbf{B}(S) \cup \{S\}$ *contains an LS for all nontrivial states* $S \in \mathbf{T}(n, m)$, *i.e., the sharp upper bound on the length of a path from a nontrivial state to its nearest LS is 1.*

Proof: The theorem is proved if S is an LS. Suppose that state S is not an LS. There are two cases to consider.

(a) If $\min(S) = 0$: This implies that $s_j = 0$, $s_{j+1} = 1$, and $s_{j+2} = 1$ or 2 for some j. f_2^{-1} can be performed in such a way as to place a coin at level 2 of position $j + 2$ (if $s_{j+2} = 2$, then raise no coins in position j; if $s_{j+2} = 1$, then raise the coin up to level 2). Hence the preimage state $T = f^{-1}(S)$ is an LS, since position $j + 1$ in state T has at least two coins and position j is empty.

(b) If $\min(S) > 0$: Suppose $\max(S) = \min(S)$; then $(n, 0, \ldots, 0)$ is a preimage of S, which is clearly an LS. Now suppose $\max(S) > \min(S)$. Let x be the top-level coin at position j, where $j \in \mathrm{Max}(S)$ and $j - 1 \notin \mathrm{Max}(S)$. In performing the first step of a backward move on state S, we lift the top-level coin in position $j+1$ to level $s_j + 1$. Then the corresponding preimage state $T \in \mathbf{B}(S)$ is an LS, since the size of the position of coin x in T is greater than the size of its previous position plus 1 in state T. ■

We would also like to have a sharp upper bound on the length of a path from a nontrivial state to its farthest LS.

DEFINITION 12. *Let* $S \in \mathbf{T}(n, m)$. *Then* $d(i, S) = \infty$ *if* $\min(S) \geq s_i$, *otherwise* $d(i, S) = \max\{k \mid s_{i-1} \leq \min\{s_{i-1}, s_{i-2}, \ldots s_{i-k}\} + 1$.

In other words, $d(i, S)$ is the distance from position i to the closest previous position with $s_i - 2$ or fewer coins. By Theorem 5, $\min\{d(i, S) \mid 1 \leq i \leq m\} = 1$ if and only if S is an LS. By Theorem 1, $\min\{d(i, S) \mid 1 \leq i \leq m\} = \infty$ if and only if S is a CS.

The following definitions are used in part (iii) of the proof of Lemma 5.

DEFINITION 13. *Let line L be the diagonal line that passes through level k of position j, level k − 1 of position j − 1, etc. We refer to lines that are parallel to L as being in a right-diagonal position. We denote* $b(L) = j - k + 1$, *i.e., the position at which line L passes through level 1.* $|L|$ *denotes the number of coins that line L passes through.*

DEFINITION 14. *Let L and M be two lines in right-diagonal position. Then* $\|b(L) - b(M)\| = k$, *where k is the smallest non-negative integer such that* $k = b(L) - b(M)$ *(modulo m).*

Note: We could as well have characterized k by saying that it equals the smallest non-negative integer such that if M passes through level t of position j, then L passes through level t of position $j + k$ (modulo m) for any t.

We employ the convention that $\min \emptyset = 0$ in the following.

LEMMA 5. *If* $S \in \mathbf{T}(n, m)$, *then*

$$\min\{d(i, S) \mid 1 \leq i \leq m\} - 1 = \max\{\min\{d(i, T) \mid 1 \leq i \leq m\} \mid T \in \mathbf{B}(S)\}.$$

Proof: We prove Lemma 5 in three stages. First (i) we eliminate the easy special cases when S is a CS or an LS. Then (ii) we show that for the remaining cases,

$$\min\{d(i, S) \mid 1 \leq i \leq m\} - 1 \leq \max\{\min\{d(i, T) \mid 1 \leq i \leq m\} \mid T \in \mathbf{B}(S)\}.$$

Finally (iii), we prove that

$$\min\{d(i, S) \mid 1 \leq i \leq m\} - 1 \geq \max\{\min\{d(i, T) \mid 1 \geq i \leq m\} \mid T \in \mathbf{B}(S)\}.$$

(i). Suppose that S is a leaf state. Then $\min\{d(i, S) \mid 1 \leq i \leq m\} - 1 = 0$, and $\max\{\min\{d(i, T) \mid 1 \leq i \leq m\} \mid T \in \mathbf{B}(S)\} = \max \emptyset = 0$, so the lemma holds. Suppose now that $d(i, S) = \infty$ for all i. This is the case iff S is a CS, which implies that there is a state $T \in \mathbf{B}(S)$ which is also a CS. Therefore $\min\{d(i, S) \mid$

$1 \le i \le m\} - 1 = \infty = \max\{\min\{d(i,T) \mid 1 \le i \le m\} \mid T \in \mathbf{B}(S)\}$, and the lemma holds.

(ii). Now suppose that S is neither LS nor CS. Since S is not LS, by Theorem 4 we know that $s_j - s_{j-1} \le 1$ for all j; Lemma 1 then tells us that every coin in S is slanted. This means we can perform a backmove on S by carrying out only f_2^{-1} without lifting any coins in the first step of the backward move, i.e., f_2^{-1} = identity. We call T' the preimage of S which is obtained in this way. Let $\alpha = \min\{d(i,T') \mid 1 \le i \le m\}$, let j be such that $d(j,T') = \alpha$, and let x be the top-level coin in position j of state T'. We let t_j = the height of position j in state T'. Since $d(j,T')$ is at a minimum, we know there are $t_j - 1$ coins in positions $j - \alpha + 1$ through $j - 1$, and $t_j - 2$ coins in position $j - \alpha$. In $S = f(T')$, by Definition 5, $(t_j - 1)$–level coins have moved over one position with respect to the $(t_j - 2)$–level coins, and the t_j-level coin x has moved two positions with respect to the $(t_j - 2)$–level coins. (No coins can drop by virtue of how T' was chosen.) It follows that $d(i,S) = \alpha + 1$, with $i = j + t_j - 1$ (= the position of coin x in state S). Therefore, there exists an i such that $d(i,S) = \alpha + 1 = \min\{d(i,T') \mid 1 \le i \le m\} + 1$. Consequently,

$$\min\{d(i,S) \mid 1 \le i \le m\} - 1 \le \alpha = \min\{d(i,T') \mid 1 \le i \le m\}, \text{ so}$$

$$\min\{d(i,S) \mid 1 \le i \le m\} - 1 \le \max\{\min\{d(i,T) \mid 1 \le i \le m\} \mid T \in \mathbf{B}(S)\}.$$

(iii). Let j be such that $d(j,S)$ is minimized. As in (ii), since S is not an LS or a CS, there are $s_j - 1$ coins in positions $j - d(j,S) + 1$ through $j - 1$, and $s_j - 2$ coins in position $j - d(j,S)$. Now let us perform the first step of an arbitrary backmove f_2^{-1} on S, so some coins may be lifted. We are now in the configuration $f_2^{-1}(S) \in \mathcal{C}(n,m)$, in which every coin is slanted.

Let P be the line in right-diagonal position which passes through the lowest empty space (whose level we call k) in position $j - d(j,S)$. There can be no coins in P whose level is $\ge k$, since every coin in $f_2^{-1}(S)$ is slanted. Let Q be the right-diagonal line which passes through level k of position j; so by the note following Definition 14, $\|b(Q) - b(P)\| = d(j,S)$. Let v = the number of coins in position j which are above P. Each of these coins must be "supported" by a coin in position $j - d(j,S)$, which is also above P, so there are at least v coins above P in position $j - d(j,S)$. Consequently, the top coin below right-diagonal line P in position $j - d(j,S)$ is at level $k - 1 \le s_j - v - 2$, which implies that $|P| \le s_j - v - 2$. Let x be the top coin which is below P in position j. The level of x is at least $s_j - v \ge k + 1$; so letting L_x be the line in right-diagonal position which passes through x, we know that $|L_x| \ge s_j - v$ and that $\|b(L_x) - b(P)\| < \|b(Q) - b(P)\|$, i.e., L_x is located between P and Q traveling clockwise from P to Q through the positions (see Figure 5). Lines P and L_x correspond to columns of coins in positions $b(P)$ and $b(L_x)$ of the preimage T, which is obtained by performing f_2^{-1} on the present configuration. So we have

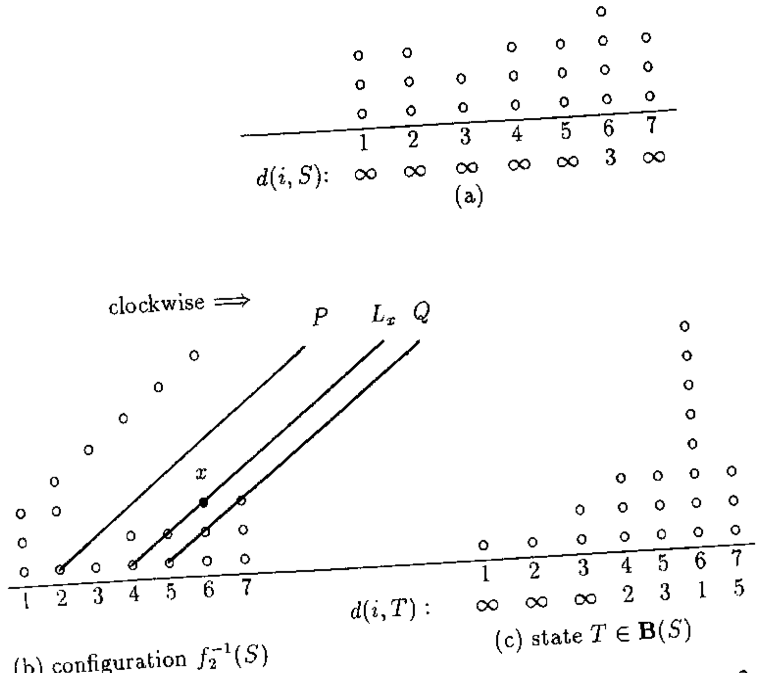


Figure 5. $d(b(L_x), T) = 2 \leq \|b(L_x) - b(P)\| = 2 < \|b(Q) - b(P)\| = 3$.

$d(b(L_x), T) \leq \|b(L_x) - b(P)\| < \|b(Q) - b(P)\| = d(j, S)$. Consequently,

$$\begin{aligned} d(j, S) - 1 &= \min\{d(i, S) \mid 1 \leq i \leq m\} - 1 \\ &\geq \min\{d(i, T) \mid 1 \leq i \leq m\} \forall T \in \mathbf{B}(S), \end{aligned}$$

so

$$\min\{d(i, S) \mid 1 \leq i \leq m\} - 1 \geq \max\{\min\{d(i, T) \mid 1 \leq i \leq m\} \mid T \in \mathbf{B}(S)\}.$$ ■

With Lemma 7 at our disposal, it is a simple matter to prove the following theorem, which gives a sharp upper bound on the length of a path from a given element to its farthest LS in $\mathbf{T}(n, m)$.

THEOREM 7. *Let* $S \in \mathbf{T}(n, m)$. *Then*

$$\min\{d(i, S) \mid 1 \leq i \leq m\} = \max\{k \mid k \textit{ backward moves can be performed on state } S\}.$$

Proof: If $\min\{d(i, S) \mid 1 \leq i \leq m\} = \infty$, then S is a CS and the result holds. For the finite case, we prove the theorem by induction. Let $r = \min\{d(i, S) \mid$

$1 \le i \le m\}$. If $r = 0$, then S is an LS and the lemma holds. Assume that this theorem is true for all $r < d$ for some $d > 0$. Now let $r = d$. By Lemma 7,

$$\begin{aligned} d - 1 &= \min\{d(i, S) \mid 1 \le i \le m\} - 1 \\ &= \max\{\min\{d(i, T) \mid 1 \le i \le m\} \mid T \in \mathbf{B}(S)\} \\ &= \max\{\max\{k \mid T \text{ can be performed } k \text{ moves backward}\} \mid T \in \mathbf{B}(S)\}. \end{aligned}$$

So

$$d = \max\{k \mid S \text{ can be performed } k \text{ moves backward}\}$$

and the theorem is proved. ■

We can now establish a theorem for the maximum number of moves that can be performed to reach a cycle state.

THEOREM 8. *Let* $S \in \mathbf{T}(n, m)$. *Then* $\min\{k \mid f^{(k)}(S)$ *is a CS*$\} \le m - 1$.

Proof: If $S \in \mathbf{T}(n, m)$ is not a CS, then $d(i, S)$ is finite and $\le m - 1$. Hence, any non-CS state can be performed $(m - 2)$ moves backward at most since $m - 2 \ge \min\{d(i, S) \mid 1 \le i \le m\} - 1$. Let $q = \min\{k \mid f^{(k)}(S)$ is a CS$\}$. So $f^{(q-1)}(S) = U$ (i.e., U is not a CS). This implies that $q \le (m-2) + 1 = m - 1$. Hence $\min\{k \mid f^{(k)}(S)$ is a CS$\} \le m - 1$. ■

COROLLARY 3. *Regardless of the initial state, one is sure to reach a cyclic state, which has only the values* $[n/m]$ *and* $[(n + m - 1)/m]$ *at all positions, in at most* $m - 1$ *moves.*

COROLLARY 4. *Let* $n = km$. *Regardless of the initial state* $S \in \mathbf{T}(n, m)$, *one is sure to reach the state* $(k, k, \ldots, k)$ *in* $m - 1$ *moves at most.*

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