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Cyclic Tableaux and Symmetric Functions

By William Y. C. Chen, Ko–Wei Lih, and Yeong–Nan Yeh

We introduce the notion of cyclic tableaux and develop involutions for Waring's formulas expressing the power sum symmetric function p_n in terms of the elementary symmetric function e_n and the homogeneous symmetric function h_n. The coefficients appearing in Waring's formulas are shown to be a cyclic analog of the multinomial coefficients, a fact that seems to have been neglected before. Our involutions also spell out the duality between these two forms of Waring's formulas, which turns out to be exactly the "duality between sets and multisets." We also present an involution for permutations in cycle notation, leading to probably the simplest combinatorial interpretation of the Möbius function of the partition lattice and a purely combinatorial treatment of the fundamental theorem on symmetric functions. This paper is motivated by Chebyshev polynomials in connection with Waring's formula in two variables.

1. Introduction

While the classical enumerative combinatorics has become more and more involved with other branches of mathematics, it is still tempting to think that a truly combinatorial problem deserves a truly combinatorial treatment. This is

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probably why today's combinatorialists are used to asking "what" instead of "why" is the "combinatorial" interpretation of a "combinatorial" identity and sometimes regard it of more intrinsic interest than the proofs obtained otherwise. This program has been carried out with great success in the past two decades. In the bijective approach to combinatorics, involutions are often desired to deal with the minus signs in combinatorial sums, whereas bijections are purely for positive situations. The first step to reach a bijective interpretation of a combinatorial identity is, of course, to establish a combinatorial setup which explains the numbers involved in the identity. The second step is to find the appropriate combinatorial operations that would translate the identity into something obvious as far as the combinatorial operations are concerned.

In this paper we will be concerned with Waring's formulas, which are among the oldest and the most fundamental formulas in the theory of symmetric functions. The usual forms of Waring's formulas are written in terms of generating functions which relate the power sum symmetric functions p_n to elementary symmetric function e_n and homogeneous symmetric function h_n [1–5]. Waring's formulas have many applications in combinatorial enumeration. We learned from the physicist J. D. Louck that they are also of interest in physics and chemistry, especially for problems in quantum mechanics in which the eigenvalue of a Hamiltonian such as the asymmetric rotator have no exact expressions, but one can still obtain useful relations between the roots [6].

For convenience, we use (H-P) to denote Waring's formula which expresses h_n in terms of p_n, and (E-P), (P-H), and (P-E) can be understood in a similar way. The first attempt to bring Waring's formulas into the combinatorial framework was due to Doubilet [7] (see also [8]), with his setup in terms of the partition lattice along with its Möbius function. However, by today's standard of bijective combinatorialists, there are still gaps in ultimately understanding Waring's formulas in such a playground that only combinatorial operations (or "combinatorial surgeries") are allowed.

We shall take a different approach to a bijective understanding of Waring's formulas. First, we choose the two forms (P-E) and (P-H), instead of (E-P) and (H-P), as our point of departure. To the best of our knowledge, no combinatorial efforts have been made to the forms (P-E) and (P-H) in the literature. Our combinatorial setup is based on a new notion of cyclic tableaux. We observe that the coefficients appearing in Waring's formulas (P-E) and (P-H) are a cyclic analog of the multinomial coefficients, a fact that seems to have been overlooked before. We further provide sign-reversing involutions on cyclic tableaux leading up to interpretations of both forms (P-E) and (P-H). Interestingly enough, these two involutions are not only based on the same underlying structure (cyclic tableaux), but also dual to each other, exemplifying the "duality between sets and multisets" in the perspective of Rota (e.g. [9]).

We also present an involution on permutations in cycle notation, which leads to probably the simplest explanation of the Möbius function of the partition lattice. Together with the involutions for Waring's formulas, we obtain a purely combinatorial treatment of the fundamental theorem on symmetric functions.

2. Waring's formulas

Throughout we shall use e_n, h_n, and p_n to denote the nth elementary, homogeneous, and power sum symmetric functions in $x_1, x_2, \ldots$. Waring's formulas are usually presented in the form of generating functions, relating p_n to e_n and h_n. As mentioned in Section 1, we shall use (H-P) to denote Waring's formula expressing h_n in terms of p_n; (E-P), (P-E) and (P-H) have similar meanings.

THEOREM 2.1 (Waring's Formulas).

$$\sum_{n=0}^{\infty} e_n t^n = e^{p_1 t - p_2 t^2/2 + p_3 t^3/3 - \cdots}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} h_n t^n = e^{p_1 t + p_2 t^2/2 + p_3 t^3/3 + \cdots}. \tag{2.2}$$

For easy reference we shall write down all the forms of Waring's formulas. By expansion we have the forms (E-P) and (H-P):

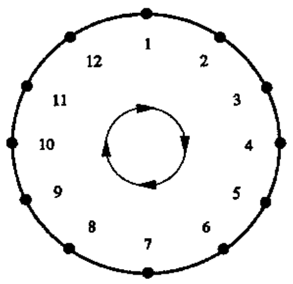
$$e_n = \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_2+k_4+\cdots} \frac{1}{1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!} p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, \tag{2.3}$$

$$h_n = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{1}{1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!} p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}. \tag{2.4}$$

Taking logarithm on (2.1) and (2.2), one obtains

$$p_n = \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_2+k_4+\cdots} \frac{n}{k_1+\cdots+k_n} \times \binom{k_1+\cdots+k_n}{k_1, \ldots, k_n} e_1^{k_1} e_2^{k_2} \cdots e_n^{k_n}, \tag{2.5}$$

$$p_n = \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+\cdots+k_n-1} \frac{n}{k_1+\cdots+k_n} \times \binom{k_1+\cdots+k_n}{k_1, \ldots, k_n} h_1^{k_1} h_2^{k_2} \cdots h_n^{k_n}. \tag{2.6}$$



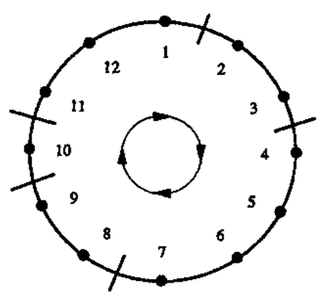


Figure 1. Cycle dissections and types. (a) 12 cycle; (b) cycle dissection of type $1^1 2^2 3^1 4^1$.

In the next section, we shall provide a combinatorial setup for the forms (P-E) and (P-H), based on the notion of cyclic tableaux. It is worthwhile to notice that the forms (P-E) and (P-H) chosen in this paper are much more natural than the forms (E-P) and (H-P) because the former are purely transformation formulas with integers as connections coefficients, whereas the latter have to be multiplied by $n!$ and consequently $n!e_n$ and $n!h_n$ have to be dealt with.

3. Cycle dissections and cyclic tableaux

As often happens in the theory of generating functions, the algebraic operation of taking logarithm is always associated with something about cycles in certain combinatorial structures [10]. An n-cycle can be either viewed as a cyclic permutation or a directed cycle in the sense of graph theory, with edges (or arcs) $(1,2), (2,3), \ldots, (n-1,n)$ and $(n,1)$. The main idea of this paper is the notion of cycle dissections and cyclic tableaux based on a cycle dissection.

DEFINITION 3.1 (Cycle Dissections and Types). *A dissection of an n-cycle is a decomposition of the cycle into straight segments, which can be viewed by putting cutting bars on some edges of the cycle. A dissection of an n-cycle is said of type* $1^{k_1}2^{k_2}\cdots n^{k_n}$ *if there are* k_i *segments of i elements in the dissection. A segment in a dissection is also called a block.*

Note that at least one cut is needed to dissect a cycle.

LEMMA 3.1 (Cyclic Multinomial Coefficients). *The number of dissections of type* $1^{k_1}2^{k_2}\cdots n^{k_n}$ *of an n-cycle equals*

$$\frac{n}{k_1+\cdots+k_n}\binom{k_1+\cdots+k_n}{k_1,\ldots,k_n}. \tag{3.1}$$

Proof: Let $f(k_1, k_2, \ldots, k_n)$ be the desired number of dissections. Then

$$(k_1 + \cdots + k_n) f(k_1, \ldots, k_n)$$

corresponds to the number of dissections with a distinguished cut. Let i be the position in the cycle that is to the right of the distinguished cut (or the endpoint of the edge having the distinguished cut). Now we open up the cycle along the distinguished cut, and straighten it into a line such that the position i is at the beginning. Note that i can be any of the n positions. After i is fixed, all the positions in the straight line are subsequently determined by the order $i, i+1, \ldots, n, 1, 2, \ldots, i-1$. Notice that the number of ways to cut a line of n labeled points into segments of type $1^{k_1} 2^{k_2} \cdots n^{k_n}$ is the same as the number of ways to arrange $k_1 + \cdots + k_n$ elements on a line in which there are k_1 1's, k_2 2's, etc. Clearly, this number equals the multinomial coefficient; hence it follows that

$$(k_1 + \cdots + k_n) f(k_1, \ldots, k_n) = n \binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n},$$

leading to (3.1). □

Since there are $2^n - 1$ dissections of an n-cycle, one obtains the following identity:

$$\sum_{k_1 + 2k_2 + \cdots + nk_n = n} \frac{n}{k_1 + \cdots + k_n} \binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n} = 2^n - 1.$$

This identity can also be derived via algebraic method as shown in [11].

A cyclic tableaux of type $1^{k_1} 2^{k_2} \cdots n^{k_n}$ on a variable set $X = \{x_1, x_2, \ldots\}$ is a dissection of an n-cycle in which every node is assigned a variable in X. For notational convenience, we assume that the variable set X is linearly ordered as $x_1 < x_2 < x_3 < \cdots$. Throughout, a cyclic tableau is always based on an n-cycle, and a tableau always means a cyclic tableau. A tableau T on X is said to be *strict* if for every block the assigned variables are increasing along the direction of the block. For example, the tableaux in Figure 2 are strict. Moreover, we define the sign of a strict tableau of type $1^{k_1} 2^{k_2} \cdots n^{k_n}$ to be $(-1)^{k_2 + k_4 + \cdots}$. A strict tableau is said to be degenerate if every block contains only one element and all the assigned variables are the same.

The following is an involution on nondegenerate strict tableaux, which leads to Waring's formula (P-E). For this reason, we call it Involution (P-E). For a position i in a tableau T, we use $T(i)$ to denote the variable assigned to position i. Note that $T(n+1)$ coincides with $T(1)$ because the positions of T are cyclically arranged. Suppose x_k is the smallest variable assigned to T. Then the *critical position* of T is defined as the position i $(1 \le i \le n)$ such that i is the smallest integer satisfying $T(i) = x_k$ and $T(i) < T(i+1)$. The *critical block* of T is

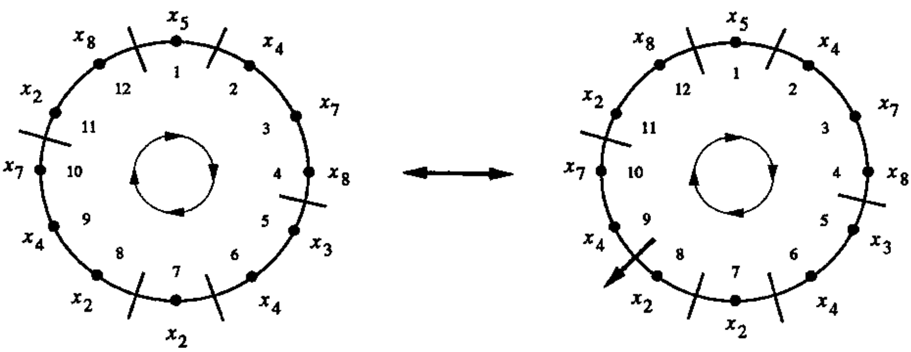


Figure 2. Involution (P-E).

defined as the block containing the critical position. It is clear that the critical position of T is independent of the dissection of T and the critical block always starts with the critical position (namely, there is a cutting bar preceding the critical position).

Given any nondegenerate strict tableau T, we define the map $f : T \to T'$ as follows:

DEFINITION 3.2 (Involution (P-E)). *Let B be the critical block of T.*

- *If B contains only one element, then T' is obtained from T by combining B with its next (along the direction of the cycle) block.*
- *If B contains more than one element, then T' is obtained from T by splitting B into two blocks such that the first one contains only one element.*

See Figure 2.

THEOREM 3.2. *The above map $f : T \to T'$ is a sign-reversing involution on the set of nondegenerate strict tableaux.*

Proof: Suppose that T is a nondegenerate strict tableau with critical block B. We say that B is a singleton if it contains only one element. We first consider the case when B is singleton. Let C be the block next to B in T. Since T is nondegenerate and B is the critical block, the first variable in C is greater than that in B. Hence we can combine B with C, and the resulting tableau T' is still strict and clearly nondegenerate. Since B is singleton, it does not contribute to the sign of T. However, the parity of the combination of B and C differs from that of C. It follows that T and T' have different signs. Let B' be the new block in T' obtained by combining B and C in T. Notice that B' is the critical block of T'. If we split B' into a singleton block and the rest as another block, then T is recovered. Hence, f is a sign-reversing involution for the case when B is a singleton.

It remains to consider the case when B is not a singleton. However, the argument for this case is simply the reverse of that for the first case. □

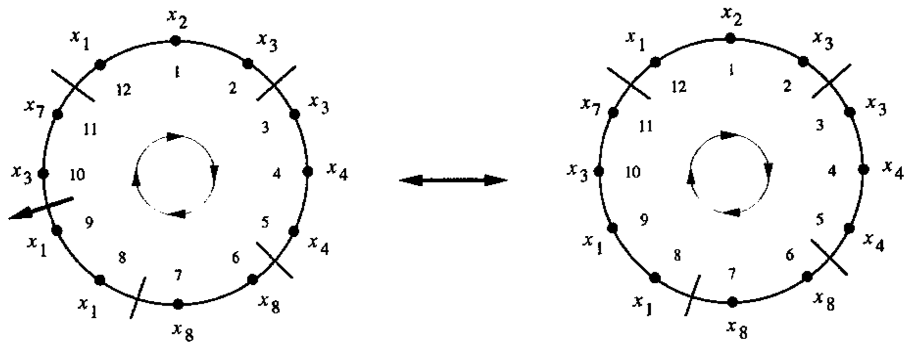


Figure 3. Involution (P-H) for nonuniform tableaux.

we notice that the Involution (P-H) consists of two involutions: the first part is an involution on the set of uniform nondegenerate standard tableaux; the second part is an involution on the set of nonuniform standard tableaux (note that a nonuniform standard tableau is always nondegenerate). The involution for uniform nondegenerate standard tableaux yields the identity

$$\sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+\cdots+k_n-1} \frac{n}{k_1+\cdots+k_n} \binom{k_1+\cdots+k_n}{k_1,\ldots,k_n} = 1,$$

which is also derived algebraically in [11].

A strict tableau T is degenerate if and only if it is uniform. This is why there is only one case for Involution (P-E), while Involution (P-H) deals with two cases. However, the Involutions (P-E) and (P-H) are essentially dual to each other, one involves sets whereas the other involves multisets. Similar to the case for formula (P-E), it is easy to see that Involution (P-H) leads to Waring's Formula (P-H).

4. The fundamental theorem

The fundamental theorem on symmetric functions asserts any symmetric function can be expressed in terms of the elementary symmetric functions. Equivalently, the monomial symmetric function can be expressed by the elementary symmetric functions. Let us use m_λ to denote the monomial symmetric function, where λ is an integer partition. Doubilet [7] gives an explicit formula for m_λ in terms of the elementary symmetric functions (and the homogeneous symmetric functions). His idea is to express the monomial symmetric function in terms of power sum symmetric functions, then by Waring's formula (P-E) it follows the formula (M-E). Since we have already provided a purely combinatorial treatment of Waring's formulas, it suffices to present a combinatorial interpretation of the formula (M-P). It turns out that the formula (M-P) is essentially a combinatorial statement about the Möbius function of the partition lattice. For this reason, we

It is straightforward to see that the Involution (P-E) implies Waring's formula (P-E). For a degenerate strict tableau T, the sign is clearly positive. We define the weight a tableau T as the product of the variables assigned to it. In this way, all the degenerate strict tableaux on X contribute to the power sum symmetric function p_n. Moreover, given a cycle dissection of type $1^{k_1}2^{k_2}\cdots n^{k_n}$, all the possible strict tableaux based on the given dissection would contribute to the weight sum $e_1^{k_1}e_2^{k_2}\cdots e_n^{k_n}$. Therefore, Waring's formula (P-E) follows from Lemma 3.1 and Definition 3.2.

The remaining part of this section aims at a combinatorial setup for the form (P-H), which turns out to be the dual form of the above treatment of (P-E). A tableau T is called *uniform* if all the variables assigned to T are the same. We take the following parallel steps to reach our objective for (P-H).

- First, we define a *standard* tableau on X to be a tableau on X in which the variables assigned to each block are nondecreasing along the direction of the cycle. The reason for this definition is obvious since we have in mind the homogeneous symmetric functions h_n.
- Second, the sign of a standard tableau based on a dissection of type $1^{k_1}2^{k_2}\cdots n^{k_n}$ is defined to be $(-1)^{k_1+\cdots k_n-1}$.
- Third, we should take special care for degenerate standard tableaux which contribute to p_n. A standard tableau T is said to be degenerate if it is uniform and has only one block with a cutting bar between position 1 and position n.
- Fourth, the critical position and the critical block of a nonuniform tableaux T are defined in the same way as those for strict tableaux.

We now describe the involution for (P-H). Given any nondegenerate standard tableaux T, we define the map $g\colon T \to T'$ by the following operations.

DEFINITION 3.3 (Involution (P-H)). *First, for the case when T is uniform:*

- *If there is a cutting bar between position 1 and position n in T, then remove that cutting bar.*
- *If there is no cutting bar between position 1 and position n in T, then add a cutting bar between position 1 and position n.*

Now, for the case when T is not uniform: Let B be the critical block of T.

- *If all the variables in B are the same, then T' is obtained from T by combining B with its next (along the direction of the cycle) block.*
- *If B contains at least two different variables, say some appearances of x followed by a variable y such that $x < y$, then T' is obtained from T by splitting B into two blocks by adding a cutting bar between the last x and y (strictly speaking, their host positions).*

It is not hard to see that the above map $g\colon T \to T'$ is a sign-reversing involution on the set of nondegenerate standard tableaux. Moreover, as in Figure 3 will digress on the Möbius function of the partition lattice, and then return to the formula (M-P).

The Möbius function of the partition lattice can be computed in various ways. Let us first mention an almost trivial proof. Given any set S, let $\mu(S) = (-1)^{|S|-1}(|S|-1)!$. For a permutation π on S, the cycle decomposition determines a partition of S if two elements are regarded to be in the same block when they are on the same cycle. On the other hand, given a set of k elements, there are $(k-1)!$ ways to form a cycle on this set. Moreover, $(-1)^{k-1}$ is the sign of a k-cycle in the sense of the sign of a permutation. If one writes a permutation as a sequence (or a linear order), say, on $\{1, 2, \ldots, n\}$, then exchanging 1 and 2 gives an involution on the set L_n of permutations on $\{1, 2, \ldots, n\}$ for $(n \geq 2)$, which changes the number of inversions by one. Since any permutation can be uniquely written in cycle notation, this leads to the following identity:

$$\sum_{\{B_1,\ldots,B_i\}} \mu(B_1)\cdots\mu(B_i) = 0, \tag{4.1}$$

where the summation runs over all the partitions of $\{1, 2, \ldots, n\}$. Notice that the above identity is essentially what is needed to reach the conclusion that the Möbius function of the partition lattice equals $\mu(0, S) = (-1)^{|S|-1}(|S|-1)!$.

We now present a more direct involution on the set S_n of permutations in cycle notation on $\{1, 2, \ldots, n\}$ for $n \geq 2$, and we shall apply it to the formula (M-P). The key idea is as follows: Given a cycle C with at least two elements, we can always remove one element from the cycle and then put it back in its original position. Such an element can be chosen as the one next to the minimum element in the cycle. The following is an example:

$$(4\,8\,6\,9\,5\,7) \Leftrightarrow (4\,6\,9\,5\,7)(8).$$

We now make this idea rigorous. For a permutation $\pi \in S_n$ in cycle notation, let us linearly order (from left to right) the cycles of π according to their minimum elements.

DEFINITION 4.1 (Involution). *The following operations on $\pi \in S_n$ lead to an involution on S_n for $n \geq 2$.*

- *If the last cycle (in the linear order) contains only one element, then put it back into the cycle on its left-hand side by inserting the element next to the minimum element.*
- *If the last cycle contains more than one element, then remove the element next to the minimum element in the last cycle and form a 1-cycle.*

The above two cases are obviously "involutional" to each other. It is also clear that this involution is sign-reversing. Hence the Möbius function of the partition lattice follows immediately. We may employ the above involution to give a combinatorial demonstration of the formula (M-P). Note that a monomial symmetric function may not be expressed by the elementary symmetric functions via integer coefficients. We need to work with the monomial symmetric function index by a partition of a set introduced by Doubilet [7]. Let f be a function from $\{1, 2, \ldots, n\}$ to $X = \{x_1, x_2, \ldots\}$. The kernel of f is a partition π of $\{1, 2, \ldots, n\}$ such that i and j are in the same block of π if $f(i) = f(j)$. The generating function or the weight of f is defined as

$$w(f) = \prod_{1 \le i \le n} f(i).$$

The monomial symmetric function m_π indexed by a partition π is defined as

$$m_\pi = \sum_{\ker f = \pi} w(f).$$

If π is of type $\lambda = 1^{k_1} 2^{k_2} \cdots n^{k_n}$, then m_π is related to the ordinary monomial symmetric function by

$$m_\pi = k_1! k_2! \cdots k_n! m_\lambda.$$

We may also write λ as $\lambda_1 \lambda_2 \ldots$, and use p_π to denote the product $p_{\lambda_1} p_{\lambda_2} \ldots$ of power sum symmetric functions. Then we have the following relation:

$$m_\pi = \sum_{\tau \ge \pi} \mu(\pi, \tau) p_\tau, \tag{4.2}$$

where $\tau \ge \pi$ means that τ is a partition based on the blocks of π. We now make the above identity into a combinatorial form. Let $C = \{C_1, C_2, \ldots, C_i\}$ be a permutation on the blocks of π in cycle notation. We say that a function f from $\{1, 2, \ldots, n\}$ to X is consistent with C if for any two elements i and j belonging to the blocks of π which are in the same block of C (note that this does not necessarily mean i and j are in the same block of π), we have $f(i) = f(j)$. The sign of C is defined in the usual way. Then (4.2) translates into

$$m_\pi = \sum_C \operatorname{sign}(C) \sum_f w(f), \tag{4.3}$$

where C ranges over all permutations on the blocks of π in cycle notation and f ranges over all functions consistent with C. The desired involution is built on the set of configurations (C, f) such that $\ker f > \pi$. For such a configuration, let x_i be the smallest variable that is assigned to at least two blocks of π. When we apply Definition 4.1 to cycles associated with variable x_i, we obtain the desired sign-reversing involution. When π is the partition of $\{1, 2, \ldots, n\}$ in which every block contains only one element, e_π reduces to $n!e_n$. Thus, from (4.2) it follows Waring's formula (E-P):

$$e_\pi = \sum_{\sigma \le \pi} \mu(0,\sigma) p_\sigma, \tag{4.4}$$

where π is of type $\lambda_1\lambda_2\cdots$ and

$$e_\pi = \lambda_1!\lambda_2!\cdots e_{\lambda_1}e_{\lambda_2}\cdots.$$

The dual form of (4.4) is the following:

$$h_\pi = \sum_{\sigma \le \pi} |\mu(0,\sigma)| p_\sigma, \tag{4.5}$$

where

$$h_\pi = \lambda_1!\lambda_2!\cdots h_{\lambda_1}h_{\lambda_2}\cdots.$$

A combinatorial interpretation of (4.5) has been given in [1] based on the notion of dispositions. Combining the involution for the formula (M-P) and the involution (P-E), one obtains a purely combinatorial treatment of the fundamental theorem on symmetric functions (M-E), whose algebraic expression has been given by Doubilet [7]:

$$m_\pi = \sum_{\tau \ge \pi} \frac{\mu(\pi,\tau)}{\mu(0,\tau)} \sum_{\sigma \le \tau} \mu(\sigma,\tau) e_\sigma. \tag{4.6}$$

Similarly, one obtains a purely combinatorial treatment of the formula (M-H):

$$m_\pi = \sum_{\tau \ge \pi} \frac{\mu(\pi,\tau)}{|\mu(0,\tau)|} \sum_{\sigma \le \tau} \mu(\sigma,\tau) h_\sigma. \tag{4.7}$$

5. A remark

We remark that this work was motivated by Waring's formula (P-E) in two variables [11, 12]:

$$x^n + y^n = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (x+y)^{n-2k}(xy)^k. \tag{5.1}$$

The above identity is related to Chebyshev polynomials [11, 12]:

$$T_n(x) = \frac{1}{2} \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}. \tag{5.2}$$

The connection between Chebyshev polynomials and Waring's formula has been extensively studied in number theory where Dickson polynomials are used as a two variable variation of Chebyshev polynomials [11]. It is well known that the above coefficient

$$\frac{n}{n-k}\binom{n-k}{k} \tag{5.3}$$

has a classical combinatorial interpretation: the number of ways of choosing k nonconsecutive elements out of n elements arranged on a cycle [13]. In the perspective of cycle dissection, (5.3) is viewed as the number of type $1^{n-2k}2^k$ dissections of an n-cycle. Our efforts then started with a combinatorial explanation of the identity (5.1).

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References

1. F. Bonetti, G. C. Rota, D. Senato, and A. M. Venezia, On the foundation of combinatorial theory, X, A categorical setting for symmetric functions, *Stud. Appl. Math.* 86:1–29 (1992).
2. F. Bonetti, G.-C. Rota, A. M. Venezia, and D. Senato, Symmetric functions and symmetric species, *Ann. Discrete Math.* 30:107–114 (1986).
3. W. Y. C. Chen, Context-free grammars, differential operators, and formal power series, *Theoret. Comp. Sci.* 117:113–129 (1993).
4. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, Oxford, 1979.
5. G.-C. Rota, Baxter algebras and combinatorial identities, Parts I and II, *Bull. Amer. Math. Soc.* 75:325–329, 330–334 (1969).
6. J. D. Louck, Trace formulas for a rigid asymmetric rotator-type Hamiltonian, *J. Molecular Spectroscopy* 10:263–277 (1963).
7. P. Doubilet, On the foundations of combinatorial theory VII: Symmetric functions through the theory of distribution and occupancy, *Stud. Appl. Math.* 51:377–396 (1972).
8. G. E. Andrews, *The Theory of Partitions, Encyclopedia of Mathematics and Its Applications*, Vol. 2 (G.-C. Rota, Ed.), Addison–Wesley, 1976.
9. N. Metropolis, G. Nicoletti, and G.-C. Rota, A new class of symmetric functions, *Adv. in Math, Suppl. Stud.* 7B:563–575 (1981).
10. I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, Wiley, New York, 1983.
11. Y. H. Tang, *The Binomial Theorem and Leibniz's Theorem—Generalizations and Equivalent Forms* (in Chinese), Hunan Univ. Press, China, 1989.

12. R. LIDL, G. L. MULLEN, and G. TURNWALD, *Dickson Polynomials, Pitman Monographs and Surveys in Pure and Applied Mathematics*, Vol. 65, Wiley, New York, 1993.

13. R. P. STANLEY, *Enumerative Combinatorics*, Vol. I, Wadsworth and Brooks/Cole, Monterey, California, 1986.

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