A Nim-like Game and Dynamic Recurrence Relations

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The nim-like game ⟨n, f; X, Y⟩ is defined by an integer $n \geq 2$, a constraint function f, and two players X and Y. Players X and Y alternate taking coins from a pile of n coins, with X taking the first turn. The winner is the one who takes the last coin. On the kth turn, a player may remove t_k coins, where $1 \leq t_1 \leq n-1$ and $1 \leq t_k \leq \max\{1, f(t_{k-1})\}$ for $k > 1$.

Let the set $S_f = \{1\} \cup \{n |$ there is a winning strategy for Y in the nim-like game $\langle n, f; X, Y\rangle\}$. In this paper, an algorithm is provided to construct the set $S_f = \{a_1, a_2, \ldots\}$ in an increasing sequence when the function $f(x)$ is monotonic. We show that if the function $f(x)$ is linear, then there exist integers n_0 and m such that $a_{n+1} = a_n + a_{n-m}$ for $n > n_0$, and we give upper and lower bounds for m (dependent on f). A duality is established between the asymptotic order of the sequence of elements in S_f and the degree of the function $f(x)$. A necessary and sufficient condition for the sequence $(a_0, a_1, a_2, \ldots)$ of elements in S_f to satisfy a regular recurrence relation is described as well.

1. Introduction

One of the oldest and most popular games in the world is the Chinese game of fan-tan [1], better known as the Chinese game of nim. In this paper, we introduce

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a nim-like game played with one pile of coins.

The nim-like game ⟨n, f; X, Y⟩ is defined by an integer $n \geq 2$, a constraint function f, and two players X, Y. Players X and Y alternate taking coins from a pile of n coins, with X taking the first turn. The winner is the one who takes the last coin. On the kth turn, a player may remove t_k coins, where $1 \leq t_1 \leq n-1$ and $1 \leq t_k \leq \max\{1, f(t_{k-1})\}$ for $k > 1$.

Let $[x]$ be the greatest integer $\leq x$. Throughout this paper, because the number t_k is obviously a positive integer, any function $f(x)$ should be replaced by the function $f^*(x)$, where

$$f^*(x) = \begin{cases} 1, & \text{if } f(x) \leq 1; \\ [f(x)], & \text{if } f(x) > 1. \end{cases}$$

For convenience, without wishing to cause confusion, we use $f(x)$ in this paper.

The Fibonacci nim game, $\langle n, 2x; A, B\rangle$, was invented by Dr. R. E. Gaskell of Oregon State University [2] and has been discussed by many mathematicians [3]–[9]. While playing the nim-like game $\langle n, f; X, Y\rangle$, we let the set $S_f = \{1\} \cup \{n \mid$ there is a winning strategy for Y in the nim-like game $\langle n, f; X, Y\rangle\}$.

EXAMPLE 1. *Let $f(x) = c$. Then $S_f = \{1, 2, \ldots, c, c+1\}$. If we add the requirement that $t_1 \leq c$, then we obtain the well-known game of nim for which $S_f = \{1, 2, \ldots, c\} \cup \{j(c+1) \mid j \in \mathrm{N}\}$.*

A. J. Schwenk [8] studied the set S_f when $f(x) = \alpha x$, and R. J. Epp and T. S. Ferguson [10] continued this study when $f(x)$ is a nondecreasing function. E. Berlekamp et al. [3] showed that the sequence $\{a_n\}$ of all elements in S_f satisfies the recurrence relation

$$a_{n+1} = a_n + a_{n-m}$$

for sufficiently large n, when $f(x) = \alpha x$, but they did not establish any relation α and m.

In Section 2 of this paper, an algorithm for constructing the set $S_f = \{a_1, a_2, \ldots\}$ in increasing order is provided. In Section 3 we prove that integers n_0 and m exist such that $a_{n+1} = a_n + a_{n-m}$ for $n > n_0$ when the function $f(x)$ is linear. Furthermore, upper and lower bounds for m (depending on f) are given. In Section 4 we study the duality between the asymptotic order of the sequence of elements in S_f and the degree of the function $f(x)$. A necessary and sufficient condition for the sequence $(a_0, a_1, a_2, \ldots)$ of elements in S_f to satisfy a regular recurrence relation is provided in Section 5.

2. Construction of S_f

The number of coins a player takes in the last turn, while playing the game $\langle n, f; X, Y\rangle$, is denoted either Last(X) or Last(Y), depending on which player wins. In this section we show how to construct S_f for the game $\langle n, f; X, Y\rangle$ when $f(x)$ is a monotonic function. We first study the case when $f(x)$ is an increasing function.

THEOREM 2.1. *Let $f(x)$ be an increasing function. Then S_f can be constructed by the following algorithm.*

Step 1. Set $a_1 = 1$.
Step 2. *If $a_1, \ldots, a_k \in S_f$, and if $T_k = \{s \mid s \leq k$ and $f(a_s) \geq a_k\} \neq \emptyset$, then set*

$$a_{k+1} = a_k + a_{\mu(k)}$$

where $\mu(k) = \min T_k$. Add a_{k+1} to S_f, and repeat Step 2 until $T_k = \emptyset$.
Step 3. *If $a_1, \ldots, a_k \in S_f$, and if $T_k = \emptyset$, then $S_f = \{a_1, \ldots, a_k\}$.*

If $n \notin S_f$, then there exists a strategy for the first player A to win the game $\langle n, f; A, B\rangle$.

Proof: We have the following three cases to consider: (1) $n = a_{m+1}$ for some $m \geq 1$; (2) $n = a_m + u$ and $u < a_{\mu(m)}$; and (3) $n = a_m + u$, and $\mu(m)$ is not defined, i.e., $f(a_m) < a_m$. In each case, we find a strategy for the winner W (A or B) of the game $\langle n, f; A, B\rangle$ with $Last(W) \leq a_m$. ■

We consider these cases separately.

Case 1: If $m = 1$, then $n = 2$. A must take one coin on the first play and B takes the remaining one. Hence the statement for case 1 is true, since $a_1 = 1$. Suppose Case 1 is true for $1 \leq k \leq m$. If $k = m+1$, then $n = a_{m+1} = a_m + a_{\mu(m)}$. Suppose A first takes t_1 coins, $1 \leq t_1 \leq n-1$. There are two possibilities: (a) $t_1 \geq a_{\mu(m)}$, or (b) $t_1 < a_{\mu(m)}$.

(a) If $t_1 \geq a_{\mu(m)}$, B can remove all the remaining coins since $f(t_1) \geq f(a_{\mu(m)}) \geq a_m = n - a_{\mu(m)} \geq n - t_1$. Thus, B wins $\langle a_{m+1}, f; A, B\rangle$ with $Last(B) = t_2 = n - t_1 \leq a_m$.
(b) If $t_1 < a_{\mu(m)}$, then by the induction hypothesis, B will first win the short game $\langle a_{\mu(m)}, f; A, B\rangle$ with $Last(B) \leq a_{\mu(m)-1}$. A and B will then continue to play $\langle a_m, f; A, B\rangle$ since $f(a_{\mu(m)-1}) < a_m$. By induction, it follows that B will win $\langle a_m, f; A, B\rangle$ with $Last(B) \leq a_{m-1} < a_m$.

Case 2: Here we also have two possibilities: (a) $n = a_m + a_i$, where $\mu(m) > i$; or (b) $n = a_{m_1} + a_{m_2} + \cdots + a_{m_r}$, where $\mu(m_i) > m_{i+1}$, $i = 1, 2, \ldots (r-1)$.

(a) A can first take a_i coins. Then players B and A will finish the game $\langle a_m, f; B, A\rangle$ (note that the order of players A and B is reversed) since $f(a_i) < a_m$. As proven in Case 1, A will win the game $\langle n, f; A, B\rangle$, with $Last(A) \leq a_m$.

(b) A can start by taking a_{m_r} coins. Since $a_{m_r} < a_{\mu(m_{r-1})}$, A will win the short game $\langle a_{m_{r-1}}, f; B, A\rangle$ with $Last(A) \leq a_{m_{r-1}}$. Then A and B will continue playing the short game $\langle a_{m_i}, f; B, A\rangle$. $Last(A)$ from the previous short game $\langle a_{m_{i+1}}, f, B, A\rangle$ is $\leq a_{m_{i+1}}$, so $f(Last(A)) \leq f(a_{m_{i+1}}) < a_{m_i}$, $i = (r-2), (r-3), \ldots, 1$ Thus, using Case 1, we have proven Case 2.

Case 3: Again we have two possibilities: (a) $u < a_m$; or (b) $u \geq a_m$.

(a) A first takes $t_1 = u$ coins. A and B then continue to play the game $\langle a_m, f; B, A\rangle$ since $f(u) \leq f(a_m) < a_m$. Case 1 implies that A will win the game $\langle n, f; A, B\rangle$ with $Last(A) \leq a_m$.

(b) Let us write $n = ka_m + r$ where $1 \leq r \leq a_m$. A will first take $t_1 = r$ coins. Then A and B will play the short game $\langle a_m, f; B, A\rangle k$ times since $f(r) \leq f(a_m) < a_m$. So Case 3 follows from Case 1 as well.

The following corollaries are obvious from the description of the algorithm in theorem 2.1.

COROLLARY 2.1. *Let T_k, $\mu(k)$ and S_f be defined as above. The following statements are equivalent.*

(i) *The set $T_k \neq \emptyset$.*
(ii) *$\mu(k)$ is well defined.*
(iii) *The set S_f contains at least $k+1$ elements.*

Furthermore, $\mu(n) \leq n$ for all n such that $\mu(n)$ is well defined.

COROLLARY 2.2. *If $f(x)$ is an increasing function and $f(n) < n$ for n sufficiently large, then S_f is a finite set.*

EXAMPLE 2. *If $f(x) = 2x$, then S_f is the set of all Fibonacci numbers, i.e., $S_f = \{1, 2, 3, 5, 8, 13, \ldots\}$.*

EXAMPLE 3. *If $f(x) = x$, then $S_f = \{2^n \mid n \in \mathbf{N} \cup \{0\}\}$.*

EXAMPLE 4. *If $f(x) = \frac{x}{2} + 1$, then $S_f = \{1, 2, 4\}$.*

In the following theorem, we study the set S_f when $f(x)$ is a decreasing function.

THEOREM 2.2. *Let $f(x)$ be a decreasing function and let $U = \{s + f(s) \mid s \in \mathbf{N}\}$. Then $S_f = \{1, 2, \ldots, t\}$, where $t = \min U$.*

Proof: We consider the following two cases: (i) $t \geq n$; (ii) $n > t$. ■

Case 1: Suppose A first takes t_1 coins from the pile where $1 \le t_1 \le n - 1$. B can remove all of the remaining coins, since $f(t_1) \ge t - t_1 \ge n - t_1$.

Case 2: Let $V = \{s \mid s + f(s) = t\}$ and let $k = \min V$, so $k + f(k) = t$. Let $n = t + \omega$ where $\omega \in \mathbf{N}$. Supopse that player A first takes $k + \omega - 1$ coins. Then B's first play on the short game $\langle t - k + 1, f;\, B, A\rangle$ cannot take all remaining coins because $t - k + 1 = f(k) + 1 > f(k + \omega - 1)$ (since $f(x)$ is a decreasing function). So players A and B continue to play the short game $\langle t - k + 1, f;\, B, A\rangle$. As proven in Case 1, A will win the game $\langle n, f;\, A, B\rangle$.

3. The limit value of $n - \mu(n)$

We now study the asymptotic behavior of the sequence $n - \mu(n)$ when S_f is an infinite set. Let $f(x) = \alpha x + \beta$ where either $\alpha < 1$, or $\alpha = 1$ and $\beta < 0$. From Corollary 2.2, we know that the set S_f must be finite, so we do not discuss this case further. That is, we consider a linear function $f(x) = \alpha x + \beta$; it should be assumed that either $\alpha > 1$ or $\alpha = 1$ and $\beta \ge 0$.

If $f(1), f(2) < 2$, then $S_f = \{1, 2\}$ since the first player A will win the game by taking either 1 or 2 coins in his first move, depending on whether n is odd or even. Only $f(x)$ with $f(2) \ge 2$ is discussed in the rest of this paper.

In this section we prove that integers n_0 and m exist such that $a_{n+1} = a_n + a_{n-m}$ for $n > n_0$ when the functions $f(x)$ is linear, and we give upper and lower bounds for m. We study the set S_f in the following two cases separately: (i) $\alpha = 1$ and $\beta \ge 0$; and (ii) $\alpha > 1$. The following fact is used in the proof for case (i):

Fact 1: For a linear function $f(x) = \alpha x + \beta$, $\mu(n)$ is well defined for all n and

$$\lim_{n\to\infty} a_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \mu(n) = \infty.$$

PROPOSITION 3.1. *Let* $f(x) = x + \beta$ *where* $\beta \ge 0$. *Then* $\lim_{n\to\infty}(n - \mu(n)) = 0$.

Proof: Fact 1 tells us that a sufficiently large $N > 0$ exists such that $a_{\mu(n-1)} > \beta$ for all $n \ge N$. This implies that

$$a_n = a_{n-1} + a_{\mu(n-1)} > a_{n-1} + \beta = f(a_{n-1}).$$

Since $f(a_{n-1}) < a_n$, it follows from the definition of $\mu(n)$ that $\mu(n) > n - 1$, so $\mu(n) = n$ for all $n > N$. ∎

To study case (ii), we need the following definition and facts:

DEFINITION 3.1. *The sequence* $p(i, n)$ *is defined as follows: Let* $p(1, n) = \mu(n) - 1$ *and* $p(i + 1, n) = p(1, p(i, n)) = \mu(p(i, n)) - 1$ *if* $p(i, n)$ *is well defined and* > 1.

Fact 2: If i, n are such that $p(i+1, n)$ is well defined, then

$$a_{p(i,n)+1} = a_{p(i,n)} + a_{p(i+1,n)+1} \text{ and } a_{\mu(p(i,n))} = a_{\mu(p(i+1,n))} + a_{p(i+1,n)}.$$

Fact 3: For any $i \in \mathbf{N}$, there exists M_i such that $p(i, n)$ is well defined for all $n \geq M_i$. Furthermore, for any fixed i, we have

$$\lim_{n\to\infty} p(i, n) = \infty.$$

For any fixed n, $p(i, n)$ is a strictly decreasing function of i, since $p(i+1, n) < \mu(p(i, n)) \leq p(i, n)$.

LEMMA 3.1. *Let $f(x) = \alpha x + \beta$ where $\alpha > 1$. Then*

$$L(\alpha) \leq \underline{\lim}_{n\to\infty} [n - \mu(n)] \leq \overline{\lim}_{n\to\infty} [n - \mu(n)] \leq U(\alpha),$$

where $L(\alpha) = (\log \alpha / \log \frac{\alpha}{\alpha-1}) - 1$ *and* $U(\alpha) = (\log \alpha)/(\log \frac{\alpha+1}{\alpha})$.

Proof: Since $p(1, n) = \mu(n) - 1$, Fact 1 implies that given any $\epsilon > 0$ there exists $N > 0$ such that

$$|\beta| < \epsilon a_{p(1,m)} < \epsilon a_{\mu(m)} \tag{3.1}$$

for $m \geq N$. By the definition of $\mu(m)$, we have $a_m \leq f(a_{\mu(m)}) = \alpha a_{\mu(m)} + \beta$. This implies that

$$\frac{1}{\alpha}(a_m - \beta) \leq a_{\mu(m)}.$$

Therefore,

$$\frac{a_{m+1}}{a_m} = \frac{a_m + a_{\mu(m)}}{a_m} \geq \frac{a_m + \frac{1}{\alpha}(a_m - \beta)}{a_m} \geq 1 + \frac{1}{\alpha} - \epsilon.$$

Now we choose n such that $\mu(n) \geq N$. By (3.1) and the definition of $\mu(n)$,

$$a_n \leq f(a_{\mu(n)}) = \alpha a_{\mu(n)} + \beta \leq \alpha a_{\mu(n)} + \epsilon a_{\mu(n)},$$

so consequently

$$\alpha + \epsilon \geq \frac{a_n}{a_{\mu(n)}} = \frac{a_n}{a_{n-1}} \frac{a_n}{a_{n-2}} \cdots \frac{a_{\mu(n)+1}}{a_{\mu(n)}} \geq \left(1 + \frac{1}{\alpha} - \epsilon\right)^{n-\mu(n)}$$

for $\mu(n) \geq N$. It follows easily that

$$\overline{\lim_{n\to\infty}}(n - \mu(n)) \leq \log\alpha / \log\left(\frac{\alpha+1}{\alpha}\right).$$

Now let us turn to the other inequality. We first have $f(a_{p(1,m)}) < a_m$ if $p(1, m)$ is well defined; this fact, taken along with (3.1), implies that

$$a_{p(1,m)} < \frac{1}{\alpha}(a_m - \beta) < a_m\left(\frac{1}{\alpha} + \epsilon\right)$$

for sufficiently large m. Since $p(i+1, n) = \mu(p(i, n)) - 1$, we have by induction

$$a_{p(i,n)} < a_n\left(\frac{1}{\alpha} + \epsilon\right)^i. \tag{3.2}$$

Since $\alpha > 1$, we can choose ϵ sufficiently small so that $\frac{1}{\alpha} + \epsilon < 1$. Then there exists $k > 0$ such that $(\frac{1}{\alpha} + \epsilon)^k < \epsilon$. Using Fact 3, we can also choose n sufficiently large such that $p(k, n)$ is well defined and $p(k, n) \geq N$. Then using Fact 2, we have

$$\begin{aligned} a_{n+1}/a_n &= (a_n + a_{p(1,n)+1})/a_n \\ &= (a_n + a_{p(1,n)} + a_{p(2,n)+1})/a_n \\ &= \cdots = (a_n + a_{p(1,n)} + \cdots + a_{p(k-1,n)} + a_{p(k,n)+1})/a_n. \end{aligned}$$

Since $a_{m+1} \leq 2a_m$ for all m, using (3.2), we have

$$\begin{aligned} a_{n+1}/a_n &\leq (a_n + a_{p(1,n)} + \cdots + a_{p(k-1,n)} + 2a_{p(k,n)})/a_n \\ &< 1 + \left(\frac{1}{\alpha} + \epsilon\right) + \cdots + \left(\frac{1}{\alpha} + \epsilon\right)^{k-1} + 2\left(\frac{1}{\alpha} + \epsilon\right)^k \\ &< \frac{1}{1 - \left(\frac{1}{\alpha} + \epsilon\right)} + \left(\frac{1}{\alpha} + \epsilon\right)^k < \frac{\alpha}{\alpha(1-\epsilon) - 1} + \epsilon. \end{aligned}$$

By (3.1), we have $|\beta| < \epsilon\alpha a_{p(1,n)} < \epsilon\alpha a_n$, hence

$$\begin{aligned} \alpha - \epsilon &= \frac{\alpha a_{p(1,n)} - \epsilon a_{p(1,n)}}{a_{p(1,n)}} \\ &< \frac{\alpha a_{p(1,n)} + \beta}{a_{p(1,n)}} < \frac{a_n}{a_{p(1,n)}} \\ &= \frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_{\mu(n)}}{a_{\mu(n)-1}} \\ &< \left(\frac{\alpha}{\alpha(1-\epsilon) - 1} + \epsilon\right)^{n-\mu(n)+1}. \end{aligned}$$

Thus we have proven that

$$\varliminf_{n\to\infty} (n - \mu(n)) \geq \left(\log\alpha / \log\frac{\alpha}{\alpha - 1}\right) - 1.$$

■

COROLLARY 3.1. *$\lim_{n\to\infty}(n - \mu(n)) = 0$ when $f(x) = \alpha x + \beta$ with $1 \leq \alpha < (1 + \sqrt{5})/2$.*

Proof: $U(\alpha) = (\log\alpha)/(\log\frac{\alpha+1}{\alpha}) < 1$ if $\alpha < \frac{\alpha+1}{\alpha}$, i.e. if $\alpha^2 - \alpha - 1 < 0$. Since $n - \mu(n)$ is always a non-negative integer, it follows that the limit is 0 for $1 \leq \alpha < (1 + \sqrt{5})/2$. ■

EXAMPLE 4. *The following table provides the limit value of $n - \mu(n)$ when $f(x) = \alpha x + \beta$, where $\alpha = 2, 3, \ldots, 10$ and $\beta = -1, 0, 1, 2,$ and 3.*

$\alpha =$	2	3	4	5	6	7	8	9	10
$\beta = -1$	0	2	5	7	10	13	15	19	22
$\beta = 0$	1	3	5	7	10	13	16	19	22
$\beta = 1$	1	3	6	8	10	13	16	19	22
$\beta = 2$	1	3	6	8	11	14	16	19	22
$\beta = 3$	1	3	6	8	11	14	17	20	23

DEFINITION 3.2. *The sequences $g(i, n)$ and $Aux(n)$ are defined as follows:*

(1) *Let $Aux(n) = \alpha a_{\mu(n)} + \beta - a_n = f(a_{\mu(n)}) - a_n$.*
(2) *Let $g(1, n) = \mu(n)$. If $g(i, n)$ is well defined and > 2, then*

$$g(i + 1, n) = \mu(g(i, n) - 1).$$

Fact 4: For any $i \in \mathbf{N}$, there exists M_i such that $g(i, n)$ is well defined for all $n \geq M_i$. Furthermore, for any fixed i, we have $\lim_{n\to\infty} g(i, n) = \infty$. For any fixed $n \geq M_i$, $g(i, n)$ is a strictly decreasing function of i, since $g(i + 1, n) = \mu(g(i, n) - 1) \leq g(i, n) - 1 < g(i, n)$.

Fact 5: For i, n such that $g(i + 1, n)$ is well defined, we have

$$a_{g(i+1,n)} = a_{g(i,n)} - a_{g(i,n)-1}.$$

PROPOSITION 3.2. *Let $f(x) = \alpha x + \beta$ where $\alpha > 1$. Then $\lim_{n\to\infty}(n - \mu(n))$ exists.*

Proof: We consider two cases separately: (1) $\beta > 0$, and (2) $\beta \leq 0$. ■

Case 1: Suppose $\mu(n) < \mu(n+1)$ for any sufficiently large n. Then the proposition holds, since $n - \mu(n)$ is a decreasing and bounded function.

Otherwise, $\mu(n) = \mu(n+1)$ for some arbitrarily large n. Let

$$L = \max_{n\geq 1}(n - \mu(n)) + 1 \text{ and } M = \max_{1\leq n\leq 1+L} Aux(n).$$

Using Fact 4, there must exist $l > 2M/\beta$ and N such that

(a) $\mu(N) = \mu(N+1)$;
(b) $g(l, N)$ is well defined with $g(l, N) \leq 1 + L$.

Let $c = Aux(N)$. We show that

$$Aux(g(l, N)) \geq c + l\beta \geq 2M, \tag{3.3}$$

which contradicts the definition of M and thereby proves Case 1.

The proof is by induction. In fact, we prove two statements: $Aux(g(k, N)) > c + k\beta$, and $g(k+1, N) = \mu(g(k, N)$ for all $k \geq 1$.

If $k = 1$, then by Fact 5 we have

$$\begin{aligned} f(a_{g(2,N)}) &= \alpha a_{g(2,N)} + \beta \\ &= \alpha(a_{g(1,N)} - a_{g(1,N)-1}) + \beta \\ &= f(a_{g(1,N)}) - \alpha a_{g(1,N)-1}. \end{aligned}$$

We know that $c = f(a_{\mu(N+1)}) - a_{N+1}$, so by the definition of μ and the fact that $\mu(N+1) = \mu(N)$, we have

$$c + a_{N+1} = f(a_{\mu(N+1)}) = f(a_{\mu(N)}) = f(a_{g(1,N)}).$$

Therefore

$$\begin{aligned} f(a_{g(2,N)}) &= c + a_{N+1} - \alpha a_{g(1,N)-1} \\ &= c + a_{\mu(N)} + (a_N - \alpha a_{g(1,N)-1}) \\ &= c + a_{g(1,N)} + (a_N - \alpha a_{\mu(N)-1}) \\ &\geq c + a_{g(1,N)} + \beta, \end{aligned}$$

since $a_N > \alpha a_{\mu(N)-1} + \beta$. This implies $f(a_{g(2,N)}) > a_{g(1,N)}$, which in turn implies $g(2, N) \geq \mu(g(1, N))$, which implies $g(2, N) = \mu(g(1, N))$. Therefore $Aux(g(1, N)) = f(a_{\mu(g(1,N))}) - a_{g(1,N)} = f(a_{g(2,N)}) - a_{g(1,N)} \geq c + \beta$, so the two statements are true for the case $k = 1$.

Now suppose that the lemma holds for $k = 1, \ldots, m-1$. The proof for $k = m$ is very similar to the proof for 1.

By Fact 5 we have

$$\begin{aligned} f(a_{g(m+1,N)}) &= \alpha a_{g(m+1,N)} + \beta \\ &= \alpha(a_{g(m,N)} - a_{g(m,N)-1}) + \beta \\ &= f(a_{g(m,N)}) - \alpha a_{g(m,N)-1}. \end{aligned} \tag{3.4}$$

By the induction hypothesis for $m - 1$, we know that

$$\begin{aligned} f(a_{g(m,N)} &= f(a_{\mu(g(m-1,N))}) \\ &= Aux(g(m-1, N)) + a_{g(m-1,N)}. \\ &\geq c + (m-1)\beta + a_{g(m-1,N)}. \end{aligned}$$

From (3.4) and Fact 5, we obtain

$$\begin{aligned} f(a_{g(m+1,N)}) &\geq c + (m-1)\beta + a_{g(m-1,N)} - \alpha a_{g(m,N)-1} \\ &\geq c + (m-1)\beta + a_{g(m,N)} + a_{g(m-1,N)-1} - \alpha a_{g(m,N)-1} \\ &\geq c + m\beta + a_{g(m,N)}. \end{aligned} \tag{3.5}$$

By the definitions of μ and g, we have

$$a_{g(m-1,N)-1} \geq f(a_{g(m,N)-1}) = \alpha a_{g(m,N)-1} + \beta.$$

This implies $f(a_{g(m+1,N)}) > a_{g(m,N)}$, which in turn implies $g(m+1, N) \geq \mu(g(m, N))$, which implies $g(m+1, N) = \mu(g(m, N))$. Using (3.5), we have $Aux(g(m, N)) = f(a_{\mu(g(m,N))}) - a_{g(m,N)} = f(a_{g(m+1,N)}) - a_{g(m,N)} > c + m\beta$, and (3.3) is proved.

Case 2: We only have to prove that $n - \mu(n)$ is an increasing function, since $n - \mu(n)$ is a bounded function by Lemma 3.1. We have $a_{n+1} = a_n + a_{\mu(n)}$ for all n, and $\beta \leq 0$, then

$$\begin{aligned} f(a_{\mu(n)+1}) &= \alpha a_{\mu(n)+1} + \beta \\ &= \alpha(a_{\mu(n)} + a_{\mu(\mu(n))}) + \beta \\ &\geq \alpha a_{\mu(n)} + \beta + \alpha a_{\mu(\mu(n))} + \beta \\ &> a_n + a_{\mu(n)} = a_{n+1}. \end{aligned}$$

This implies that $\mu(n) + 1 \geq \mu(n+1)$. Therefore, $n - \mu(n)$ is an increasing function.

The following theorem summarizes the results established in this section.

THEOREM 3.1. *Let* $f(x) = \alpha x + \beta$. *Then*

(i) *if* $\alpha = 1$, *and* $\beta \geq 0$, *then* $\lim_{n\to\infty}(n - \mu(n)) = 0$;
(ii) *if* $\alpha > 1$, *then* $\lim_{n\to\infty}(n - \mu(n))$ *exists and*

$$L(\alpha) \leq \lim_{n\to\infty}(n - \mu(n)) \leq U(\alpha)$$

where $L(\alpha) = \log\alpha/\log(\frac{\alpha}{\alpha-1}) - 1$ *and* $U(\alpha) = \log\alpha/\log(\frac{\alpha+1}{\alpha})$.

4. Duality

In this section, let $p, q > 1$ denote two positive real numbers that satisfy

$$\frac{1}{p} + \frac{1}{q} = 1.$$

LEMMA 4.1. *Let*

$$f(n) = (n+1)^q - n^q - qn^{q-1} \quad \text{and} \quad g(n) = (n+1)^q - n^q - (1-\epsilon)^{-1}qn^{q-1}$$

where $0 < \epsilon < 1$. *Then*

(a) $f(n) \geq 0$ *for all* $n > 0$, *and*
(b) *there exists* $m > 0$ *such that* $g(n) < 0$ *for all* $n > m$.

Proof: (a) $f(n) = (n+1)^q - n^q - qn^{q-1} \geq 0$ for all $n > 0$, since $n^q + qn^{q-1}$ is the sum of the first two terms in the binomial expansion of $(n+1)^q$.

(b) $0 < 1 - \epsilon < 1$, so $(1-\epsilon)^{-1} > 1$. Consequently, $(n+1)^q - n^q - (1-\epsilon)^{-1}qn^{q-1}$ is a polynomial of degree $q - 1$ with negative leading coefficient $1 - (1-\epsilon)^{-1}$. ■

The following lemma is straightforward but useful later. We omit the proof.

LEMMA 4.2. *Let* $f \geq 0$ *be an increasing integer function. Let* $(d_0, d_1, d_2, \ldots)$, $(b_0, b_1, b_2, \ldots)$, *and* $(c_0, c_1, c_2, \ldots)$ *be the sequences that satisfy the following conditions:*

$$\begin{aligned} d_{n+1} &= d_n + f(d_n), \\ b_{n+1} &\geq b_n + f(b_n), \\ c_{n+1} &\leq c_n + f(c_n). \end{aligned}$$

For all, $n > 0$. *If* $b_r \geq d_s \geq c_t$ *for some nonnegative* r, s, t, *then* $b_{n+r} \geq d_{n+s} \geq c_{n+t}$ *for all* $n \geq 0$.

LEMMA 4.3. *Choose $\alpha > 0$, $0 < \epsilon < 1$, and $a_0 > 0$. Let the sequence $(a_0, a_1, a_2, \ldots)$ be defined by the relation*

$$a_{n+1} = a_n + \gamma a_n^{1/p}$$

for all $n \geq 0$. Then there exists $k > 0$ such that

$$a_{n-k} \leq \left(\frac{\gamma}{q}\right)^q n^q \qquad \text{and} \qquad a_{n+k} \geq (1-\epsilon)^q \left(\frac{\gamma}{q}\right)^q n^q$$

for all $n > k$.

Proof: For any $n \geq 0$, $a_{n+1} - a_n = \gamma a_n^{1/p} \geq \gamma a_0^{1/p}$. Consequently $\{a_n\}$ is a strictly increasing sequence and $\lim_{n\to\infty} a_n = \infty$.

Let

$$\begin{aligned} f(n) &= (n+1)^q - n^q - qn^{q-1}, \\ g(n) &= (n+1)^q - n^q - (1-\epsilon)^{-1} qn^{q-1}, \\ b_n &= \left(\frac{\gamma}{q}\right)^q \cdot n^q, \\ c_n &= (1-\epsilon)^q \cdot \left(\frac{\gamma}{q}\right)^q (n+m)^q, \end{aligned}$$

where m is such that $g(n) < 0$ for $n < m$. One can verify that

$$b_{n+1} - b_n - \gamma b_n^{1/p} = \left(\frac{\gamma}{q}\right)^q f(n)$$

and

$$c_{n+1} - c_n - \gamma c_n^{1/p} = (1-\epsilon)^q \cdot \left(\frac{\gamma}{q}\right)^q g(n+m).$$

■

Then from Lemma 4.1 we have

$$b_{n+1} > b_n + \gamma b_n^{1/p} \qquad \text{and} \qquad c_{n+1} \leq c_n + \gamma c_n^{1/p}.$$

Given an integer k, let

$$s_n = a_{n-k} \qquad \text{and} \qquad t_n = a_{n+k} \qquad \forall n \geq k.$$

Then since $a_{n+1} = a_n + \gamma a_n^{1/p}$ for all n, we have

$$s_{n+1} = s_n + \gamma s_n^{1/p} \qquad \text{and} \qquad t_{n+1} = t_n + \gamma t_n^{1/p} \qquad \forall n \geq k.$$

Now choose an integer k large enough that

$$b_k = \left(\frac{\gamma}{q}\right)^q k^q > a_0 = s_k \qquad \text{and} \qquad t_0 = a_k > 0 = c_0.$$

Lemma 4.2 then implies that

$$b_n > s_n \qquad \text{and} \qquad t_n > c_n \qquad \forall n \geq k.$$

That is equivalent to

$$\left(\frac{\gamma}{q}\right)^q \cdot n^q > a_{n-k} \qquad \text{and} \qquad a_{n+k} \geq (1-\epsilon)^q \cdot \left(\frac{\gamma}{q}\right)^q n^q$$

for all $n \geq k$.

DEFINITION 4.1.

(a) *$f(n) \sim g(n)$ means that* $\lim_{n\to\infty} f(n)/g(n) = 1$.
(b) *$f(n) \sim o(g(n))$ means that* $\lim_{n\to\infty} f(n)/g(n) = 0$.
(c) *The asymptotic order of a sequence $\{a_n\}$ is p if $a_n \sim \alpha n^p$ where $\alpha > 0$.*

Theorem 4.1 follows immediately from Definition 4.1 and Lemma 4.3.

THEOREM 4.1. *Let $\gamma > 0$, $a_0 > 0$, and let the sequence $(a_0, a_1, \ldots)$ be defined by the relation*

$$a_{n+1} = a_n + \gamma a_n^{1/p}$$

for all $n > 0$. Then

$$n \sim \left(\frac{\gamma}{q}\right)^q n^q.$$

Now we can establish a relation between the asymptotic order of the sequence of elements in S_f and the degree of the function $f(x)$.

THEOREM 4.2. *Let f be an increasing function such that $f(x) \geq x$ for all $x > 0$. Let all the elements of S_f be listed $\{a_1, a_2, \ldots\}$ in increasing order. If $f(n) \sim \alpha n^p$ for some $\alpha > 0$, then*

$$a_n \sim \left(\frac{1}{q}\right)^q \left(\frac{1}{\alpha}\right)^{q-1} n^q.$$

Proof: According to the definition of μ, we have

$$a_n \leq f(a_{\mu(n)}) \qquad \text{and} \qquad a_n \geq f(a_{\mu(n)-1}) = f(a_{p(1,n)})$$

for all n, so

$$\lim_{n\to\infty} \frac{f(a_{\mu(n)})}{a_n} \geq 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{f(a_{p(1,n)})}{a_n} \leq 1.$$

Since $f(n) \sim \alpha n^p$, we have

$$\lim_{n\to\infty} \frac{\alpha a_{\mu(n)}^p}{a_n} \geq 1 \geq \lim_{n\to\infty} \frac{\alpha a_{p(1,n)}^p}{a_n}. \tag{4.1}$$

■

The second inequality implies that there exists some M such that for $n \geq M$, $(\frac{1}{\alpha}a_n)^{1/p} \geq a_{p(1,n)}$. It follows from this that $a_{p(1,n)} \sim \circ(a_n)$ since $1/p < 1$. Substituting $p(1,n)$ for n, we find

$$a_{p(1,p(1,n))} = a_{p(2,n)} \sim \circ(a_{p(1,n)}). \tag{4.2}$$

Since $a_{p(1,n)} \leq (\frac{1}{\alpha}a_n)^{1/p}$ for $n \geq M$, we have $a_{p(2,n)} \sim \circ((\frac{1}{\alpha}a_n)^{1/p})$.

Consequently for $n \geq M$, using the fact that $a_m \leq 2a_{m-1}$, we know that

$$\begin{aligned} a_{\mu(n)} &= a_{\mu(n)-1} + a_{\mu(\mu(n)-1)} \\ &\leq a_{\mu(n)-1} + 2a_{\mu(\mu(n)-1)-1} \\ &= a_{p(1,n)} + 2a_{p(2,n)} \\ &\leq \left(\frac{1}{\alpha}a_n\right)^{1/p} + 2a_{p(2,n)}. \end{aligned}$$

This implies that

$$\frac{a_{\mu(n)}}{(\frac{1}{\alpha}a_n)^{1/p}} \leq 1 + \frac{2a_{p(2,n)}}{(\frac{1}{\alpha}a_n)^{1/p}},$$

so $\lim_{n\to\infty} a_{\mu(n)}/(1/\alpha a_n)^{1/p} \leq 1$. Combining with the first inequality of (3.6), we have $a_{\mu(n)} \sim (1/\alpha)^{1/p}(a_n)^{1/p}$.

Since $a_{n+1} = a_n + a_{\mu(n)}$, this implies that for any two numbers s, t where $0 < t < (1/\alpha)^{1/p} < s$, there exists $k > 0$ such that

$$a_n + t a_n^{1/p} \leq a_{n+1} \leq a_n + s a_n^{1/p}$$

for all $n \geq k$. Lemma 4.3 then implies that

$$\left(\frac{t}{q}\right)^q \leq \lim_{n\to\infty} (a_n/n^q) \leq \left(\frac{s}{q}\right)^q.$$

Hence, we have

$$\lim_{n\to\infty} (a_n/n^q) = \left(\frac{(1/\alpha)^{1/p}}{q}\right)^q = \left(\frac{1}{q}\right)^q \left(\frac{1}{\alpha}\right)^{q-1}.$$

5. Dynamic recurrence relation

For a numeric function $(a_0, a_1, a_2, \ldots)$, an equation relating a_n, for any n, to $\{a_1, \ldots, a_n\}$ is called a recurrence relation. Consider the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$. This sequence can be described by the relation $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$, together with the conditions that $a_0 = 1$ and $a_1 = 1$. The coefficients of each term are always constants. A recurrence relation

$$a_{n+1} = c_1 a_n + c_2 a_{n-1} + \cdots + c_n a_1 \tag{4.3}$$

is called a regular recurrence relation (RRR) if all c_i are eventually constants; it is called a dynamic recurrence relation (DRR) if some c_i are not constants but depend on the value of n.

Let $f(x)$ be an increasing polynomial and $f(x) \geq x$ for all $x > 0$. While playing the game $\langle n, f; A, B \rangle$, all the elements of $S_f = \{a_1, a_2, \ldots\}$ can be constructed in an increasing order. According to Theorem 2.1, the sequence $\{a_n\}$ satisfies the dynamic recurrence relation (4.1) where $c_1 = 1$, $c_i = 1$ if $i = n - \mu(n) + 1$, and $c_i = 0$ if $i \neq n - \mu(n) + 1$. These coefficients are not constants but depend on the value of n. A necessary and sufficient condition for the sequence $(a_0, a_1, a_2, \ldots)$ of elements in S_f to satisfy a regular recurrence relation is described in the following theorem.

THEOREM 5.1. *Suppose $f(x)$ is an increasing polynomial and $f(x) \geq x$ for all $x > 0$. The following statements are equivalent:*

(a) *The sequence $(a_0, a_1, a_2, \ldots)$ of elements in S_f satisfies an RRR.*
(b) *The sequence $(a_0, a_1, a_2, \ldots)$ of elements in S_f grows exponentially.*
(c) *$\lim_{n\to\infty}(n - \mu(n))$ exists.*
(d) *$f(x) = \alpha x + \beta$ where $\alpha > 1$ or $\alpha = 1$ and $\beta \geq 0$.*

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