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The Cardinality of the Collection of Maximum Independent Sets of a Functional Graph

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An independent set (or stable set) of a graph $G(V, E)$ is a subset S of the vertices set V in which no two are adjacent. Let $\psi(G)$ be the number of vertices in a stable set of maximum cardinality; $\psi(G)$ is called the stability number of G. Stability numbers of a graph have been well studied, but little has been done on the number of independent subsets whose cardinality is the stability number. In this paper we will provide an algorithm to find the number of independent subsets whose cardinality is the stability number. © 1997 Academic Press

1. INTRODUCTION

Consider a graph $G = (V, E)$. A subset S of the vertex set V is an *independent set* of G if no two vertices of S are adjacent in G. An independent set is also called a stable set. An independent set S of G is called maximum if G has no independent set S' with $|S| < |S'|$. Then the number of vertices in a maximum independent set of G is called the *stability number* and is denoted by $\psi(G)$.

Many parameters associated with a graph are related to the stability number, namely rank, matching number, vertex and edge covering number [9], chromatic number, chromatic index [2], and so on. Stability numbers have many applications in such topics as information theory, extremal

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graph theory, optimization theory, and near-rings [5]. It is well known that the stable set problem is *NP-complete* and it can be solved in polynomial time if the graph G is a functional graph. Wang [11] and Bouchard and Yeh [3, 12] studied 2-free and q-free subsets and have neat results for their maximum possible cardinality. In fact, finding a f-free subset of maximum cardinality is equivalent to finding a stable set of maximum cardinality in the graph G_f (as defined below).

Throughout this paper, G_f is a functional graph, i.e., it is the graph (or digraph) $G_f = (V, E)$ of a function $f: A \to A$ where $V = A$, $E = \{\overrightarrow{af(a)} : a \in A\}$. An equivalent characterization of a functional graph is that each of its vertices has at most one subsequent vertex. Using the terminology of species, a functional graph is equal to the substitution of rooted trees into the combinatorial structure of cycles.

DEFINITION 1.1. Let $\mathscr{I}(G) = \{S \subset V : S$ is an independent subset$\}$. Given a vertex $x \in V$, let $\mathscr{I}_x(G) = \{S \subset V : S \in \mathscr{I}(G)$ and $x \in S\}$, and let $\mathscr{I}_{-x}(G) = \{S \subset V : S \in \mathscr{I}(G)$ and $x \notin S\}$. We also let $\mathscr{F}_x(G) = \{S \in \mathscr{I}(G) : x \in S, |S| = \psi(G)\}$., $\mathscr{F}_{-x}(G) = \{S \in \mathscr{I}(G) : x \notin S, |S| = \psi(G)\}$, and $\mathscr{F}(G) = \mathscr{F}_x(G) \cup \mathscr{F}_{-x}(G)$. The cardinality of the sets $\mathscr{F}_x(G)$, $\mathscr{F}_{-x}(G)$, and $\mathscr{F}(G)$ are denoted by $\phi_x(G)$, $\phi_{-x}(G)$, and $\phi(G)$, respectively.

DEFINITION 1.2. Let G_f be a functional graph with vertex set V and $x \in V$. Then

(i) x is called a *circulant point* if x lies on an n-cycle $(n \geq 2)$.

(ii) x is called a *root* if $f(x) = x$, i.e., x lies on a 1-cycle.

(iii) x is called a *terminal point* if x is not a root and $f(x)$ is a root.

(iv) x is called a *leaf* if it is an end-vertex, i.e., the indegree of x is 0.

DEFINITION 1.3. Let G_f be a functional graph with vertex set V and $x \in V$. Then x is called a *branch point* if the in-degree of x is greater than or equal to 2 and x is not a circulant point. If x is a branch point, let T_x denote the subgraph of G which terminates at x, i.e., the vertex set of T_x is $V(T_x) = \{y \in V(G) : \exists k > 0$ such that $f^k(y) = x\} \cup \{x\}$. A branch point x is a first-layer branch point, denoted by FLBP, if there are no other branch points in T_x.

DEFINITION 1.4. Let $N_0 = \{0\} \cup N = \{0, 1, 2, \ldots,\}$. Given a graph G and an associated *weight function* $w : V \to N_0 \times N_0$, which is defined by $w(x) = (C_x, N_x)\ \forall x \in V$, we define

$$\phi(w, G) = \sum_{S \in \mathscr{F}(G)} \Big(\prod_{z \in S} C_z\Big)\Big(\prod_{y \notin S} N_y\Big);$$

$$\phi_x(w,G) = \sum_{S \in \mathscr{F}_x(G)} \Big(\prod_{z\in S} C_z\Big)\Big(\prod_{y \notin S} N_y\Big);$$

$$\phi_{-x}(w,G) = \sum_{S \in \mathscr{F}_x(G)} \Big(\prod_{z\in S} C_z\Big)\Big(\prod_{y \notin S} N_y\Big);$$

where $\mathscr{F}(G)$, $\mathscr{F}_x(G)$, and $\mathscr{F}_{-x}(G)$ is defined in Definition 1.1.

The following lemmas are basic and useful.

LEMMA 1.5. *If P_n is a path of length n, then*

$$\phi(P_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 1 + \dfrac{n}{2} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Let the vertices of P_n (from left to right) be $x_1, x_2, \dots, x_n$.

If n is odd, $\psi(P_n) = \lceil n/2 \rceil = (n+1)/2$ (where $\lceil\ \rceil$ is the ceiling function), and clearly there is only one independent set $\{x_1, x_3, \dots, x_n\}$ with such maximum cardinality.

Now, suppose n is even. Clearly, $\phi(P_2) = 2$. For $n \geq 4$ and even, we have $\phi(P_n) = \phi_{x_1}(P_n) + \phi_{-x_1}(P_n) = \phi(P_{n-2}) + 1$. Iteratively, we have $\phi(P_n) = 1 + n/2$. ∎

LEMMA 1.6. *Let C_n be a cycle of n vertices. Then*

$$\phi(C_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even}. \end{cases}$$

Proof. Let x be a vertex of C_n. Then

$$\phi(C_n) = \phi_x(C_n) + \phi_{-x}(C_n) = \phi(P_{n-3}) + \phi(P_{n-1}).$$

By Lemma 1.5, we have

$$\phi(C_n) = \begin{cases} \left(1 + \dfrac{n-3}{2}\right) + \left(1 + \dfrac{n-1}{2}\right) = n & \text{if } n \text{ is odd} \\ 1 + 1 = 2 & \text{if } n \text{ is even}. \end{cases}$$ ∎

LEMMA 1.7. *Suppose G is made up of k connected components $G_1, G_2, \dots, G_k$. Then $\phi(G) = \prod_{i=1}^{k} \phi(G_i)$.*

Proof. Obvious. ∎

We adopt the convention that $\phi_x(\emptyset) = \phi_{-x}(\emptyset) = \phi(\emptyset) = 1$. We require some elementary facts about functional graphs. Given any function $f: A \to A$, the functional graph G_f is the disjoint union of a collection of components. Each component is either a tree or contains one cycle; if it contains a cycle, then any vertex of the cycle may have a tree which "flows" into it.

We can extract all roots from functional graph G_f without changing $\psi(G_f)$ or $\phi(G_f)$, since a root can never be an element of an independent set. So henceforth, for convenience, we let $G = G_f - \{\text{roots}\}$. According to Lemma 1.7, we need only concern ourselves with a connected graph G. Given the original graph G, we assume that $C_x = N_x = 1$ for all $x \in V(G)$. Hence we have $\phi(w, G) = \phi(G)$.

Let G be a connected, rootless functional graph with a weight function w. In this paper, we provide a method of calculating how many independent subsets of G whose cardinality is $\psi(G)$, i.e., evaluating the number $\phi(G)$. In Section 2, we define α-points, β-points, and γ-points for the first-layer branch points (FLBP's) of a graph G. We then have the reduced graph G', which is obtained from G by "compressing" G at all FLBP's, and w', a suitably modified weight function from the old function w such that $\phi(w, G) = \phi(w', G')$ (w' is different from w at those FLBP's of G). We also extend this result for a graph G which may contain α-, β-, or γ-points. In Section 3, if G has no circulant point (the root was extracted), then we can repeatedly apply the reducing algorithm until we end up with a point z. Otherwise, we will end up with a cycle J with modified weight function w_J. Finally, we evaluate $\phi(w_J, J)$, which is equal to $\phi(G)$.

2. α-, β, AND γ-POINTS

Notation. Let $G = (V, E)$ be a connected rootless functional graph, $x \in V$ be a *cut-vertex* if $G - \{x\}$ has more components than G. Let $B(x) = \{x\} \cup \{y \in V : \overrightarrow{xy} \in E \text{ or } \overrightarrow{yx} \in E\}$. Let $\hat{A}(x) = A \cup B(x)$ and $\check{A}(x) = A - \{x\}$ for any subgraph A of G. For convenience, let $\hat{T}(x)$ and $\check{T}(x)$ denote $\hat{T}_x(x)$ and $\check{T}_x(x)$ respectively.

DEFINITION 2.1. Let x be a FLBP of functional graph G and T_x be defined as in Definition 1.3. Then

(a) x is an α-point if $\phi_x(T_x) = 0$;

(b) x is a β-point if $\phi_x(T_x) > 0$ and $\phi_{-x}(T_x) > 0$;

(c) x is a γ-point if $\phi_{-x}(T_x) = 0$.

We will regard leaves as γ-points, since we have $\phi_{-x}(\{x\}) = 0$ if x is a leaf. Let $G = (V, E)$ be a connected rootless functional graph, and x be a FLBP of G, and T_x be defined as in Definition 1.3. Then

(a) If x is an α-point, then $\phi_x(G) = 0$, i.e., $x \notin S$ for all $S \in \mathscr{F}(G)$.

(b) If x is a β-point, then $\phi_{-x}(G) > 0$. Furthermore, if x is a terminal point, then $\phi_x(G) > 0$ as well.

(c) If x is a γ-point then $\phi_x(G) > 0$. Furthermore, if x is a terminal point, then $\phi_{-x}(G) = 0$, i.e., $x \in S$ for all $S \in \mathscr{F}(G)$.

The reader should note that if x is a γ-point but not a terminal point, then $\phi_{-x}(G)$ may or may not equal 0. Likewise, if x is a β-point but not a terminal point, then $\phi_x(G)$ may or may not be 0. We have the following important theorem.

THEOREM 2.2. *Let G be a connected rootless functional graph, and let x be a FLBP of G. Then we have*:

1. *If x is an α-point then* $\phi(G) = \phi(\check{T}_x)\phi(G - T_x)$.

2. *If x is a β-point then* $\phi(G) = \phi(\hat{T}_x)\phi_{-f(x)}(G - T_x) + \phi(\check{T}_x)\phi(G - T_x)$.

3. *If x is a γ-point then* $\phi(G) = \phi(\hat{T}_x)\phi_x(G - \check{T}_x) + \phi(\check{T}_x)\phi_{-x}(G - \check{T}_x)$.

Proof. 1. If x is an α-point, we know that $\phi_x(G) = 0$. By Lemma 1.7, any $S \in \mathscr{F}_{-x}(G)$ can be expressed as the disjoint union of S_1 and S_2, with $S_1 \in \mathscr{F}(\check{T}_x)$ and $S_2 \in \mathscr{F}(G - T_x)$. Hence we have $\phi_{-x}(G) = \phi(\check{T}_x)\phi(G - T_x)$.

2. If x is a β-point, then we have the following decompositions: any $S \in \mathscr{F}_x(G)$ can be expressed as the disjoint union of S_1 and S_2, with $S_1 \in \mathscr{F}_x(\hat{T}_x)$ and $S_2 \in \mathscr{F}_{-f(x)}(G - T_x)$, and any $S \in \mathscr{F}_{-x}(G)$ can be expressed as the disjoint union of S_1 and S_2, with $S_1 \in \mathscr{F}(\check{T}_x)$ and $S_2 \in \mathscr{F}(G - T_x)$. Hence we have $\phi(G) = \phi(\hat{T}_x)\phi_{-f(x)}(G - T_x) + \phi(\check{T}_x)\phi(G - T_x)$. Note that if G is such that $\phi_x(G) = 0$, then $\phi_{-f(x)}(G - T_x)$ is 0 as well.

3. If x is a γ-point, then we have the following decompositions: any $S \in \mathscr{F}_x(G)$ can be expressed as the disjoint union of S_1 and S_2, with $S_1 \in \mathscr{F}_x(\hat{T}_x)$ and $S_2 \in \mathscr{F}_x(G - \check{T}_x)$, and any $S \in \mathscr{F}_{-x}(G)$ can be expressed as the disjoint union of S_1 and S_2, with $S_1 \in \mathscr{F}(\check{T}_x)$ and $S_2 \in \mathscr{F}_{-x}(G - \check{T}_x)$. Hence we have $\phi(G) = \phi(\hat{T}_x)\phi_x(G - \check{T}_x) + \phi(\check{T}_x)\phi_{-x}(G - \check{T}_x)$. Note that if G is such that $\phi_{-x}(G) = 0$, then $\phi_{-x}(G - \check{T}_x)$ is 0 as well. In particular, if x is a terminal point, then our expression for $\phi(G)$ reduces to $\phi(\check{T}_x)$. ∎

Now we specify exactly how to "update" the weight function w. Let w' be the weight function on G', which is the same as w except at the FLBP's of G; we will update the weight function at the FLBP x of G:

$$w' = (C'_x, N'_x)$$

with

$$C'_x = \begin{cases} 0 & \text{if } x \text{ is an } \alpha\text{-point} \\ \phi_x(\hat{T}_x) & \text{otherwise (i.e. } x \text{ is chosen)} \end{cases}$$

$$N'_x = \begin{cases} 0 & \text{if } x \text{ is both a } \gamma\text{-point and a terminal point} \\ \phi(\check{T}_x) & \text{otherwise (i.e. } x \text{ is not chosen).} \end{cases}$$

If we assign $w' = (C'_x, N'_x)$ to each FLBP x of G, we will prove that $\phi(w, G) = \phi(w', G')$ in the following theorem:

THEOREM 2.3. *Let graphs G, G' and weight functions w, w' be defined as above. Then $\phi(w, G) = \phi(w', G')$.*

Proof. By Definition 1.4, we have

$$\begin{aligned} \phi(w', G') &= \sum_{S \in \mathscr{F}(G')} \Big(\prod_{z \in S} C'_z\Big)\Big(\prod_{y \notin S} N'_y\Big) \\ &= \sum_{S \in \mathscr{F}_x(G')} \Big(\prod_{z \in S} C'_z\Big)\Big(\prod_{y \notin S} N'_y\Big) + \sum_{S \in \mathscr{F}_{-x}(G')} \Big(\prod_{z \in S} C'_z\Big)\Big(\prod_{y \notin S} N'_y\Big). \end{aligned}$$

If x is an α-point, then we have

$$\begin{aligned} \phi(w', G') &= N'_x \sum_{S \in \mathscr{F}(G - T_x)} \Big(\prod_{z \in S} C'_z\Big)\Big(\prod_{y \notin S} N'_y\Big) \\ &= \phi(\check{T}_x)\phi(w, G - T_x) \\ &= \phi(w, G) \end{aligned}$$

by Theorem 2.2.

If x is a β-point, then we have

$$\begin{aligned} &\phi(w', G') \\ &\quad= \sum_{S \in \mathscr{F}_x(G')} \Big(\prod_{z \in S} C'_z\Big)\Big(\prod_{y \notin S} N'_y\Big) + \sum_{S \in \mathscr{F}_{-x}(G')} \Big(\prod_{z \in S} C'_z\Big)\Big(\prod_{y \notin S} N'_y\Big) \\ &\quad= C'_x \sum_{S \in \mathscr{F}_{-f(x)}(G - T_x)} \Big(\prod_{z \in S} C'_z\Big)\Big(\prod_{y \notin S} N'_y\Big) \\ &\qquad + N'_x \sum_{S \in \mathscr{F}(G - T_x)} \Big(\prod_{z \in S} C'_z\Big)\Big(\prod_{y \notin S} N'_y\Big) \\ &\quad= \phi_x(\hat{T}_x)\phi_{-f(x)}(w, G - T_x) + \phi(\check{T}_x)\phi(w, G - T_x) \\ &\quad= \phi(w, G). \end{aligned}$$

If x is a γ-point, then we have

$$\begin{aligned}
&\phi(w',G') \\
&\quad = \sum_{S\in\mathscr{F}_x(G')}\Big(\prod_{z\in S}C'_z\Big)\Big(\prod_{y\notin S}N'_y\Big) + \sum_{S\in\mathscr{F}_{-x}(G')}\Big(\prod_{z\in S}C'_z\Big)\Big(\prod_{y\notin S}N'_y\Big) \\
&\quad = C'_x \sum_{S\in\mathscr{F}_x(G-\check{T}_x)}\Big(\prod_{z\in S-\{x\}}C'_z\Big)\Big(\prod_{y\notin S}N'_y\Big)C_x \\
&\qquad + N'_x \sum_{S\in\mathscr{F}_{-x}(G-\check{T}_x)}\Big(\prod_{z\in S}C'_z\Big)\Big(\prod_{y\in S-\{x\}}N'_y\Big)N_x \\
&\quad = \phi_x\big(\hat{T}_x\big)\phi_x\big(w,G-\check{T}_x\big) + \phi\big(\check{T}_x\big)\phi_{-x}\big(w,G-\check{T}_x\big) \\
&\quad = \phi(w,G).
\end{aligned}$$

The second equality holds since $C_x = N_x = 1$. Note that as in Theorem 2.2, if x is a terminal point then the second term on the right side is 0, since $N'_x = 0$. ∎

Suppose x is a FLBP of G and $G - T_x \neq \emptyset$. Let G' be the graph obtained from G by replacing T_x by x for all FLBPs x of G. When we execute the procedure of reducing G to G' to G'' to $\dots$, the graph $G^{(i)}$ contains some α-leaves, β-leaves, or γ-leaves. Since the branch points of $G^{(i)}$ $(i = 1, 2, \dots)$ have the same look, we are facing the same problem as in G' in determining types and weights of those FLBPs of $G^{(i)}$. We require some more definitions:

DEFINITION 2.4. The *virtual* points of a weighted graph (w, G) are its α-points and β-points. Let $V^r \subset V$ be the subset of vertices which are not virtual points (so points of V^r either are an internal vertices (not leaves) or are γ-points). Let G^r be the subgraph of G which has V^r as its vertex set. Given a branch point x of the graph G, let T_x^r be the subgraph of G^r which terminates at x (note that x is not necessarily a branch point of the graph G^r).

DEFINITION 2.5. Let $\mathscr{K}_x(G)$ and $\mathscr{K}_{-x}(G)$ be defined as follows:

1. $\mathscr{K}_x(G) = \{S : S \in \mathscr{I}_x(G)$ and $|S| = \max\{|R| : R \in \mathscr{I}_x(G)\}\}$.
2. $\mathscr{K}_{-x}(G) = \{S : S \in \mathscr{I}_{-x}(G)$ and $|S| = \max\{|R| : R \in \mathscr{I}_{-x}(G)\}\}$.

$\mathscr{K}_x(G)$ is the collection of maximum independent subsets which contain x. Hence, $\mathscr{F}_x(G) \subset \mathscr{K}_x(G)$ and $\mathscr{F}_{-x}(G) \subset \mathscr{K}_{-x}(G)$. We have $\mathscr{K}_x(G) = \mathscr{F}_x(G)$ if $\mathscr{F}_x(G) \neq \emptyset$; and $\mathscr{K}_{-x}(G) = \mathscr{F}_{-x}(G)$ if $\mathscr{F}_{-x}(G) \neq \emptyset$.

In this paper, an *α-leaf* (*β-leaf* and *γ-leaf*, respectively) is a leaf of *α-type* (*β-type* and *γ-type* respectively). Turning back to the general case of a graph G which may or may not contain α-, β-, and γ-points, and x is a FLBP of G. Let x be a FLBP of G. Let T_x be the subgraph of G which terminates at x as shown in Fig. 1. Let O_{α_i}, O_{β_i}, and O_{γ_i} represent an *α-leaf*, *β-leaf*, and *γ-leaf* of a branch (path) of T_x with odd lengths o_{α_i}, o_{β_i}, and o_{γ_i}, respectively. Similarly, E_{α_i}, E_{β_i}, and E_{γ_i} represent an *α-leaf*, *β-leaf*, and *γ-leaf* of a branch (path) of T_x with even lengths e_{α_i}, e_{β_i}, and e_{γ_i}, respectively. Let a, c, and e be the number of odd-branches with α-, β-, and γ-leaves, respectively, and b, d, and f be the number of even-branches with α-, β-, and γ-leaves, respectively. The following lemma will tell us how to determine the type of x.

LEMMA 2.6. *Let x be a FLBP of graph G which may or may not contain α-, β-, and γ-points. Let T_x be the subgraph of G which terminates at x as*

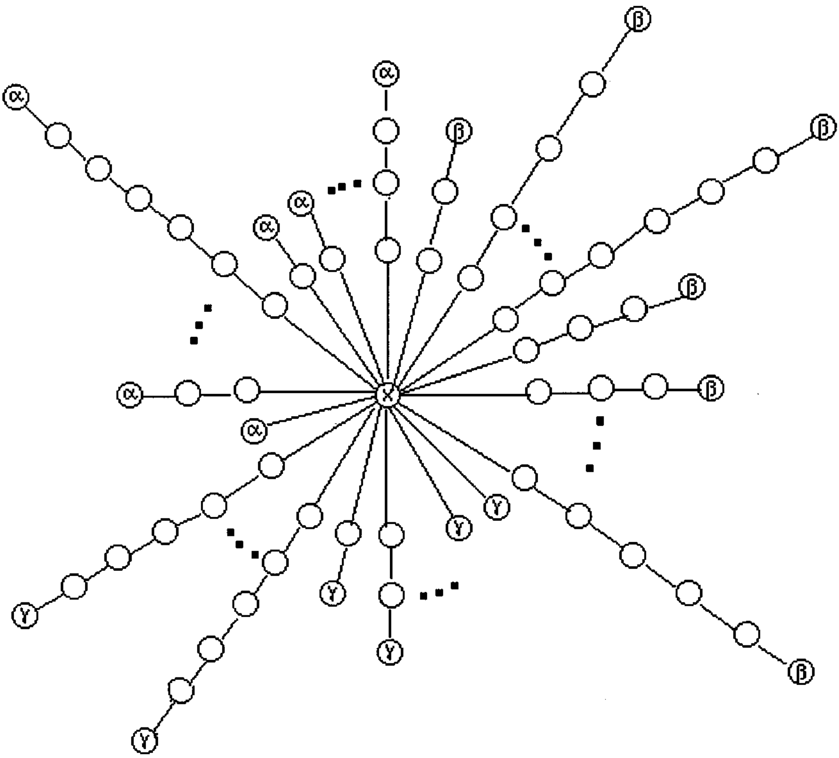

FIG. 1. Rooted tree T_x.

shown in Fig. 1. *Let* $H_x \in \mathscr{K}_x(T_x)$ *and* $H_{-x} \in \mathscr{K}_{-x}(T_x)$. *Then*

$$|H_{-x}| - |H_x| = b + d + e - 1.$$

Proof. By inspection, we have

$$|H_x| = 1 + \sum_1^a \left\lceil \frac{o_{\alpha_i} - 2}{2} \right\rceil + \sum_1^b \left\lceil \frac{e_{\alpha_i} - 2}{2} \right\rceil + \sum_1^c \left\lceil \frac{o_{\beta_i} - 2}{2} \right\rceil$$

$$+ \sum_1^d \left\lceil \frac{e_{\beta_i} - 2}{2} \right\rceil + \sum_1^e \left(1 + \left\lceil \frac{o_{\gamma_i} - 3}{2} \right\rceil\right) + \left(\sum_1^f \left(1 + \left\lceil \frac{e_{\gamma_i} - 3}{2} \right\rceil\right)\right).$$

$$|H_{-x}| = \sum_1^a \left\lceil \frac{o_{\alpha_i} - 1}{2} \right\rceil + \sum_1^b \left\lceil \frac{e_{\alpha_i} - 1}{2} \right\rceil + \sum_1^c \left\lceil \frac{o_{\beta_i} - 1}{2} \right\rceil$$

$$+ \sum_1^d \left\lceil \frac{e_{\beta_i} - 1}{2} \right\rceil + \sum_1^e \left(1 + \left\lceil \frac{o_{\gamma_i} - 2}{2} \right\rceil\right) + \sum_1^f \left(1 + \left\lceil \frac{e_{\gamma_i} - 2}{2} \right\rceil\right).$$

Note that if n is odd, then $\lceil n/2 \rceil = \lceil (n-1)/2 \rceil + 1$, and if n is even, $\lceil n/2 \rceil = \lceil (n-1)/2 \rceil$. Thus, we have $|H_{-x}| - |H_x| = b + d + e - 1$.

COROLLARY 2.7. *Let x be a FLBP of G. Let* T_x *be the subgraph of G which terminates at x as shown in Fig*. 1. *Then*

$$x \text{ is a } \begin{cases} \alpha\text{-point} & \text{if } b + d + e \geq 2 \\ \beta\text{-point} & \text{if } b + d + e = 1 \\ \gamma\text{-point} & \text{if } b + d + e = 0 \end{cases}$$

When we are executing the procedure of reducing G to G' by replacing T_x by x for all FLBPs of G, we have to find the modified weight function w' such that $\phi(w', G') = \phi(w, G)$. Let $\mathscr{P}$ ($\mathscr{Q}$ and $\mathscr{R}$ respectively) be the set of all *α-leaves* (*β-leaves* and *γ-leaves*, respectively) of graph G'.

When selecting a maximum independent set from G', the α-points can be omitted from consideration, since they cannot be selected if the independent set is to be maximum. And it is possible to not select any β-points and still obtain a maximum independent set. Hence, for any $I \in \mathscr{F}(G)$, we have

$$\psi(G) = |I| = |I \cap G'^r| + \sum_{x \in \mathscr{P}} \psi\left(\check{T}_x\right)$$

$$+ \sum_{x \in I \cap (\mathscr{Q} \cup \mathscr{R})} \psi\left(\hat{T}_x\right)^{+1} + \sum_{x \in (\mathscr{Q} \cup \mathscr{R}) - I} \psi\left(\check{T}_x\right),$$

and by definition of ψ, $|I \cap G''| = \psi(G'')$, i.e., $I \cap G'' \in \mathscr{F}(G'')$. This suggests a decomposition of a maximum independent set of G into the disjoint union of some maximum independent set of G'' and an independent set of $G - G''$.

Conversely, for any $I^r \in \mathscr{F}(G'')$, we may extend I^r to a new set $\mathscr{B}$ by adding some "qualified" β-leaves of G' (i.e., by picking those β-leaves which maintain the independence of $\mathscr{B}$). If x is a FLBP of G, let $S_x \in \mathscr{F}_x(T_x)$ and $S_{-x} \in \mathscr{F}_{-x}(T_x)$. Let

$$S = I^r \cup \left(\bigcup_{x \in \mathscr{P}} S_{-x} \right) \cup \left(\bigcup_{x \in \mathscr{B} \cap (\mathscr{Q} \cup \mathscr{R})} S_x \right) \cup \left(\bigcup_{x \in (\mathscr{Q} \cup \mathscr{R}) - \mathscr{B}} S_{-x} \right).$$

Then S is independent and

$$|S| = |\mathscr{B}| + \sum_{x \in \mathscr{B} \cap (\mathscr{Q} \cup \mathscr{R})} (\psi(T_x) - 1) + \sum_{x \in (\mathscr{P} \cup \mathscr{Q} \cup \mathscr{R}) - \mathscr{B}} \psi\left(\check{T}_x\right) = \psi(G).$$

Clearly, an element I^r of $\mathscr{F}(G'')$ may induce many elements of $\mathscr{F}(G)$.

In the following theorem, we let $\phi(P_0) = \phi(P_{-1}) = 1$ and $\phi(P_{-n}) = 0$ $\forall n \geq 2$.

THEOREM 2.8. *Let G be a graph of α-, β-, and γ-points with weight function w. Let x be a FLBP of G and T_x be as shown in Fig. 1. Let G' be the graph obtained from G by replacing T_x by x, and let w' be the weight function on G' which is the same as w except at every FLBP x of G, where $w' = (C'_x, N'_x)$. Then*

$$C'_x = \begin{cases} 0 & \text{if } x \text{ is a } \alpha\text{-point} \\ ABC & \text{otherwise} \end{cases}$$

$$N'_x = \begin{cases} 0 & \text{if } x \text{ is a } \gamma\text{-point and terminal} \\ DEF & \text{otherwise,} \end{cases}$$

where

$$A = \left[\prod_1^a N_{O_{\alpha_i}} \right] \left[\prod_1^b N_{E_{\alpha_i}} \phi\left(P_{e_{\alpha_i} - 2}\right) \right];$$

$$B = \left[\prod_1^c N_{O_{\beta_i}} \right] \left[\prod_1^d \left(C_{E_{\beta_i}} + N_{E_{\beta_i}} \phi\left(P_{e_{\beta_i} - 2}\right) \right) \right];$$

$$C = \left[\prod_1^e \left(C_{O_{\gamma_i}} \phi\left(P_{o_{\gamma_i} - 3}\right) + N_{O_{\gamma_i}} \right) \right] \left[\prod_1^f C_{E_{\gamma_i}} \right];$$

$$D = \left[\prod_1^a N_{O_{\alpha_i}} \phi\left(P_{o_{\alpha_i} - 1}\right) \right] \left[\prod_1^b N_{E_{\alpha_i}} \right];$$

$$E = \left[\prod_1^c \left(C_{O_{\beta_i}} + N_{O_{\beta_i}}\phi\left(P_{o_{\beta_i}-1}\right)\right)\right]\left[\prod_1^d N_{E_{\beta_i}}\right];$$

$$F = \left[\prod_1^e C_{O_{\gamma_i}}\right]\left[\prod_1^f \left(C_{E_{\gamma_i}}\phi\left(P_{e_{\gamma_i}-2}\right) + N_{E_{\gamma_i}}\right)\right],$$

and

$$\phi(w,G) = \phi(w',G').$$

Proof. If x is selected, then no other member of $N(x)$ can be selected. Since each O_{α_i} and E_{α_i} is a α-point, they are not members of any maximum independent sets. There are

$$\left[\prod_1^a N_{O_{\alpha_i}}\phi\left(P_{o_{\alpha_i}-2}\right)\right]\left[\prod_1^b N_{E_{\alpha_i}}\phi\left(P_{e_{\alpha_i}-2}\right)\right] = A$$

choices for these two groups of branches.

Since each O_{β_i} and E_{β_i} is a β-point, in order to have maximum cardinality no O_{β_i} can be chosen. There are

$$\left[\prod_1^c N_{O_{\beta_i}}\right]\left[\prod_1^d \left(C_{E_{\beta_i}}\phi\left(P_{e_{\beta_i}-3}\right) + N_{E_{\beta_i}}\phi\left(P_{e_{\beta_i}-2}\right)\right)\right] = B$$

choices for these two groups of branches since $\phi(P_{e_{\beta_i}-3}) = 1$.

Finally, since each O_{γ_i} and E_{γ_i} is a γ-point, there are

$$\left[\prod_1^e \left(C_{O_{\gamma_i}}\phi\left(P_{o_{\gamma_i}-3}\right) + N_{O_{\gamma_i}}\phi\left(P_{o_{\gamma_i}-2}\right)\right)\right]\left[\prod_1^f C_{E_{\gamma_i}}\right] = C$$

choices for these two groups of branches since $\phi(P_{o_{\gamma_i}-2}) = 1$.

If x is not selected, we have an argument similar to the above. By the statement before this theorem and Theorem 2.2, we have proven

$$\phi(w,G) = \phi(w',G'). \qquad \blacksquare$$

3. PATH, POINT, OCTOPUS, AND CYCLE

By Theorem 2.8, we can successively reduce the graph and updating the weight function w, we will end up with a path or an "octopus."

We will first discuss the case that G is a path of n vertices with a weighted leaf x, of weight (C_x, N_x), in the following:

LEMMA 3.1. *If G is a n-path with a weighted leaf x, of weight (C_x, N_x), then G can be reduced to a single weighted point z with weight $w(z) = (C_z, N_z)$*

defined as below:

Case 1: n is odd.

$$z \text{ is } \begin{cases} \text{a } \gamma\text{-point and } w(z) = (\phi(P_{n-3})N_x, N_x) & \text{if } x \text{ is an } \alpha\text{-leaf} \\ \text{a } \gamma\text{-point and } w(z) = (\phi(P_{n-3})N_x + C_x, N_x) & \text{if } x \text{ is a } \beta\text{-leaf} \\ \text{a } \beta\text{-point and } w(z) = (C_x, 0) & \text{if } x \text{ is a } \gamma\text{-leaf} \end{cases}$$

Case 2: n is even.

$$z \text{ is } \begin{cases} \text{a } \beta\text{-point and } w(z) = (N_x, 0) & \text{if } x \text{ is an } \alpha\text{-leaf} \\ \text{a } \beta\text{-point and } w(z) = (N_x, 0) & \text{if } x \text{ is a } \beta\text{-leaf} \\ \text{a } \gamma\text{-point and } w(z) = (N_x + \phi(P_{n-4})C_x, C_x) & \text{if } x \text{ is a } \gamma\text{-leaf} \end{cases}$$

Proof. By inspection. ∎

It is very easy to have the result in the following theorem by the above lemma.

THEOREM 3.2. *If G is a point graph z and $w(z) = (C_z, N_z)$, then $\phi(G) = N_z$ if z is an α-point*; *$\phi(G) = C_z + N_z$ if z is a β-point*; *and $\phi(G) = C_z$ if z is a γ-point*.

Now, we will discuss the case that G is an "octopus" with a weighted function w. As in Lemma 3.1, the "legs" of the octopus can be reduced to weighted α-points, β-points, or γ-points. It is then reduced to a cycle J of weighted α-points, β-points, and/or γ-points. So what we are left is to evalute $\phi(J)$. The argument is similar to that of the paragraph following Corollary 2.7. A maximum independent set of G contains no α-points and as many γ-points as possible. As to the β-points, some of them may be chosen as long as the independence of the whole set is maintained.

THEOREM 3.3. *Let J be a cycle of α-, β-, and γ-points with weight function w_J. Let $\mathscr{A} = \{\alpha$-points of the cycle $J\}$, $\mathscr{B} = \{\beta$-points of the cycle $J\}$, $\mathscr{C} = \{\gamma$-points of the cycle $J\}$. Then*

$$\phi(w_J, J) = \begin{cases} \Sigma_{I \in \mathscr{F}(\mathscr{C})} (\Pi_{x \in I} C_x)(\Pi_{y \notin I} N_y) & \text{if } \mathscr{C} = J; \\ (\Pi_{x \in \mathscr{A}} N_x)\left[\Sigma_{S \in \mathscr{I}(\mathscr{B})} (\Pi_{y \in S} C_y)(\Pi_{z \in \mathscr{B} - S} N_z)\right] & \text{if } \mathscr{C} = \phi; \\ (\Pi_{x \in \mathscr{A}} N_x)\left\{\Sigma_{I \in \mathscr{F}(\mathscr{C})}\left[(\Pi_{x \in I} C_x)(\Pi_{y \in \mathscr{C} - I} N_y)\right.\right. & \\ \quad \left.\left. \cdot \left(\Sigma_{I \subseteq I' \in \mathscr{I}(\mathscr{B} \cup \mathscr{C})} (\Pi_{x \in I' - I} C_x)(\Pi_{y \in \mathscr{B} - I'} N_y)\right)\right]\right. & \\ & \text{if } \phi \neq \mathscr{C} \neq J. \end{cases}$$

Proof. 1. If $\mathscr{C} = \emptyset$, i.e., J contains no γ-points, then $\psi(G) = \psi(G - J)$. A maximum independent set of G may contain any independent subset (including the empty set) of $\mathscr{B}$, so

$$\phi(w_J, J) = \left(\prod_{x\in\mathscr{A}} N_x\right)\left(\sum_{S\in\mathscr{I}(\mathscr{B})}\left(\prod_{y\in S} C_y\right)\left(\prod_{z\in\mathscr{B}-S} N_z\right)\right).$$

2. If $\mathscr{C} = J \neq \emptyset$, i.e., J contains γ-points only, then

$$\phi(w_J, J) = \sum_{I\in\mathscr{F}(\mathscr{C})}\left(\prod_{x\in I} C_x\right)\left(\prod_{y\notin I} N_y\right).$$

3. If $\emptyset \neq \mathscr{C} \neq J$, i.e., J contains not only γ-points but also some α-points and/or β-points. As in the case of G'^r, at first, we choose a maximum independent set I of $\mathscr{C}$, then extend I to independent set $I' \in \mathscr{I}(\mathscr{B} \cup \mathscr{C})$ if $\mathscr{B} \neq \emptyset$. Hence we have

$$\phi(w_J, J) = \left(\prod_{x\in\mathscr{A}} N_x\right)\left\{\sum_{I\in\mathscr{F}(\mathscr{C})}\left[\left(\prod_{x\in I} C_x\right)\left(\prod_{y\in\mathscr{C}-I} N_y\right)\right.\right.$$
$$\left.\left.\cdot\left(\sum_{I\subseteq I'\in\mathscr{I}(\mathscr{B}\cup\mathscr{C})}\left(\prod_{x\in I'-I} C_x\right)\left(\prod_{y\in\mathscr{B}-I'} N_y\right)\right)\right]\right\}. \qquad \blacksquare$$

CONCLUSION

In this paper, we only deal with functional graphs, but actually the argument used in this paper can be extended to solve those graphs which are "similar" to functional graphs. Since the modification process is very tedious, it is difficult to set forth the material compactly. We can obtain beautiful formulas for some sets of functional graphs; interested readers may try some examples.

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