Random Perturbation in Games of Chance

Cheng-Der Fuh and Yeong-Nan Yeh

In this article, we consider a problem in games of chance. Our result shows that two losing games (A and B, in the sense of a negative expectation) can become a **winning game** (in the sense of a positive expectation), when the two are played in a suitable alternating order; for example, $ABBABB\ldots$. By using a regrouping technique in Automata and the concept of Markov chain embedding, we give proof of this gambling result. A signal-to-noise ratio is also presented to explain this counterintuitive phenomenon.

1. Introduction

Let A be a game such that

$$P\{\text{winning a dollar}\} = a_1 := \frac{1}{2} - \varepsilon$$
$$P\{\text{losing a dollar}\} = a_0 := \frac{1}{2} + \varepsilon, \tag{1}$$

where ε is a given positive number. Note that game A can be interpreted as tossing a weighted coin or going on a biased random walk. Simple calculations show that it has negative expected value -2ε.

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Game B is a little more complex and can be described as follows. Let S_n be the capital at time n. When S_n is a multiple of 3, game B is described by (2)

$$P\{\text{winning a dollar } |S_n = 0 \text{ mod } 3\} = b_1 := \frac{1}{10}$$
$$P\{\text{losing a dollar } |S_n = 0 \text{ mod } 3\} = b_0 := \frac{9}{10}, \qquad (2)$$

where "mod" refers to the notation of module. When S_n is not a multiple of 3, game B is described by (3)

$$P\{\text{winning a dollar}|S_n \neq 0 \text{ mod } 3\} = c_1 := \frac{1}{2} + \delta$$
$$P\{\text{losing a dollar}|S_n \neq 0 \text{ mod } 3\} = c_0 := \frac{1}{2} - \delta, \qquad (3)$$

where δ is a given positive number. Here, we refer to *capital* and *winning* (or *losing*) as if anyone playing this game is against a common opponent: for example, the bank, and the capital can be negative as a natural interpretation. The behavior of game B differs from game A, because one is likely to win or lose a small amount, depending on the starting capital. If the starting capital is a multiple of 3, then we lose a little or vice versa. The concept of what it means for game B to be winning, losing, or fair can be defined precisely in terms of expected value of a given Markov chain. From (6) of Section 2, it follows that game B has expected value $-(4\delta^2 - 5\delta + 1)/(20\delta^2 - 16\delta + 45)$ and hence, is a fair game if $\delta = 0.25$, a losing game if $0 < \delta < 0.25$ and a winning game if $0.25 < \delta < 0.5$.

For given $\varepsilon > 0$ and $0 < \delta < 0.25$, it is clear that both game A and game B are losing games. Consider the scenario if we consider a new game G of switching between the two losing games A and B; play one game of A, two games of B, one of A, two of B, and so on. (The act of playing a game can not be broken). The result, which is counterintuitive, is that we start winning. That is, we can play the two losing games A and B in such a way as to produce a positive expected value. The concept that two losing games can produce a winning game was devised by Parrondo as a pedagogical illustration of the Brownian ratchet. Further numerical analysis was given by Harmer and Abbott [1], based on the criterion of whether the probability of moving up n states is greater, less than, or equal to the probability of moving down n states as n becomes large. In this article, we give a general framework of games of chance, using a regrouping technique in Automata to define an appropriate Markov chain, and then provide a probabilistic argument to resolve this gambling result under the criterion of expected value. Our result shows that the alternating order game $ABBABB\ldots$ has positive expected value for any given $\varepsilon > 10^{-4}$ and $0.1636 < \delta < 0.25$. The proof is described in detail in Section 2. Other possible alternating orders of games A and B, $BABBAB\ldots$, $BBABBA\ldots$, and $ABAB\ldots$, are also investigated. We show that both game $BABBAB\ldots$ and game $BBABBA\ldots$ have the same phenomena as game $ABBABB\ldots$; whereas, game $ABAB\ldots$ cannot produce a positive expectation if both games A and game B are losing games. A signal-to-noise ratio is also presented to explain this counterintuitive phenomenon.

2. The proof and a heuristic explanation

Consider games A and B described in (1), (2), and (3). We first compute the expected value of game B as a solution of the transition probabilities of a given Markov chain. Because game B is played according to the current value of the capital S_n module 3, for $n = 0, 1, 2, \ldots$, we let $B_n := S_n$ module 3, and observe that $\{B_n, n \geq 0\}$ is a Markov chain on the state space $D := \{0, 1, 2\}$, with transition probability matrix

$$P^B = \begin{pmatrix} 0 & b_1 & b_0 \\ c_0 & 0 & c_1 \\ c_1 & c_0 & 0 \end{pmatrix}. \tag{4}$$

Note that $\{B_n, n \geq 0\}$ is an ergodic (aperiodic, irreducible, and positive recurrent) Markov chain, the invariant distribution $\pi^B = (\pi_0^B, \pi_1^B, \pi_2^B)$ of $\{B_n, n \geq 0\}$ exists, and satisfies the balance equation $\pi^B P = \pi^B$ (cf. Karlin and Taylor, [2]). Simple calculations and substituting the values from (1)–(3) lead to

$$\begin{aligned} \pi_0^B &= \frac{20\delta^2 + 15}{20\delta^2 - 16\delta + 45}, \\ \pi_1^B &= \frac{11 - 18\delta}{20\delta^2 - 16\delta + 45}, \\ \pi_2^B &= \frac{19 + 2\delta}{20\delta^2 - 16\delta + 45}. \end{aligned} \tag{5}$$

Hence, the expected value of game B is

$$\begin{aligned} EB &= (-8/10)\pi_1^B + 2\delta(\pi_2^B + \pi_3^B) \\ &= -(4\delta^2 - 5\delta + 1)/(20\delta^2 - 16\delta + 45). \end{aligned} \tag{6}$$

When $\delta = 0$, we have $\pi_0^B = 1/3$ and $\pi_1^B + \pi_2^B = 2/3$. That is, if (3) is a fair game, we have a chance of one in three to play (2) and a chance of two in three to play (3) in game B. Because as $0.25 < \delta < 0.5$, and the expected value of game B is positive, we consider only the case of $0 < \delta \leq 0.25$ in the remainder of this article.

Next, we compute the expected value of game G. Because the game is played according to the order $ABBABB\ldots$, we first regroup the plays in game G by defining $H := ABB$ and then calculate the expectation of game G in terms of that of H. Second, we note that the reward of each play H depends on the starting capital S_{3n} module 3, for each $n = 0, 1, 2, \ldots,$ and can only take values in the set $S := \{-3, -1, 1, 3\}$. The key idea in the analysis is to let $G_n := S_{3n}$ and observe that $\{G_n, n \geq 0\}$ is a Markov chain on the state space $D := \{0, 1, 2\}$. For each $i, j = 0, 1, 2$, denote $p_{ij} := P\{G_n = j | G_{n-1} = i\}$ as the transition probability. For the computation of p_{00}, we note that in each play H the probability from state 0 to state 0 can take place only when one has either 3 dollars winning or 3 dollars losing; i.e., $p_{00} = a_1c_1c_1 + a_0c_0c_0$. Similar arguments can be applied to achieve the transition probabilities p_{ij} for $i, j = 0, 1, 2$.

From Figure 1, we see that the transition probability matrix P of the Markov chain $\{G_n, n \geq 0\}$ is given by

$$\begin{pmatrix} a_1c_1c_1 + a_0c_0c_0 & a_0c_1b_1 + a_1c_0b_1 + a_1c_0c_1 & a_0c_1c_0 + a_0c_1b_0 + a_1c_0b_0 \\ a_0b_1c_0 + a_1c_0c_0 + a_0b_0c_1 & a_1b_1c_1 + a_0b_0c_0 & a_1c_0c_1 + a_1b_0c_1 + a_0b_1c_1 \\ a_0c_1c_1 + a_1b_0c_1 + a_1b_1c_0 & a_1b_0c_0 + a_0c_1c_0 + a_0c_0b_1 & a_1b_1c_1 + a_0b_0c_0 \end{pmatrix}. \quad (7)$$

Because $\{G_n, n \geq 0\}$ is an ergodic Markov chain, for any given $0 < \varepsilon < 0.5$ and $0 < \delta < 0.25$, the invariant distribution $\pi = (\pi_0, \pi_1, \pi_2)$ of $\{G_n, n \geq 0\}$ exists, and satisfies the balance equation $\pi P = \pi$. Simple calculations and substituting the values from (1)–(3) entail

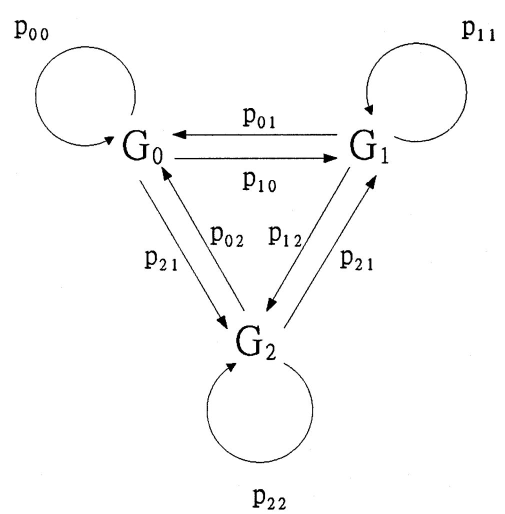

Figure 1. A diagram for the transition probability.

$$
\begin{aligned}
&(\pi_0, \pi_1, \pi_2) \\
&= \left(\frac{\frac{27}{64} + \frac{3\delta}{5} + \frac{157\delta^2}{200} - \frac{\delta^4}{4} - \frac{3\varepsilon}{10} + \frac{93\delta\varepsilon}{100} - \frac{2\delta^2\varepsilon}{5} + \delta^3\varepsilon + \frac{\varepsilon^2}{16} - \frac{2\delta\varepsilon^2}{5} + \frac{57\delta^2\varepsilon^2}{50} - \frac{8\delta^3\varepsilon^2}{5} + \delta^4\varepsilon^2}{\frac{81}{64} + \frac{7\delta}{10} - \frac{143\delta^2}{200} - \frac{2\delta^3}{5} + \frac{\delta^4}{4} - \frac{4\varepsilon}{5} + 4\delta\varepsilon + \frac{3\varepsilon^2}{16} - \frac{6\delta\varepsilon^2}{5} + \frac{171\delta^2\varepsilon^2}{50} - \frac{24\delta^3\varepsilon^2}{5} + 3\delta^4\varepsilon^2}, \right. \\
&\quad \frac{\frac{111}{320} - \frac{87\delta}{400} - \frac{17\delta^2}{20} + \frac{\delta^3}{20} + \frac{\delta^4}{4} - \frac{\varepsilon}{4} + \frac{247\delta\varepsilon}{200} - \frac{87\delta^2\varepsilon}{100} - \frac{9\delta^3\varepsilon}{10} + \delta^4\varepsilon + \frac{\varepsilon^2}{16} - \frac{2\delta\varepsilon^2}{5} + \frac{57\delta^2\varepsilon^2}{50} - \frac{8\delta^3\varepsilon^2}{5} + \delta^4\varepsilon^2}{\frac{81}{64} + \frac{7\delta}{10} - \frac{143\delta^2}{200} - \frac{2\delta^3}{5} + \frac{\delta^4}{4} - \frac{4\varepsilon}{5} + 4\delta\varepsilon + \frac{3\varepsilon^2}{16} - \frac{6\delta\varepsilon^2}{5} + \frac{171\delta^2\varepsilon^2}{50} - \frac{24\delta^3\varepsilon^2}{5} + 3\delta^4\varepsilon^2}, \\
&\quad \left. \frac{\frac{159}{320} + \frac{127\delta}{400} - \frac{13\delta^2}{20} - \frac{9\delta^3}{20} + \frac{\delta^4}{4} - \frac{\varepsilon}{4} + \frac{367\delta\varepsilon}{200} + \frac{127\delta^2\varepsilon}{100} - \frac{\delta^3\varepsilon}{10} - \delta^4\varepsilon + \frac{\varepsilon^2}{16} - \frac{2\delta\varepsilon^2}{5} + \frac{57\delta^2\varepsilon^2}{50} - \frac{8\delta^3\varepsilon^2}{5} + \delta^4\varepsilon^2}{\frac{81}{64} + \frac{7\delta}{10} - \frac{143\delta^2}{200} - \frac{2\delta^3}{5} + \frac{\delta^4}{4} - \frac{4\varepsilon}{5} + 4\delta\varepsilon + \frac{3\varepsilon^2}{16} - \frac{6\delta\varepsilon^2}{5} + \frac{171\delta^2\varepsilon^2}{50} - \frac{24\delta^3\varepsilon^2}{5} + 3\delta^4\varepsilon^2} \right).
\end{aligned}
$$

The expected value of H for given G_0 can be expressed as

$$
\begin{aligned}
E(H|G_0 = 0) &= -3a_0c_0c_0 - (a_0c_1c_0 + a_0c_1b_0 + a_1c_0b_0) \\
&\quad + (a_0c_1b_1 + a_1c_0b_1 + a_1c_0c_1) + 3a_1c_1c_1, \qquad (8) \\
E(H|G_0 = 1) &= -3a_0b_0c_0 - (a_0b_1c_0 + a_1c_0c_0 + a_0b_0c_1) \\
&\quad + (a_1c_0c_1 + a_1b_0c_1 + a_0b_1c_1) + 3a_1b_1c_1, \qquad (9) \\
E(H|G_0 = 2) &= -3a_0b_0c_0 - (a_1b_0c_0 + a_0c_1c_0 + a_0c_0b_1) \\
&\quad + (a_0c_1c_1 + a_1b_0c_1 + a_1b_1c_0) + 3a_1b_1c_1. \qquad (10)
\end{aligned}
$$

Hence, the expected value EH of H is given by

$$
\begin{aligned}
&P\{G_0=0\}E(H|G_0=0)+P\{G_0=1\}E(H|G_0=1)+P\{G_0=2\}E(H|G_0=2) \\
&\quad = \pi_0E(H|G_0=0)+\pi_1E(H|G_0=1)+\pi_2E(H|G_2=2) \\
&\quad = -6\big(180-836\delta-800\delta^2+144\delta^3+320\delta^4+547\varepsilon+1440\delta\varepsilon-3144\delta^2\varepsilon \\
&\qquad +640\delta^3\varepsilon+1200\delta^4\varepsilon-400\varepsilon^2+1744\delta\varepsilon^2+640\delta^2\varepsilon^2-4672\delta^3\varepsilon^2 \\
&\qquad +3840\delta^4\varepsilon^2+100\varepsilon^3-640\delta\varepsilon^3+1824\delta^2\varepsilon^3-2560\delta^3\varepsilon^3+1600\delta^4\varepsilon^3\big) \\
&\qquad \big(2025+1120\delta-1144\delta^2-640\delta^3+400\delta^4-1280\varepsilon+6400\delta\varepsilon+300\varepsilon^2 \\
&\qquad -1920\delta\varepsilon^2+5472\delta^2\varepsilon^2-7680\delta^3\varepsilon^2+4800\delta^4\varepsilon^2\big)^{-1}. \qquad (11)
\end{aligned}
$$

Table 1 and Figure 2 show the values of the expected values EH for suitably chosen ε and δ. Note that game G becomes a winning game when $0 < \varepsilon < 10^{-4}$, and $0.1636 < \delta < 0.25$. The notation E in Tables 1–4 denotes the power of a given number.

The same argument can be applied to analyze the alternating order game $BABBAB\ldots$ and game $BBABBA\ldots$. Let $\{G_n^{(bab)}, n \geq 0\}$ and $\{G_n^{(bba)}, n \geq 0\}$ be the associated Markov chains constructed from games

Table 1
Expected Values of Game $ABBABB\ldots$

$\varepsilon\backslash\delta$	0.1634590477	0.1634600013	0.1634695381	0.1635648906
10^{-7}	$0.2806843646E-07$	$0.1832916382E-05$	$0.1995796811E-04$	$0.2011908655E-03$
10^{-6}	$-0.1794787408E-05$	$0.2665204235E-07$	$0.1814369716E-04$	$0.1993707701E-03$
10^{-5}	$-0.1990754026E-04$	$-0.1809727655E-04$	$0.1440393582E-07$	$0.1812535484E-03$
10^{-4}	$-0.2011725155E-03$	$-0.1993537735E-03$	$-0.1812056871E-03$	$0.3051937369E-07$

$BABBAB\ldots$ and $BBABBA\ldots$, with transition probability matrices

$$P^{(bab)} = \begin{pmatrix} b_1a_1c_1 + b_0a_0c_0 & b_0a_1b_1 + b_1a_0b_1 + b_1a_1c_0 & b_1a_0a_0 + b_0a_1b_0 + b_0a_0c_1 \\ c_1a_0c_0 + c_0a_1c_0 + c_0a_0c_1 & c_1a_1b_1 + c_0a_0c_0 & c_0a_1c_1 + c_1a_0c_1 + c_1a_1b_0 \\ c_0a_1c_1 + c_1a_0c_1 + c_1a_1c_0 & c_1a_0c_0 + c_0a_1c_0 + c_0a_0b_1 & c_1a_1c_1 + c_0a_0b_0 \end{pmatrix}, \quad (12)$$

and

$$P^{(bba)} = \begin{pmatrix} b_1c_1a_1 + b_0c_0a_0 & b_0c_1a_1 + b_1c_0a_1 + b_1c_1a_0 & b_0c_0a_1 + b_0c_1a_0 + b_1c_0a_0 \\ c_0b_0a_1 + c_0b_1a_0 + c_1c_0a_0 & c_1c_1a_1 + c_0b_0a_0 & c_0b_1a_1 + c_1c_0a_1 + c_1c_1a_0 \\ c_0c_1a_1 + c_1b_0a_1 + c_1b_1a_0 & c_1c_0a_1 + c_0c_1a_0 + c_1b_0a_0 & c_1b_1a_1 + c_0c_0a_0 \end{pmatrix}, \quad (13)$$

respectively. The corresponding invariant distributions $\pi^{(bab)}$ and $\pi^{(bba)}$ can be derived via the balance equations, and the expected value of each game can be evaluated as that of (11).

Tables 2 and 3 give the values of the expected values $E(BAB)$ and $E(BBA)$ for suitably chosen ε and δ. Note that both games $BABBAB\ldots$ and game $BBABBA\ldots$ become winning games when $0 < \varepsilon < 10^{-4}$ and $0.1844 < \delta < 0.25$.

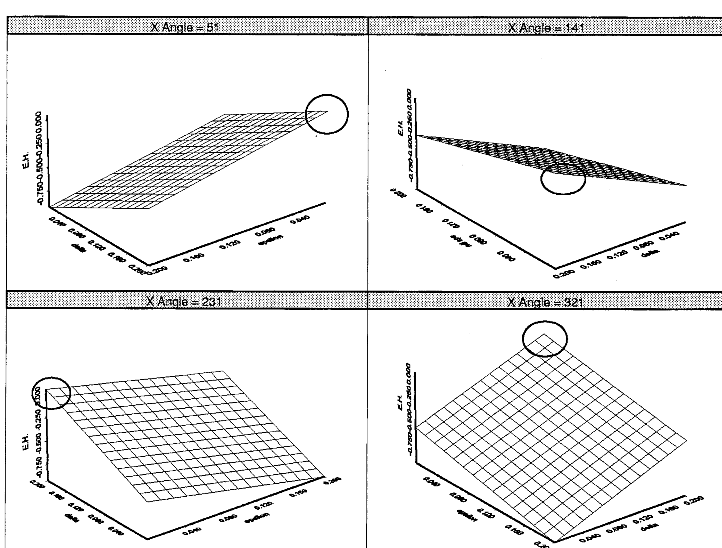

Figure 2. Three-dimensional plot for values of EH. Note: Each plot in Figure 2 is a rotation of a 3-D plot for the value of EH. The circuliar part in the figure indicates the positive values.

Table 2
Expected Values of Game *BABBAB*...

$\varepsilon\backslash\delta$	0.1843205094	0.1843211055	0.1843270063	0.1843846291
10^{-7}	$0.7048026163E-07$	$0.1763128694E-05$	$0.1981854803E-04$	$0.1948323188E-03$
10^{-6}	$-0.1687636541E-05$	$0.4315722002E-08$	$0.1804233034E-04$	$0.1930621802E-03$
10^{-5}	$-0.1921516377E-04$	$-0.1742093809E-04$	$0.5069173312E-06$	$0.1755430130E-03$
10^{-4}	$-0.1946203411E-03$	$-0.1928358251E-03$	$-0.1748949289E-03$	$0.1662782836E-06$

Table 3
Expected Values of Game *BBABBA*...

$\varepsilon\backslash\delta$	0.1843205094	0.1843211055	0.1843268573	0.1843846142
10^{-7}	$0.7571608052E-07$	$0.1843105224E-05$	$0.1938158857E-04$	$0.1947593264E-03$
10^{-6}	$-0.1750207957E-05$	$0.7671628310E-07$	$0.1761043859E-04$	$0.1929443679E-03$
10^{-5}	$-0.1928434904E-04$	$-0.1746201815E-04$	$0.6633780458E-07$	$0.1754333498E-03$
10^{-4}	$-0.1946483681E-03$	$-0.1928230631E-03$	$-0.1752977405E-03$	$0.5817637572E-07$

Finally, we consider the game $ABAB\ldots$. Let $\{G_n^{(ab)}, n \geq 0\}$ be the associated Markov chain constructed from game $ABAB\ldots$, with transition probability matrix

$$P^{(ab)} = \begin{pmatrix} a_1c_0 + a_0c_1 & a_0c_0 & a_1c_1 \\ a_1c_1 & a_1c_0 + a_0b_1 & a_0b_0 \\ a_0c_0 & a_1b_1 & a_1b_0 + a_0c_1 \end{pmatrix}, \tag{14}$$

and invariant distribution $\pi^{(ab)}$

$$(\pi_0^{(ab)}, \pi_1^{(ab)}, \pi_2^{(ab)}) = \Bigg(\frac{\frac{3}{16} - \frac{\delta}{5} - \frac{\delta^2}{4} + \frac{2\varepsilon}{5} - \delta\varepsilon + \frac{\varepsilon^2}{4} - \frac{4\delta\varepsilon^2}{5} + \delta^2\varepsilon^2}{\frac{9}{16} - \frac{\delta}{5} + \frac{\delta^2}{4} + \frac{4\varepsilon}{5} - 4\delta\varepsilon + \frac{3\varepsilon^2}{4} - \frac{12\delta\varepsilon^2}{5} + 3\delta^2\varepsilon^2},$$
$$\frac{\frac{7}{80} - \frac{\delta}{4} + \frac{\delta^2}{4} + \frac{\varepsilon}{5} - \frac{11\delta\varepsilon}{10} + \delta^2\varepsilon + \frac{\varepsilon^2}{4} - \frac{4\delta\varepsilon^2}{5} + \delta^2\varepsilon^2}{\frac{9}{16} - \frac{\delta}{5} + \frac{\delta^2}{4} + \frac{4\varepsilon}{5} - 4\delta\varepsilon + \frac{3\varepsilon^2}{4} - \frac{12\delta\varepsilon^2}{5} + 3\delta^2\varepsilon^2},$$
$$\frac{\frac{23}{80} + \frac{\delta}{4} + \frac{\delta^2}{4} + \frac{\varepsilon}{5} - \frac{19\delta\varepsilon}{10} - \delta^2\varepsilon + \frac{\varepsilon^2}{4} - \frac{4\delta\varepsilon^2}{5} + \delta^2\varepsilon^2}{\frac{9}{16} - \frac{\delta}{5} + \frac{\delta^2}{4} + \frac{4\varepsilon}{5} - 4\delta\varepsilon + \frac{3\varepsilon^2}{4} - \frac{12\delta\varepsilon^2}{5} + 3\delta^2\varepsilon^2} \Bigg).$$

When $\varepsilon = 0$ in game $ABAB\ldots$, the transition probability matrix $P^{(ab)}$ becomes

$$\begin{pmatrix} \frac{\frac{1}{2}-\delta}{2} + \frac{\frac{1}{2}+\delta}{2} & \frac{\frac{1}{2}-\delta}{2} & \frac{\frac{1}{2}+\delta}{2} \\ \frac{\frac{1}{2}+\delta}{2} & \frac{1}{20} + \frac{\frac{1}{2}-\delta}{2} & \frac{9}{20} \\ \frac{\frac{1}{2}-\delta}{2} & \frac{1}{20} & \frac{9}{20} + \frac{\frac{1}{2}+\delta}{2} \end{pmatrix},$$

and the invariant distribution $\pi^{(ab)}$ becomes

$$\left(\frac{\frac{3}{16} - \frac{\delta}{5} - \frac{\delta^2}{4}}{\frac{9}{16} - \frac{\delta}{5} + \frac{\delta^2}{4}}, \quad \frac{\frac{7}{80} - \frac{\delta}{4} + \frac{\delta^2}{4}}{\frac{9}{16} - \frac{\delta}{5} - \frac{\delta^2}{4}}, \quad \frac{\frac{23}{80} + \frac{\delta}{4} + \frac{\delta^2}{4}}{\frac{9}{16} - \frac{\delta}{5} + \frac{\delta^2}{4}} \right).$$

Because $E(AB|G_0^{(ab)} = 0) = 2\delta$, $E(AB|G_0^{(ab)} = 1) = -\frac{2}{5} + \delta$ and $E(AB|G_0^{(ab)} = 2) = -\frac{2}{5} + \delta$, the expected value of game AB is

$$\frac{2\delta\left(\frac{3}{16} - \frac{\delta}{5} - \frac{\delta^2}{4}\right)}{\frac{9}{16} - \frac{\delta}{5} + \frac{\delta^2}{4}} + \frac{\left(-\frac{2}{5} + \delta\right)\left(\frac{7}{80} - \frac{\delta}{4} + \frac{\delta^2}{4}\right)}{\frac{9}{16} - \frac{\delta}{5} + \frac{\delta^2}{4}} + \frac{\left(-\frac{2}{5} + \delta\right)\left(\frac{23}{80} + \frac{\delta}{4} + \frac{\delta^2}{4}\right)}{\frac{9}{16} - \frac{\delta}{5} + \frac{\delta^2}{4}},$$

which equals 0 only when $\delta = 0.25$. That is, game $ABAB\ldots$ is a fair game when both games A and B are fair games.

Table 4 presents the values of the expected values $E(AB)$ for suitably chosen ε and δ. Note that for any given $\varepsilon > 0$, game AB becomes a winning game only when $\delta > 0.25$. In this case, B itself is a winning game.

Table 4

Expected Values of Game $ABAB\ldots$

$\varepsilon\backslash\delta$	0.2500001490	0.2500011921	0.2500112951	0.2501125038
10^{-7}	$0.3077663280E-07$	$0.9681737083E-06$	$0.9540223800E-05$	$0.9579032485E-04$
10^{-6}	$-0.8845274238E-06$	$0.2788311981E-07$	$0.8631241144E-05$	$0.9491912351E-04$
10^{-5}	$-0.9463574315E-05$	$-0.8559142771E-05$	$0.2683429834E-07$	$0.8627864736E-04$
10^{-4}	$-0.9572972340E-04$	$-0.9486274939E-04$	$-0.8620460721E-04$	$0.8057462964E-07$

Recall from Table 1 that both games A and B are losing games, when $0 < \varepsilon < 10^{-4}$ and $0.1636 < \delta < 0.25$; whereas, game $ABBABB\ldots$ becomes a winning game. We now provide a heuristic explanation of this gambling phenomenon from a signal-to-noise ratio, which may give some insight into what is happening between our two games. To be more precise, we first treat game B as a signal and note that it is a losing game for $0 < \delta < 0.25$. The reason that game B is a losing game is that play (2) has a comparably larger negative expectation than that of (3). When game A is involved in game $ABBABB\ldots$, and played alternatively with game B, it can be regarded as a noise-to-signal (game B.) This random perturbation, attributable to game A, affects the frequency of playing (2) in mixed game G, and it has a comparably smaller negative expectation than that of (2) for any given small $\varepsilon > 0$, such as $\varepsilon = 10^{-6}$. That makes game $ABBABB\ldots$ a winning game. A similar interpretation can also be given for both games $BABBAB\ldots$ and $BBABBA\ldots$. Note that game $ABAB\ldots$ is still a losing game for any given $0 \le \varepsilon < 0.5$ and $0 < \delta < 0.25$. It becomes a fair game only when $\varepsilon = 0$ and $\delta = 0.25$, and, in this case, both games A and B are fair games. This result can also be interpreted by saying that the noise (game A) is too large (because of the 1/2 frequency) as compared to the signal (game B). That makes game $ABAB\ldots$ a losing game, for any given $0 < \varepsilon < 10^{-4}$, unless game B becomes a winning game, i.e., $0.25 < \delta < 0.5$. Finally, we speculate that increased understanding of this random perturbation phenomenon may have applications in statistical mechanics, signal processing, biology, and perhaps in financial engineering.

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