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Taylor expansions for Catalan and Motzkin numbers

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Abstract

In this paper we introduce two new expansions for the generating functions of Catalan numbers and Motzkin numbers. The novelty of the expansions comes from writing the Taylor remainder as a functional of the generating function. We give combinatorial interpretations of the coefficients of these two expansions and derive several new results. These findings can be used to prove some old formulae associated with Catalan and Motzkin numbers. In particular, our expansion for Catalan number provides a simple proof of the classic Chung–Feller theorem; similar result for the Motzkin paths with flaws is also given.

Keywords: Catalan and Motzkin numbers; Dyck and Motzkin paths; Taylor expansion

1. Introduction

The usual *Taylor expansion* of a function F has the form

$$F(x) = \sum_{i=0}^{n-1} \frac{F^{(i)}(0)}{i!} x^i + F_n(x), \tag{1}$$

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where $F_n(x)$ is the nth remainder.

Traditionally, the remainders of Taylor expansions play a central role in theory of functions, numerical approximations, asymptotic expansions, etc. They are used mainly for *quantitative* or *numerical* purposes. We are concerned in this paper with *combinatorial* interpretations of remainders of the usual Taylor expansions for the generating functions of two of the most widely studied numbers: Catalan and Motzkin numbers. This perspective leads to several intriguing findings.

Consider the simple example of rooted unitary tree, which is nothing but a "line" of nodes. The generating function A satisfies the functional equation $A = 1 + xA$, which, after iterating, yields the Taylor expansion $A = \sum_{i=0}^{n-1} x^k + x^n A$. Note that, unlike the usual Taylor expansions, the remainders in such expansions involve the generating function itself. Such expansions are quite different from the usual binomial expansions or continued-fraction expansions (see [9]) but are not exceptions for combinatorial structures.

We consider in Section 2 the Catalan numbers c_n defined by $c_n = \binom{2n}{n}/(n+1)$. It is well known that the generating function $C(x) := \sum_{n\geqslant 0} c_i x^i$ satisfies $C = 1 + xC^2$. We show that

$$C = \sum_{i=0}^{n-1} c_i x^i + x^n f_n(C) \quad (n \geqslant 1),$$

where the f_n's are polynomials that can be computed recursively. We will give different proofs to illustrate the multifacets of such expansions.

Similarly, we consider in Section 3 the Motzkin numbers whose generating function $M(x)$ satisfies $M = 1 + xM + x^2M^2$ and prove that

$$M = \sum_{i=0}^{n-1} m_i x^i + x^n g_n(M) + x^{n+1} h_n(M),$$

where $m_i := M^{(i)}(0)/i!$ and the g_n's and the h_n's are recursively-defined polynomials.

Since Catalan and Motzkin numbers naturally appear in a large number of combinatorial objects (see [2–4,8,15,18,19]), our Catalan and Motzkin expansions can be interpreted in many different ways.

As a nontrivial application of our Catalan and Motzkin expansions we consider the enumeration of paths with flaws. It is well known that the Catalan number c_n enumerates the *Dyck paths* of semilength n. Furthermore, by a *Dyck path with flaws*, we allow some flaw steps falling under the x-axis. In 1949 Chung and Feller [6] proved that the number of Dyck paths with flaws is independent of the number of flaws; see also Narayana [16,17] for more information. In Section 4, we give a simple proof of the Chung–Feller theorem by the Catalan expansion. In Section 5 we give a "Motzkin version" by giving two formulae for enumerating

Motzkin paths with flaws. Our results also shed new light on other classical structures (like ballot problem, binary trees, etc.) in connection with Catalan numbers and admit natural extensions to structures related to Motzkin numbers.

Finally, our approach in this paper is likely to be applied to other combinatorial structures.

2. Catalan–Taylor expansions

Starting from the equation $C = 1 + xC^2$, we successively obtain, by iterating, the following chain of equations:

$$\begin{aligned} C &= 1 + xC^2 = 1 + x + x^2\bigl(C^2 + C^3\bigr) \\ &= 1 + x + 2x^2 + x^3\bigl(2C^2 + 2C^3 + C^4\bigr) \\ &= 1 + x + 2x^2 + 5x^3 + x^4\bigl(5C^2 + 5C^3 + 3C^4 + C^5\bigr). \end{aligned}$$

The general pattern is described as follows.

Theorem 2.1. *The generating function of the Catalan numbers satisfies*

$$C = \sum_{i=0}^{n-1} c_i x^i + x^n f_n(C), \tag{2}$$

where

$$f_n(y) = \sum_{2\leqslant k\leqslant n+1} \frac{k-1}{n}\binom{2n-k}{n-1} y^k.$$

We will refer to (2) as the *Catalan–Taylor expansion with remainder* (*Catalan expansion* for short).

Algebraic proof. Write $f_n(y) = \sum_{2\leqslant k\leqslant n+1} f_{n,k} y^k$. Observe that

$$C^k = 1 + x\sum_{j=2}^{k+1} C^j.$$

Substituting this relation into the $(n-1)$th remainder, we get

$$\sum_{k=2}^{n} f_{n-1,k} C^k = \sum_{k=2}^{n} f_{n-1,k} + x\sum_{j=2}^{n+1}\left(\sum_{k=j-1}^{n} f_{n-1,k}\right)C^j$$

by interchanging the summations. Note that $\sum_{k=2}^{n} f_{n-1,k} = c_{n-1}$, by considering the coefficients of x^{-n} on the both sides of (2). Thus

$$x^n\sum_{j=2}^{n+1}\left(\sum_{k=j-1}^{n} f_{n-1,k}\right)C^j$$

is the remainder of the nth expansion; it follows that $f_{n,j} = \sum_{k=j-1}^{n} f_{n-1,k}$. Consequently,

$$f_{n,j} - f_{n,j+1} = \sum_{k=j-1}^{n} f_{n-1,k} - \sum_{k=j}^{n} f_{n-1,k} = f_{n-1,j-1}.$$

After a routine check we prove that

$$f_{n,k} = \frac{k-1}{n}\binom{2n-k}{n-1}. \qquad \square$$

In fact the number

$$f_{n,k} = \frac{k-1}{n}\binom{2n-k}{n-1}$$

is the ballot number which counts the paths above the x-axis starting form $(0,0)$ to $(2n-k, k-2)$ with rise and fall steps [12]. Such paths can also be interpreted as n-Dyck paths which begins with exactly $k-1$ rise steps (followed by a fall step immediately).

By a *block* of a Dyck path we mean a sequence of steps starting from and ending at the x-axis, without any step falling on the x-axis in between. A block can be made by a rise step followed by a "Dyck path" and then a fall step. Given an n-Dyck paths which begins with exactly $k-1$ rises, let us mark the first $k-1$ steps black and, after the $(k-1)$th step, mark the first step falls from the level $y = i$ white for $i = 1, \dots, k-1$. Separated by white steps, the unmarked steps form $k-1$ disjoint "Dyck paths" (some of them might be of length zero). By rearranging these rise steps, fall steps and "Dyck paths," we obtain an n-Dyck path with exactly $k-1$ blocks. Clearly, this is a bijection. Such relation between consecutive rise steps and blocks will be used again for Motzkin paths. For the completeness and convenience to this paper, we single out the fact:

Lemma 2.2 [12]. *The coefficient $f_{n,k}$ is the number of n-Dyck paths that begins with exactly $k-1$ rise steps; it is also the number of n-Dyck paths with $k-1$ blocks.*

Combinatorial proof of Theorem 2.1. Let $\mathcal{D}$ be the set of all Dyck paths, $\mathcal{D}_n$ the set of n-Dyck paths, and $(\mathcal{D}^k)_n$ the set of k disjoint Dyck paths with the total length of these k paths equal to n. Clearly, $|(\mathcal{D}^k)_n| = [x^n]C^k$, where $[x^n]C^k$ means the coefficient of x^n in C^k. Also let $\mathcal{F}_{n,k}$ the set of n-Dyck paths that begins with exactly $k-1$ rises. By Lemma 2.2,

$$|\mathcal{F}_{n,k}| = f_{n,k} = \frac{k-1}{n}\binom{2n-k}{n-1}.$$

To prove Eq. (2), we need to show

$$c_m = \sum_{k=2}^{n+1} f_{n,k}\big[x^{m-n}\big]C^k \quad \text{for } m \geqslant n.$$

It suffices to give the bijection between $\mathcal{D}_m$ and the set

$$\bigcup_{k=2}^{n+1}\big(\mathcal{F}_{n,k} \times \big(\mathcal{D}^k\big)_{m-n}\big)$$

as follows.

Let D be a Dyck path of length m and $D(i)$ denote the ith step in D. Note that every lattice point (x, y) on a Dyck path has $x + y$ even; so the first meet of D and the line $x + y = 2n$ is at the end of a rise step, say $D(j)$. (The meet is not a crossing.) Clearly, $n \leqslant j \leqslant 2n - 1$ and $D(j)$ rises to the level $y = 2n - j$. Let us mark the first j steps black, and after jth step mark the first fall step from the level $y = i$ white for $i = 1, 2, \ldots, 2n - j$. Concatenating these marked steps in order one see an n-Dyck which has the jth step being rise and all steps behind it being fall. Set $k = 2n - j + 1$; so $2 \leqslant k \leqslant n + 1$. The dual of this path just begins with exactly $k - 1$ rises. Also note that the unmarked steps of D are separated by the white steps into k disjoint Dyck paths of total length $m - n$. The reverse of this bijection is a routine check. □

The basic idea of this combinatorial proof is to count paths in a digraph according to their intersections with some "cut set" [5].

Also if we arrange $f_{n,k}$ like Pascal's triangle, we get

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 1 & & & \\
 & & 5 & & 5 & & 3 & & 1 & & \\
 & 14 & & 14 & & 9 & & 4 & & 1 & \\
42 & & 42 & & 28 & & 14 & & 5 & & 1
\end{array}$$

It is known to be the Catalan triangle [10].

3. Motzkin–Taylor expansion

We now turn to Motzkin numbers. We define the *nth Motzkin–Taylor expansion* (*Motzkin expansion* for short) by

$$M = \sum_{k=0}^{n-1} m_k x^k + x^n g_n(M) + x^{n+1} h_n(M), \tag{3}$$

where g_n and h_n are polynomials, and $x^n g_n(M) + x^{n+1} h_n(M)$ is the nth remainder.

To prove the uniqueness of g_n and h_n, it suffices to show that $a(M) + xb(M) = 0$ only has the trivial solution for polynomials $a(y)$ and $b(y)$. Note that

$$M(x) = \frac{(1 - x - \sqrt{1 - 2x - 3x^2})}{2x^2}$$

is a continuous function in $[-1, 1/3]$ with $M(-1) = M(0) = 1$ and $\min\{M(x) \mid -1 \leqslant x \leqslant 0\} = 3/4$. For any $y \in (3/4, 1]$, there exist distinct $\alpha, \beta \in [-1, 0]$ such that $M(\alpha) = M(\beta) = y$. By plugging $x = \alpha, \beta$ into $a(M) + xb(M) = 0$, one obtains $a(y) = b(y) = 0$. As polynomials, a and b must be constant zero.

Here we list the 2nd to 4th Motzkin expansions:

$$\begin{aligned} M &= 1 + x + x^2\bigl(M + M^2\bigr) + x^3 M^2 \\ &= 1 + x + 2x^2 + x^3\bigl(2M + 2M^2\bigr) + x^4\bigl(2M^2 + M^3\bigr) \\ &= 1 + x + 2x^2 + 4x^3 + x^4\bigl(4M + 4M^2 + M^3\bigr) + x^5\bigl(4M^2 + 2M^3\bigr). \end{aligned}$$

To obtain the $(n + 1)$th Motzkin expansion from the last one, we replace those M^k in g_n, but not in h_n, by

$$1 + x\sum_{i=1}^{k} M^i + x^2 \sum_{i=1}^{k} M^{i+1}.$$

The last term of the substitute contributes to h_{n+1} and the second term (combined with h_n) belongs to g_{n+1}. The constant is extracted out of the remainder so that

$$g_n(1) = m_n. \tag{4}$$

One can also obtain this identity by considering the coefficient of x^n in (3). It is easy to show that g_n is of degree $\lceil (n + 1)/2\rceil$ without constant term and h_n is of degree $\lfloor (n + 3)/2\rfloor$ without constant and linear terms. For convenience, we set

$$g_n(M) = \sum_{k=1}^{n} g_{n,k} M^k \quad \text{and} \quad h_n(M) = \sum_{k=1}^{n} h_{n,k} M^{k+1}$$

with $g_{n,i} = h_{n,j} = 0$ for $i > \lceil (n + 1)/2\rceil$ and $j > \lfloor (n + 1)/2\rfloor$. The following theorem deals with the coefficients $g_{n,k}$ and $h_{n,k}$.

Theorem 3.1. *Let*

$$M = \sum_{k=0}^{n-1} m_k x^k + x^n \sum_{k=1}^{n} g_{n,k} M^k + x^{n+1} \sum_{k=1}^{n} h_{n,k} M^{k+1}$$

be the nth Motzkin expansion. Then

(a) $g_{n,1} = h_{n,1} = m_{n-1}$ *for* $n \geqslant 1$;

(b) $g_{n,2} = m_{n-1}$ *and* $h_{n,2} = m_{n-1} - m_{n-2}$ *for* $n \geqslant 2$;
(c) $g_{n,k} = h_{n,k} + h_{n-1,k-1}$ *for* $n \geqslant 2$ *and* $k \geqslant 1$, *where* $h_{n-1,0} = 0$;
(d) *both* $g_{n,k}$ *and* $h_{n,k}$ *satisfy the following recurrence relation*

$$a_{n,k} = a_{n,k+1} + a_{n-1,k} + a_{n-2,k-1} \quad \text{for } n \geqslant 3 \text{ and } 2 \leqslant k \leqslant n-1,$$

with initial condition given in part (a), $g_{m+1,m+1} = h_{m,m} = 0$ *for* $m \geqslant 2$, *and* $g_{2,2} = 1$.

Proof. Parts (a) and (b) are true for initial n. Let $n \geqslant 2$. Replacing each M^k in g_{n-1} with $1 + x\sum_{i=1}^{k} M^i + x^2 \sum_{i=1}^{k} M^{i+1}$, we get

$$g_{n-1}(M) = m_{n-1} + x\sum_{i=1}^{n-1}\sum_{k=i}^{n-1} g_{n-1,k}M^i + x^2\sum_{i=1}^{n-1}\sum_{k=i}^{n-1} g_{n-1,k}M^{i+1}.$$

The last two terms together with $h_{n-1}(M)$ form the new remainder; hence

$$g_{n,i} = \sum_{k=i}^{n-1} g_{n-1,k} + h_{n-1,i-1} \quad \text{where } h_{n-1,0} = 0; \tag{5}$$

$$h_{n,i} = \sum_{k=i}^{n-1} g_{n-1,k}. \tag{6}$$

These two identities imply part (c) directly. In particular, $g_{n,1} = h_{n,1} = g_{n-1}(1) = m_{n-1}$, $g_{n+1,2} = g_n(1) - g_{n,1} + h_{n,1} = m_n$, and $h_{n+1,2} = g_n(1) - g_{n,1} = m_n - m_{n-1}$ by (4). Thus the proof of (a) and (b) follows.

The recurrence relation of $g_{n,k}$ can derived as follows:

$$\begin{aligned} g_{n,i} - g_{n,i+1} &= \sum_{k=i}^{n-1} g_{n-1,k} + \sum_{k=i-1}^{n-2} g_{n-2,k} - \sum_{k=i+1}^{n-1} g_{n-1,k} - \sum_{k=i}^{n-2} g_{n-2,k} \\ &= g_{n-1,i} + g_{n-2,i-1}. \end{aligned}$$

The same relation of $h_{n,k}$ can be obtained by the similar way with the help of part (c). □

In the proof above, identities (5) and (6) form a recurrence relation of $g_{n,k}$ and $h_{n,k}$ with $g_{1,1} = h_{1,1} = 1$ initially. As a consequence of part (c), we find a relation between $g_n(M)$ and $h_n(M)$.

Corollary 3.2. *The two functions* g_n *and* h_n *satisfy* $g_n(M) = h_n(M)/M + h_{n-1}(M)$ *for* $n \geqslant 1$ *with* $h_0(M) = 0$.

Also the following result is easily derived by parts (a), (b), and (d) of the last theorem.

Corollary 3.3. *The generating functions*

$$G(x,y)=\sum_{n\geqslant k\geqslant 1} g_{n,k}x^{n-1}y^{k-1} \quad \textit{and} \quad H(x,y)=\sum_{n\geqslant k\geqslant 1} h_{n,k}x^{n-1}y^{k-1}$$

have closed forms:

$$H=\frac{M(x)-y}{1-y+xy+x^2y^2} \quad \textit{and} \quad G=(1+xy)H.$$

We can arrange $g_{n,k}$ and $h_{n,k}$ into the following triangles, and call them *Motzkin g-triangle* and *Motzkin h-triangle*, respectively:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 0 & & & \\
 & & 4 & & 4 & & 1 & & 0 & & \\
 & 9 & & 9 & & 3 & & 0 & & 0 & \\
21 & & 21 & & 8 & & 1 & & 0 & & 0,
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 0 & & & & \\
 & & & 2 & & 1 & & 0 & & & \\
 & & 4 & & 2 & & 0 & & 0 & & \\
 & 9 & & 5 & & 1 & & 0 & & 0 & \\
21 & & 12 & & 3 & & 0 & & 0 & & 0.
\end{array}$$

The h-triangle is known, in literary, to be the Motzkin triangle [18], but the g-triangle is new. Furthermore, the sequence

$$\left\langle \sum_{k=1}^{n+1} h_{n,k} \right\rangle_{n=1}^{\infty} = \langle 1, 1, 3, 6, 15, 36, \ldots\rangle$$

is known to be the Riordan numbers [4]. Hence the Motzkin expansion reveal some structural relation between Motzkin h-triangle and Riordan numbers.

We now give combinatorial interpretations for $g_{n,k}$ and $h_{n,k}$.

Lemma 3.4. (a1) *The coefficient $h_{n,k}$ enumerates the number of $(n+1)$-Motzkin paths with k rise steps followed by a fall step, and also*

(a2) *the number of $(n+1)$-Motzkin paths with k blocks and no level step on x-axis.*

(b) *The coefficient $g_{n,k}$ enumerates the number of n-Motzkin paths with $k-1$ rise steps followed by a level or a fall step.*

Proof. The bijection between Motzkin paths with consecutive rise steps in (a1) and blocks in (a2) is a routine check. So we only focus on the proof of (a1) and (b). In the following we give a simple sketch of a bijection for the equation

$$m_t = \sum_{k=1}^{\lceil (n+1)/2 \rceil} g_{n,k} \left[x^{t-n}\right] M^k + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} h_{n,k} \left[x^{t-n-1}\right] M^{k+1}$$

for $t \geqslant n \geqslant 1$. This bijection is similar to the combinatorial proof of Theorem 2.1.

Let Z be a Motzkin path with length t and $Z(i)$ denote the ith step of Z. The first step, say $Z(j)$, that meets the line $x + y = n$ has two possible cases:

(i) $Z(j)$ is a rise or level step and meets the line at its end, or
(ii) $Z(j)$ crosses the line.

For case (i), the meet should be the point $(j, n - j)$ and $\lceil n/2 \rceil \leqslant j \leqslant n$. Let us mark the first j steps black, and after jth step mark the first fall step from the level $y = i$ white for $i = 1, 2, \ldots, n - j$. (No step is marked white when $j = n$.) Let $k = n - j + 1$; so $1 \leqslant k \leqslant \lceil (n+1)/2 \rceil$. Clearly, the unmarked steps are separated by white steps into k disjoint Motzkin paths with total length equal to $t - n$; so the combination of these disjoint paths is corresponding to the term $[x^{t-n}]M^k$. Also the marked steps forms a dual of an n-Motzkin path that begins with $k - 1$ rise (white) steps followed by either a level or fall step. Therefore the number of such paths must be $g_{n,k}$.

For case (ii), $Z(j)$ must be a rise step which meets the line $x + y = n + 1$ at its end and $\lceil (n+1)/2 \rceil \leqslant j \leqslant n$. Again we set $k = n - j + 1$; so $1 \leqslant k \leqslant \lfloor (n+1)/2 \rfloor$. Similar to above argument, from Z, one can extract an $(n + 1)$-Motzkin path whose dual begins with k rise steps followed by a fall step and k disjoint Motzkin paths with total length equal to $t - n - 1$. Thus the proof of part (a1) follows. □

By replacing the level step that follows the consecutive rise steps in (b) with a rise step and then a fall, one obtains an $(n + 1)$-Motzkin path described in (a1). This just give a bijective proof for the identity $g_{n,k} = h_{n,k} + h_{n-1,k-1}$ in Theorem 3.1(c). The other parts in that theorem can be derived too by this combinatorial meaning of $g_{n,k}$ and $h_{n,k}$.

Given a path as described in (a2) of the last lemma. By removing those steps intersecting with the x-axis, it becomes k disjoint Motzkin paths of total length equal to $n - 2k + 1$. So we get the following identity as a direct consequence of Lemma 3.4(a2).

Corollary 3.5 [18]. *The sequence $h_{n,k}$ satisfies $h_{n,k} = [x^{n-2k+1}]M^k$ or $[x^n]M^k = h_{n+2k-1,k}$ for $k \geqslant 1$.*

4. Dyck paths with flaws and the Chung–Feller theorem

Dyck paths are among the most studied objects counted by Catalan numbers [7, 13,14]. Consider a path P from $(0, 0)$ to $(2n, 0)$ with rise and fall steps. We say P is an *n-Dyck path with m flaws* if P has m fall steps under the x-axis. Of course, there are same amount of rise steps under the x-axis. A normal Dyck path has no flaw on itself. It is surprising that the number of n-Dyck paths with m flaws is independent of m.

Theorem 4.1 (Chung–Feller theorem [6]). *The number of n-Dyck paths with exactly m flaws* $(0 \leqslant m \leqslant n)$ *is the Catalan number c_n.*

Chung and Feller originally considered the fluctuation of coin-tossing. For other proofs see [1,11,16]. Here the authors provide a new proof. An analogous approach for counting the number of Motzkin paths with k flaws will be demonstrated in the next section.

New proof of the Chung–Feller theorem. Let us adopt $\mathcal{D}$ and $(\mathcal{D}^k)_n$ from the combinatorial proof of Theorem 2.1. Also let $\mathcal{D}_{n,m}$ be the set of n-Dyck paths with m flaws and $\mathcal{F}_{m,k}$ the set of m-Dyck paths with $k-1$ blocks. Notice that $|\mathcal{F}_{m,k}| = f_{m,k}$. We shall establish a bijection between $\mathcal{D}_{n,m}$ and

$$\bigcup_{k=2}^{m+1} \bigl(\mathcal{F}_{m,k}, \bigl(\mathcal{D}^k\bigr)_{n-m}\bigr)$$

so that the proof follows by

$$|\mathcal{D}_{n,m}| = \sum_{k=2}^{m+1} f_{m,k}\bigl[x^{n-m}\bigr]C^k = c_n,$$

where the second equation is obtained by considering the x^n term of the mth Catalan expansion.

Let D be an n-Dyck path with m flaws. The flaw steps of D form an m-Dyck path if we see them upside down. Assume that such m-Dyck path has $k-1$ blocks $(2 \leqslant k \leqslant m+1)$; so it is an element of $\mathcal{F}_{m,k}$. By removing all flaw steps, one obtains k disjoint Dyck paths of total length $n-m$; so they form an element of $(\mathcal{D}^k)_{n-m}$. Obviously, this is the desired bijection. $\square$

5. Motzkin paths with flaws

We define *Motzkin paths with flaws*, but the index of flaws counts all steps under the x-axis. Let $m_{p,q}$ be the number of p-Motzkin paths with q flaws. Clearly, $m_{p,0} = m_p$ and $m_{p,1} = 0$. The following theorem gives two general formulae of $m_{p,q}$.

Theorem 5.1. *For any integers p and q with $p \geqslant q \geqslant 0$,*

$$m_{p,q} = \sum_{k=1}^{\lfloor q/2 \rfloor} h_{q-1,k} h_{p-q+2k+1,k+1} = m_p - \sum_{k=1}^{\lceil q/2 \rceil} g_{q-1,k} h_{p-q+2k,k}.$$

Proof. Let P be p-Motzkin paths with q flaws. Concatenating all flaw steps of P, we obtain a q-Motzkin path without level steps on x-axis. Assume such path has k blocks ($1 \leqslant k \leqslant \lfloor q/2 \rfloor$), hence it is the kind of path described in Lemma 3.4(a2). By removing all flaw steps from P, one obtains $k+1$ disjoint Motzkin paths of total length equal to $p-q$. By Corollary 3.5, this bijection implies that

$$\begin{aligned} m_{p,q} &= \sum_{k=1}^{\lfloor q/2 \rfloor} h_{q-1,k} \left[x^{p-q}\right] M^{k+1} \\ &= \sum_{k=1}^{\lfloor q/2 \rfloor} h_{q-1,k} h_{p-q+2k+1,k+1}. \end{aligned} \tag{7}$$

Since the right hand side of (7) also equals $[x^p](M - x^{q-1} g_{q-1}(M))$, we get

$$\begin{aligned} m_{p,q} &= m_p - \sum_{k=1}^{\lceil q/2 \rceil} g_{q-1,k} \left[x^{p-q+1}\right] M^k \\ &= m_p - \sum_{k=1}^{\lceil q/2 \rceil} g_{q-1,k} h_{p-q+2k,k}. \qquad \square \end{aligned} \tag{8}$$

For example, there are $m_6 - g_{3,1}h_{4,1} - g_{3,2}h_{6,2} - g_{3,3}h_{8,3} = 19$ different 6-Motzkin paths with 4 flaws which are illustrated in Fig. 1.

Using (7) in the last proof, we find out

$$\begin{aligned} m_{p,2} &= m_p - m_{p-1}, \\ m_{p,3} &= m_{p-1} - m_{p-2}, \\ m_{p,4} &= m_p - 2m_{p-1} + 2m_{p-2} - 2m_{p-3}. \end{aligned}$$

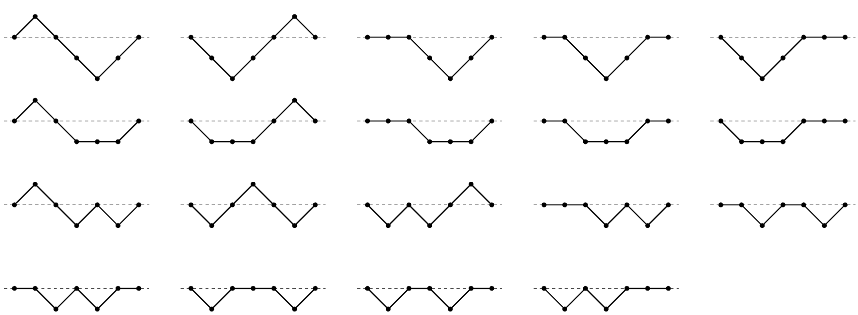

Fig. 1. 6-Motzkin paths with 4 flaws.

Actually $m_{p,q}$ can be written as an alternating sum of Motzkin numbers by the following method.

Theorem 5.2. *The sequence $m_{p,q}$ satisfies the recurrence relation*

$$m_{p,q} = m_p - m_{p,q-1} - \sum_{i=0}^{q-2} m_i\, m_{p-i-1,q-i-2} \quad \textit{for } p \geqslant q \geqslant 2, \tag{9}$$

with the initial condition $m_{p,0} = m_p$ and $m_{p,1} = 0$.

Proof. Let $z_{p,q} = [x^{p-q}]h_{q-1}(M)/M$. Referring the interpretation of $m_{p,q} = [x^{p-q}]h_{q-1}(M)$ (Eq. (7)), we know that $z_{p,q}$ counts the p-Motzkin paths with q flaws with the first step being fall. Extracting the first block of flaws, we get

$$z_{p,q} = \sum_{i=0}^{q-2} m_i m_{p-i-2,q-i-2} \quad \text{for } q \geqslant 2$$

with initial condition $z_{p,0} = z_{p,1} = 0$. Now the proof follows by

$$\begin{aligned} m_{p,q} &= m_p - \left[x^{p-q+1}\right] g_{q-1}(M) \\ &= m_p - \left[x^{p-q+1}\right] h_{q-2}(M) - \left[x^{p-q+1}\right] h_{q-1}(M)/M, \end{aligned}$$

with the help of (9) and Corollary 3.2. □

Fix p, then (9) is a linear recurrence relation with coefficients $m_0, m_1, \ldots, m_{q-2}$. Therefore, by induction, $m_{p,q}$ is an alternating sum of $m_{p-q+1}, m_{p-q+2}, \ldots, m_p$. Here we list the results of $m_{p,5}$ to $m_{p,8}$:

$$\begin{aligned} m_{p,5} &= 2m_{p-1} - 4m_{p-2} + 4m_{p-3} - 4m_{p-4}, \\ m_{p,6} &= m_p - 3m_{p-1} + 6m_{p-2} - 9m_{p-3} + 9m_{p-4} - 9m_{p-5}, \\ m_{p,7} &= 3m_{p-1} - 9m_{p-2} + 15m_{p-3} - 21m_{p-4} + 21m_{p-5} - 21m_{p-6}, \\ m_{p,8} &= m_p - 4m_{p-1} + 12m_{p-2} - 25m_{p-3} + 38m_{p-4} - 51m_{p-5} \\ &\quad + 51m_{p-6} - 51m_{p-7}. \end{aligned}$$

So far, we still looking for combinatorial meanings of these alternating coefficients.

Conclusion

Our Taylor style expansion can be applied to any generating function, $G = \sum_{i\geqslant 0} g_i x^i$, satisfying $G = 1 + \sum_{i=1}^{m} x^i p_i(G)$, where p_i are polynomials. We can set the nth expansion as

$$G = \sum_{i=0}^{n-1} g_i x^i + \sum_{j=0}^{m-1} x^{n+j} f_n^j(G),$$

where f_n^j are polynomials. All f_n^j can be obtained by substituting each G^k in f_{n-1}^0 by

$$1 + \sum_{i=1}^{m} x^i \left(p_i(G) \sum_{j=0}^{k-1} G^j \right).$$

If G is generating function according to a certain category of structures, then there must be a corresponding interpretation (called *remained structures*) for the coefficient of $x^i G^j$ in the remainder. In other words, besides those structures counted by $\sum_{i=0}^{n-1} g_i x^i$, each structure can be decomposed into several self-similar structures and a *remained structure*. Such decomposition reflects certain relationship between combinatorial objects and will offer a new point of view.

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