Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Discrete Mathematics 281 (2004) 189–196
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Odd or even on plane trees

Sen-Peng Eu[a,1], Shu-Chung Liu[b,2], Yeong-Nan Yeh[c,3]

[a] Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung, Taiwan, ROC
[b] Department of General Curriculum, Chung-Kuo Institute of Technology, Taipei, Taiwan, ROC
[c] Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan, ROC

Received 25 January 2002; received in revised form 21 July 2003; accepted 24 July 2003

Abstract

Over all plane trees with n edges, the total number of vertices with odd degree is twice the number of those with odd outdegree. Deutsch and Shapiro posed the problem of finding a direct two-to-one correspondence for this property. In this article, we give three different proofs via generating functions, an inductive proof and a two-to-one correspondence. Besides, we introduce two new sequences which enumerate plane trees according to the parity of the number of leaves. The explicit formulae for these sequences are given. As an application, the relation provides a simple proof for a problem concerning colored nets in Stanley's *Catalan Addendum*.

Keywords: Catalan numbers; Plane trees; Odd and even nets; Outdegree

1. Introduction

Among the combinatorial structures counted by the Catalan number, $c_n = (1/(n+1))\binom{2n}{n}$, the set of (rooted) plane trees with n edges (n-plane tree, for short), denoted $\mathscr{T}_n$, is one of the most well-known and well-studied structures. See [10,11] and references therein for a survey of families of objects enumerated by the Catalan numbers. Existing results have focused on finding bijections between such structures, as well as on various statistics on plane trees. In this article, we focus on two parameters of n-plane trees, namely, the degree of vertices and the number of leaves.

[1] Partially supported by NSC 92-2119-M-390-001.
[2] Partially supported by NSC 91-2115-M-163-001.
[3] Partially supported by NSC 91-2115-M-001-022.
E-mail address: mayeh@math.sinica.edu.tw (Y.-N. Yeh).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2003.07.011

Deutsch and Shapiro [3, p. 259], posed the problem of finding a direct two-to-one correspondence for proving combinatorially the following known property:

Theorem 1. *Over all plane trees with n edges, the total number of vertices with odd degree is twice the total number of vertices with odd outdegree.* [4]

This property follows readily from two papers of Meir and Moon [7, p. 321] [8, p. 413], in which the generating functions for the number of odd degree and for the number of vertices of outdegree k were pointed out, respectively. A direct algebraic proof was almost spelled out in [3, pp. 258–259]. In Section 2, the authors offer three different proofs of this theorem using (i) generating functions, (ii) induction, and (iii) a two-to-one correspondence. We mention that the first proof was kindly provided by one of the referees.

Inspired by an idea of the inductive proof, the authors introduce two new statistics, $\bar{o}_n$ and $\bar{e}_n$, which are the cardinalities of two disjoint subsets of $\mathscr{T}_n$ according to the parity of the number of leaves. In Section 3, the authors show the explicit formulae of these two statistics. Surprisingly, the relation $\bar{e}_{2n+1} - \bar{o}_{2n+1} = (-1)^{n+1}c_n$ coincides with the one in an interesting problem posed in Stanley's *Catalan Addendum* [11] stated as follows (including the theorem):

Join $4m+2$ points on the circumference of a circle with $2m+1$ non-intersecting chords. Call such a set of chords a *net*. The circle together with the chords forms a map with $2m+2$ (interior) regions. Color the regions red and blue so that adjacent regions receive different colors. Call the net *even* if an even number of regions are colored red and an even number blue, and *odd* otherwise.

Theorem 2. *Let $f_e(m)$ ($f_o(m)$) denote the number of even (odd) nets on $4m+2$ points. Then*

$$f_e(m) - f_o(m) = (-1)^{m+1}c_m. \tag{1}$$

This result was first proved by Eremenko and Gabrielov using techniques from algebraic geometry [5]. A solution using recurrence relations was also provided in [11]. Here, in Section 3, through a bijection between nets and plane trees, the relation $\bar{e}_{2n+1} - \bar{o}_{2n+1} = (-1)^{n+1}c_n$ actually offers a simple combinatorial proof of Theorem 2.

Further lines of research following this article might focus on the parity, or modularity, of certain parameters over different kinds of structures, to name a few.

2. Odd degree and odd outdegree

We first introduce some notations. The root of a plane tree with odd (even) degree is called an *odd* (*even*) *root*. Given a family of plane trees $\mathscr{A} \subseteq \mathscr{T}_n$, let $\mathscr{D}_o(\mathscr{A})$ ($\mathscr{D}_e(\mathscr{A})$) be the total number of vertices of odd (even) degree in $\mathscr{A}$. Similarly, the values $\mathscr{OD}_o(\mathscr{A})$ and $\mathscr{OD}_e(\mathscr{A})$ are defined according to the parity of the outdegree of

[4] The outdegree of a vertex makes sense after orienting every plane tree from its root to all its leaves.

vertices in $\mathscr{A}$. Theorem 1 can be restated simply as $\mathscr{D}_o(\mathscr{T}_n) = 2\mathscr{O}\mathscr{D}_o(\mathscr{T}_n)$ for any nonnegative integer n. Given a plane tree T, we always write r as the root of T. For any vertex v of T, let $\bar{v}$ and $\underline{v}$ be the parent and the leftmost child of v, if they exist. Also let T_v denote the plane subtree of T induced by v and its descendants. If T is a nonempty plane tree with root r, then we denote by $T - T_{\underline{r}}$ the tree obtained by removing $T_{\underline{r}}$ and the edge $r\underline{r}$ from T.

Proof of Theorem 1 by generating functions. Let $C(z) := \sum_{n\geqslant 0} c_n z^n$ denote the generating function of the Catalan number, $c_n = |\mathscr{T}_n|$. It is well known that $C(z) = (1 - \sqrt{1-4z})/2z$ and that $C = C(z)$ satisfies $C = 1 + zC^2$, or equivalently $1/C = 1 - zC$. For a fixed nonnegative integer k, let $a_k(m,n)$ be the number of n-plane trees with exactly m vertices of outdegree k, and $G_k(t,z)$ its generating function, i.e., $G_k(t,z) := \sum_{m,n\geqslant 0} a_k(m,n) t^m z^n$.

By considering the outdegree of the root and T_v for all children v of the root, one can easily derive that $G_k = G_k(t,z)$ satisfies

$$\begin{aligned} G_k &= (1 + zG_k + z^2(G_k)^2 + \cdots) - z^k(G_k)^k + tz^k(G_k)^k \\ &= \frac{1}{1 - zG_k} + (t-1)z^k(G_k)^k. \end{aligned}$$

Differentiating G_k with respect to t, setting $t = 1$, and making use of the facts that $G_k(1,z) = C(z)$ and $1/C = 1 - zC$, we obtain the generating function for the number of vertices of outdegree k over all plane trees in $\mathscr{T}_n$:

$$G_k'(1,z) = \frac{z^k C^k}{1 - zC^2}. \tag{2}$$

From here, we obtain that

(i) $H_1 := zCF/(1 - zC^2)$ is the generating function of $\mathscr{O}\mathscr{D}_o(\mathscr{T}_n)$, where $F = 1 + z^2C^2 + z^4C^4 + \cdots$ is the generating function of the Fine numbers [3,6].
(ii) $H_2 := F/(1 - zC^2)$ is the generating function of $\mathscr{O}\mathscr{D}_e(\mathscr{T}_n)$.

It is easy to see that

(iii) $H_3 := zC + z^3C^3 + z^5C^5 + \cdots = zCF$ is the generating function for the number of n-plane trees with odd roots.
(iv) $H_4 := 1 + z^2C^2 + z^4C^4 + \cdots = F$ is the generating function for the number of n-plane trees with even roots.

A simple computation, together with $C = 1 + zC^2$, then yields

$$H_2 + H_3 - H_4 = 2H_1.$$

The left-hand side of this equation is exactly the generating function of $\mathscr{D}_o(\mathscr{T}_n)$. This completes the proof. □

Remark. Setting $k = 0$ in (2), we obtain the generating function $1/(1 - zC^2)$ for the total number of leaves in $\mathscr{T}_n$.

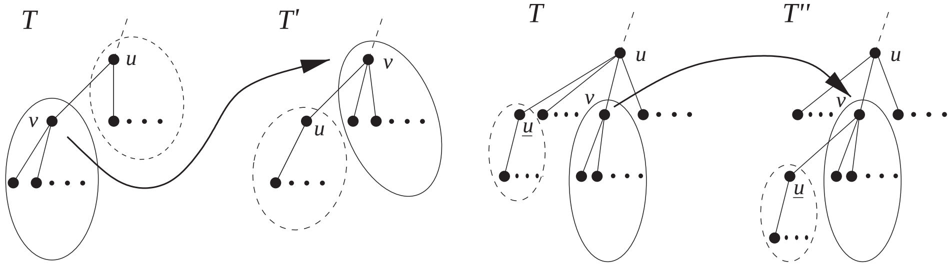

Fig. 1. The two-to-one correspondence g.

Proof of Theorem 1 by a two-to-one correspondence. Let (T, v) be an ordered pair, where T is an n-plane tree and v a vertex of T. We shall establish a two-to-one correspondence g between vertices with odd degree and vertices with odd outdegree, i.e.,

$$g : \{(T, v) : T \in \mathscr{T}_n \text{ and } v \in T \text{ is of odd degree}\}$$
$$\rightarrow \{(T, v) : T \in \mathscr{T}_n \text{ and } v \in T \text{ is of odd outdegree}\}.$$

Suppose that $v \in T$ is of odd degree. If $v = r$, then we simply let $g(T, v) := (T, v)$. Otherwise, we let u be the parent of v. If $v = \underline{u}$, then we let $g(T, v) := (T', v)$, where T' is the plane tree obtained by interchanging two subtrees T_v and $T_u - T_v$ in T. Clearly, v is of odd outdegree in T'. If $v \neq \underline{u}$, then we let $g(T, v) := (T'', v)$, where T'' is the plane tree obtained by deleting the edge $u\underline{u}$, followed by attaching $T_{\underline{u}}$ so that $\underline{u}$ becomes the leftmost child of v. See Fig. 1 for graphical descriptions of the two operations $T \rightarrow T'$ and $T \rightarrow T''$.

The inverse mapping g^{-1} is easy to figure out. Let $v \in T$ be of odd outdegree, so that $\underline{v}$ exists. If $v \neq r$, then $g^{-1}(T, v) := \{(T^*, v), (T^{**}, v)\}$ where T^* is obtained by interchanging $T_{\underline{v}}$ and $T_v - T_{\underline{v}}$ in T (i.e., the inverse of $(T, v) \rightarrow (T', v)$), and T^{**} is obtained by deleting $v\underline{v}$ and moving $T_{\underline{v}}$ so that $\underline{v}$ becomes the leftmost child of $\bar{v}$ (i.e., the inverse of $(T, v) \rightarrow (T'', v)$). If $v = r$, then $g^{-1}(T, v) := \{(T, v), (T^*, v)\}$. From here it follows that g is a two-to-one correspondence. □

Proof of Theorem 1 by induction. The case $n = 0$ holds trivially, since $\mathscr{D}_o(\mathscr{T}_0) = 2\mathscr{O}\mathscr{D}_o(\mathscr{T}_0) = 0$. So let us assume $n \geqslant 1$ and $\mathscr{D}_o(\mathscr{T}_i) = 2\mathscr{O}\mathscr{D}_o(\mathscr{T}_i)$ for $0 \leqslant i \leqslant n - 1$.

A bijection $f : \mathscr{T}_n \mapsto \bigcup_{i=0}^{n-1} (\mathscr{T}_i \times \mathscr{T}_{n-1-i})$ can be easily established by defining $f(T) = (f_1(T), f_2(T))$, where $f_1(T) = T_{\underline{r}}$ and $f_2(T) = (T - T_{\underline{r}})_r$. The inverse $f^{-1}(T', T'')$ is the plane tree obtained by introducing an edge between the roots of T' and T'', with root same as that of T'', and T' as the leftmost subtree. Such bijection is used very often in the form: "every nonempty plane tree can be decomposed into two plane subtrees by removing the leftmost edge of its root".

To prove $\mathscr{D}_o(\mathscr{T}_n) = 2\mathscr{O}\mathscr{D}_o(\mathscr{T}_n)$, it suffices to show the following lemma:

Lemma 3. *Let* $0 \leqslant i \leqslant n-1$. *Over all n-plane trees in* $f^{-1}(\mathscr{T}_i \times \mathscr{T}_{n-1-i}) \uplus f^{-1}(\mathscr{T}_{n-1-i} \times \mathscr{T}_i)$, *the total number of vertices of odd degree is twice the total number of vertices of odd outdegree, where* $\uplus$ *denotes disjoint union.*

This lemma can alternatively be proved by the two-to-one correspondence described in the previous proof; by exploiting the induction hypothesis, an algebraic proof readily obtains.

Let o_n (e_n) be the number of n-plane trees with odd (even) root. When adding an edge between the roots of T' and T'', only the parity of degrees of both roots, as well as that of the outdegree of T'' are changed; the degree and outdegree of other vertices of T' and T'' remain unchanged. Thus, one can easily establish the following identities:

$$\begin{aligned}\mathscr{D}_o(f^{-1}(\mathscr{T}_i \times \mathscr{T}_{n-1-i})) &= [\mathscr{D}_o(\mathscr{T}_i) + (e_i - o_i)\ c_{n-1-i}] \\ &\quad + [\mathscr{D}_o(\mathscr{T}_{n-1-i}) + (e_{n-1-i} - o_{n-1-i})c_i] \\ &= \mathscr{D}_o(f^{-1}(\mathscr{T}_{n-1-i} \times \mathscr{T}_i)),\end{aligned}$$

$$\mathscr{O}\mathscr{D}_o(f^{-1}(\mathscr{T}_i \times \mathscr{T}_{n-1-i})) = \mathscr{O}\mathscr{D}_o(\mathscr{T}_i) + [\mathscr{O}\mathscr{D}_o(\mathscr{T}_{n-1-i}) + (e_{n-1-i} - o_{n-1-i})c_i],$$

$$\mathscr{O}\mathscr{D}_o(f^{-1}(\mathscr{T}_{n-1-i} \times \mathscr{T}_i)) = \mathscr{O}\mathscr{D}_o(\mathscr{T}_{n-1-i}) + [\mathscr{O}\mathscr{D}_o(\mathscr{T}_i) + (e_i - o_i)c_{n-1-i}].$$

Let $T' \in \mathscr{T}_i$ and r_1 be its root, and $T'' \in \mathscr{T}_{n-1-i}$ and r_2 be its root. Since the degree of vertices other than r_1 and r_2 are preserved under f^{-1}, this accounts for the two contributions $\mathscr{D}_o(\mathscr{T}_i)$ and $\mathscr{D}_o(\mathscr{T}_{n-1-i})$. Note that r_1 has an odd (even) degree in $f^{-1}(T', T'')$ if and only if it has an even (odd) degree in T', and analogously for r_2. It follows that there are $(e_i - o_i)$ choices of T' for which r_1 has an odd degree in $f^{-1}(T', T'')$, with $T'' \in \mathscr{T}_{n-1-i}$ being arbitrary, thus accounting for the contribution $(e_i - o_i)c_{n-1-i}$. By a similar argument, we have the contribution $(e_{n-1-i} - o_{n-1-i})c_i$. Putting pieces together, we have the first identity. The remaining identities can be derived in a similar way.

By virtue of the induction hypothesis $\mathscr{D}_o(\mathscr{T}_i) = 2\mathscr{O}\mathscr{D}_o(\mathscr{T}_i)$ for $0 \leqslant i \leqslant n-1$, the above identities directly yield

$$\begin{aligned}&\mathscr{D}_o(f^{-1}(\mathscr{T}_i \times \mathscr{T}_{n-1-i})) + \mathscr{D}_o(f^{-1}(\mathscr{T}_{n-1-i} \times \mathscr{T}_i)) \\ &\quad = 2\mathscr{O}\mathscr{D}_o(f^{-1}(\mathscr{T}_i \times \mathscr{T}_{n-1-i})) + 2\mathscr{O}\mathscr{D}_o(f^{-1}(\mathscr{T}_{n-1-i} \times \mathscr{T}_i)),\end{aligned}$$

which completes the proofs of Lemma 3, as well as Theorem 1. □

3. Odd leaves and even leaves

The number o_n (e_n) defined in the inductive proof also counts the number of Dyck paths of semilength n that have an odd (even) number of returns to the x-axis. The e_n's are known as the Fine numbers. According to the parity of yet another parameter, one can partition plane trees or Dyck paths into two categories, and then obtain a pair of new sequences. For $n \geqslant 1$, let $\bar{o}_n$ $(\bar{e}_n)$ be the number of n-plane trees with odd (even) number of leaves. Also set $\bar{o}_0 := \bar{e}_0 := 0$. Note that leaves in a plane tree correspond, by virtue of a standard bijection, to peaks in a Dyck path. By considering the subtrees $T_{\underline{r}}$ and $T - T_{\underline{r}}$, one derives that the generating functions, $\bar{O}$ and $\bar{E}$, of $\bar{o}_n$ and $\bar{e}_n$ satisfy

$$\bar{O} = z(1 + \bar{O} + \bar{E} + 2\bar{O}\bar{E})$$

and

$$\bar{E} = z(\bar{O} + \bar{E} + \bar{O}^2 + \bar{E}^2).$$

A little algebra yields the quadratic equation $z(\bar{E} - \bar{O})^2 - (\bar{E} - \bar{O}) - z = 0$ whose solution is

$$\bar{E} - \bar{O} = \frac{1 - \sqrt{1 + 4z^2}}{2z} = -zC(-z^2).$$

By extracting the coefficients of z^n, we have

$$\bar{e}_{2n} - \bar{o}_{2n} = 0, \tag{3}$$

$$\bar{e}_{2n+1} - \bar{o}_{2n+1} = (-1)^{n+1} c_n, \tag{4}$$

which, together with the obvious relation $\bar{e}_n + \bar{o}_n = c_n$ for $n \geqslant 1$, enable us to derive explicit formulae of $\bar{o}_n$ and $\bar{e}_n$ as follows:

Theorem 4. *Let $\bar{o}_n$ and $\bar{e}_n$ be the numbers of n-plane trees ($n \geqslant 1$) with odd number and even number of leaves, respectively. Then,*

$$\bar{o}_{2n} = \bar{e}_{2n} = \frac{1}{4n+2}\binom{4n}{2n} = \frac{1}{2} c_{2n},$$

$$\bar{o}_{2n+1} = \frac{1}{4n+4}\binom{4n+2}{2n+1} - (-1)^{n+1}\frac{1}{2n+2}\binom{2n}{n} = \frac{1}{2}(c_{2n+1} - (-1)^{n+1} c_n),$$

$$\bar{e}_{2n+1} = \frac{1}{4n+4}\binom{4n+2}{2n+1} + (-1)^{n+1}\frac{1}{2n+2}\binom{2n}{n} = \frac{1}{2}(c_{2n+1} + (-1)^{n+1} c_n).$$

The first few terms of $\bar{o}_n$ and $\bar{e}_n$ are $0, 1, 1, 2, 7, 22, 66, 212, \ldots$ and $0, 0, 1, 3, 7, 20, 66, 217, \ldots$. These two new sequences have just been registered in [9] as “the number of plane trees having an odd number of leaves” (sequence A071684) and “the number of plane trees having an even number of leaves” (sequence A071688), respectively.

The resemblance of Eqs. (1) and (4) suggests a simple proof of Theorem 2 by exploiting a bijection between nets and plane trees. Before doing this, we refer to a relation between leaves and even-level vertices, where the *level* of a vertex is the distance between this vertex and the root.

Theorem 5 (Deutsch [2, p. 213]). *In $\mathscr{T}_n$, the number of leaves and the number of even-level vertices are equidistributed.*

The next useful corollary is a direct consequence of this theorem.

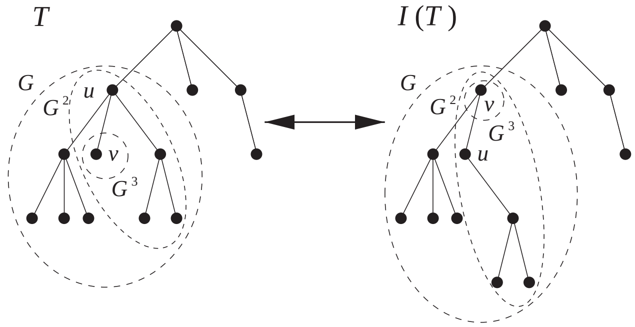

Fig. 2. Examples of involution I.

Corollary 6. *The number of n-plane trees containing an odd (even) number of even-level vertices is equal to* $\bar{o}_n$ $(\bar{e}_n)$.

Proof of Theorem 2. Taking into account Eq. (4), we need only show that $f_o(n) = \bar{o}_{2n+1}$ and $f_e(n) = \bar{e}_{2n+1}$ for $n \geqslant 0$. From Corollary 6, it suffices to show that

> the number of $(2n+1)$-plane trees containing an odd (even) number of even-level vertices is equal to $f_o(n)$ $(f_e(n))$.

Now let a net with points $v_1, v_2, \ldots, v_{4n+2}$ be given. We regard each region as a vertex and link two adjacent regions by an edge. We obtain a plane tree with $2n+1$ edges if we always set the region containing both points v_1 and v_2 to be the root. We can color the even-level vertices red and the odd-level vertices blue to form a colored net whose parity is the same as the parity of even-level vertices of the plane tree. Since this association of colored nets with plane trees is a bijection, the above result now follows from Eq. (4). □

We conclude this work by proving (3) using a sign-reversing involution. Readers are suggested to consult [1] and [4] for other proofs of (3).

Proof of Eq. (3) by an involution. Let us define an involution $I : \mathscr{T}_{2n} \to \mathscr{T}_{2n}$ for $n \geqslant 1$ by the following way. Given a plane tree T_r with even number of edges and root r, either $T_{\underline{r}}$ or $T_r - T_{\underline{r}}$ has even number of edges. So we let $G(T_r)$ denote this plane subtree with even number of edges. For any $T \in \mathscr{T}_{2n}$, there is the least k such that $G^{k+1}(T)$ is a single vertex. Suppose $G^k(T) = T_u$ and $v = \underline{u}$. The plane tree $I(T)$ is obtained from T by exchanging the subtrees T_v and $T_u - T_v$. See Fig. 2 for a graphical description of the operation $T \to I(T)$.

It is easy to check that $I^2(T) = T$ and that the parities of leaves in T and $I(T)$ are different. Thus I is the involution being sought and the proof of Eq. (3) follows. □

Acknowledgements

We thank Richard Stanley for his bringing to our attention the problem of colored nets, the referees for helpful comments and suggestions, and especially one of them for his kindly providing us the first proof of Theorem 1.

References

[1] N. Dershowitz, S. Zaks, Enumerations of ordered trees, Discrete Math. 31 (1980) 9–28.
[2] E. Deutsch, A bijection on ordered trees and its consequences, J. Combin. Theory Ser. A 90 (2000) 210–215.
[3] E. Deutsch, L. Shapiro, A survey of the fine numbers, Discrete Math. 241 (2001) 241–265.
[4] E. Deutsch, L. Shapiro, A bijection between ordered trees and 2-Motzkin paths and its many consequences, Discrete Math. 256 (2002) 655–670.
[5] A. Eremenko, A. Gabrielov, The Wronski map and real Schubert calculus preprint, http://www.math.purdue.edu/∼eremenko/newprep.html.
[6] T. Fine, Extrapolation when very little is known about the source, Inform. and Control 16 (1976) 331–359.
[7] A. Meir, J.W. Moon, Path edge-covering constants for certain families of trees, Utilitas Math. 14 (1978) 313–333.
[8] A. Meir, J.W. Moon, On nodes of out-degree one in random tree, in: Combinatorics, Colloqium Mathematical Society of János Bolyai, Vol. 52, North-Holland, Amsterdam, 1988, pp. 405–416.
[9] N.J.A. Sloane (Ed.), The On-Line Encyclopedia of Integer Sequences, 2003, published electronically at http://www.research.att.com/∼njas/sequences/.
[10] R. Stanley, Enumerative Combinatorics, Vol. II, Cambridge University Press, Cambridge, 1999.
[11] R. Stanley, Problem 6.C6, in: Catalan Addendum, version of July 1, 2002, published electronically at http://www-math.mit.edu/∼rstan/ec/catadd.pdf.