Ordering the Complements of Trees by the Number of Maximum Matchings

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ABSTRACT: A "perfect matching" of a graph G with n vertices is a set of $\lfloor n/2 \rfloor$ independent edges of G. In the present study, we succeeded in determining the trees whose complements have the extremal number of "perfect matchings" for two different group of trees. Some further problems are also posed. Int J Quantum Chem 105: 131–141, 2005

Key words: tree; complement; perfect matching; maximum matching; adjacency matrix; characteristic polynomial; transformation of trees

1. Introduction

Throughout this article, we will assume that $G = [V(G), E(G)]$ is a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of G, which will be denoted by $A(G) = (a_{ij})_{n\times n}$, is a square matrix of order n, whose entry a_{ij} is 1 if v_i and v_j are adjacent and 0 otherwise. The characteristic polynomial $\phi(G) = \det[xI - A(G)]$ of $A(G)$ is called the characteristic polynomial of the graph G, where I is a unit matrix of order n. Let e and v be an edge and a vertex of G. Denote by $G - e$ or $G - v$ the graph obtained from G by removing e or v, respectively. Denote by $\bar{G}$ the complement of a graph G. A subset $\mathcal{M}$ of $E(G)$ is called a matching of G if every vertex of G is incident with at most one edge in $\mathcal{M}$; and $\mathcal{M}$ is called a perfect matching of G if every vertex of G is incident with exactly one edge in $\mathcal{M}$; and $\mathcal{M}$ is called a maximum matching of G if G has no matching $\mathcal{M}'$ such that $|\mathcal{M}'| > |\mathcal{M}|$. If $\mathcal{M}$ is a maximum matching of G, then $|\mathcal{M}|$ is called the edge-independence number of G, denoted by $\nu(G)$. Denote by

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$m(G, i)$ the number of matchings with i edges in G, and in particular by $M(G)$ the number of perfect matchings of G if n is even and the number of matchings with $(n-1)/2$ edges of G otherwise. Hence, by definition, $M(G) = m(G, \lfloor n/2 \rfloor)$. Denote by $\mathcal{T}_n$ the set of trees with n vertices and by $\mathcal{T}_{n,p}$ the set of trees with n vertices and with the edge-independence number at least p for $p = 1, 2, \ldots, \lfloor n/2 \rfloor$. Let P_n be a path with n vertices, and $K_{1,n-1}$ a star with n vertices, and $T_{n,2}$ a tree with n vertices obtained from the star $K_{1,3}$ by attaching a path P_{n-3} to one of the pendant vertices of $K_{1,3}$ (i.e., by identifying one of the pendant vertices of $K_{1,3}$ and one of the pendant vertices of P_{n-3}), and T_n^p a tree with n vertices obtained from the star $K_{1,n-p}$ by attaching a pendant edge to each of $p-1$ pendant vertices in $K_{1,n-p}$ for $p = 1, 2, \ldots, \lfloor n/2 \rfloor$, respectively. Undefined terminology and notation of graph theory are referred to [1, 2].

The problem of ordering graphs by some indices was investigated extensively not only by mathematicians, but also by mathematical chemists (see Klein and Babić [3] and Klein [4]). The study of ordering graphs by their spectral radius (i.e., the largest eigenvalue) was first studied by Collatz and Sinogowitz [5] in 1957. For trees with n vertices, Lovász and Pelikan [6] determined the extreme cases by proving that the star $K_{1,n-1}$ has the largest spectral radius ($\sqrt{n-1}$) and the path P_n has smallest spectral radius [$2\cos(\pi/(n+1))$]. Recently, there has been great progress in this field. The reader is referred to Refs. [7–15]. In contrast, the problem of ordering graphs by some chemical indices (e.g., the Hosoya index, the Randić index, the total π-electron energy or the number of perfect matchings) has been considered. Gutman [16] proved that among all trees with n vertices the path P_n has the greatest Hosoya index and the star $K_{1,n-1}$ has the smallest Hosoya index. Hou [17] characterized the trees of a given size of matchings that have a minimal and second minimal Hosoya index. Caporossi et al. [7] proved that among trees with n vertices, path P_n has the maximum Randić index. Gutman [18] determined the trees with the maximal total π-electron energy and minimal total π-electron energy and gave some further results. Along this line a number of results have been reached (see, e.g., Refs. [19–23]).

Suppose that K_n is a complete graph with n vertices, and $\mathcal{G}_n$ is the set of simple graphs with n vertices. Let $\mathcal{E}$ be a set of subsets of the edge set $E(K_n)$ satisfying some selected properties, and let $f: \mathcal{G}_n \to R$ be a real-valued function. A natural problem of combinatorial optimization is to determine $\max\{f(K_n - E) | E \in \mathcal{E}\}$ and $\min\{f(K_n - E) | E \in \mathcal{E}\}$, where $K_n - E$ is the graph obtained from K_n by deleting edges in E.

In the present work, we restrict our attention to the case where $f(G)$ denotes the number of matchings of G with $\lfloor n/2 \rfloor$ edges and $\mathcal{E} = \{E(T) | T \in \mathcal{T}_n\}$ or $\mathcal{E} = \{E(T) | T \in \mathcal{T}_{n,p}\}$. That is, we consider the problem of ordering the complements $\bar{T}$ of trees T with n vertices by the number of matchings with $\lfloor n/2 \rfloor$ edges of $\bar{T}$ [i.e., $M(\bar{T})$]. In the next section, some basic results are introduced. Furthermore, we show that $\bar{T}$ has perfect matchings if n (>2) is even and $T \neq K_{1,n-1}$, and $\bar{T}$ has matchings with $(n-1)/2$ edges otherwise. In Section 3, six types of transformations I–VI of trees are defined, which play a key role in the proof of our main results. In Section 4, we prove that if $T \in \mathcal{T}_n$ and $n \geq 7$, $M(\bar{T}) \leq M(\overline{P_n})$ with equality if and only if $T = P_n$, and that if $T \in \mathcal{T}_n$, $n \geq 9$ and $T \neq P_n$, then $M(\bar{T}) \leq M(\overline{T_{n,2}})$ with equality if and only if $T = T_{n,2}$. We also prove that if $T \in \mathcal{T}_{n,p}$ then $M(\bar{T}) \geq M(\overline{T_n^p})$ with equality if and only if $T = T_n^p$. In Section 5, some further problems are posed.

2. Preliminaries

Lemma 2.1 If T is a tree with $2n+1$ vertices ($n \geq 1$), then $m(\bar{T}, n) > 0$.

Proof If $n = 1$, this is trivial. Now we assume that $n \geq 2$ and proceed by induction. Note that T has at least two vertices of degree one, denoted by u and v. Hence $T - u - v$ is a tree with $2n-1$ vertices. By the induction assumption, $\overline{T - u - v}$ has at least one matching with $n-1$ edges, denoted by $\mathcal{M}_1$. Then $\mathcal{M} = \mathcal{M}_1 \cup \{uv\}$ is a matching with n edges of $\bar{T}$. The lemma thus holds.

Remark 1 By Lemma 2.1, if T is a tree with n vertices and n is an odd integer > 1, then every maximum matching of $\bar{T}$ has $(n-1)/2$ edges. Hence $M(\bar{T})$ equals the number of maximum matchings of $\bar{T}$. Moreover, if $n \geq 3$ is odd, then $M(\bar{T}) > 0$.

Lemma 2.2 If T is a tree with $2n$ vertices and $T \neq K_{1,2n-1}$ ($n \geq 2$), then $m(\bar{T}, n) > 0$.

Proof If $n = 2$, this is trivial. Now we assume that $n \geq 3$ and proceed by induction. Note that $T \neq K_{1,2n-1}$. Then T has at least two vertices of degree one, denoted by u and v, such that $T - u - v$ is a tree with $2n-2$ vertices, which is not a star $K_{1,2n-3}$. By the induction assumption, $\overline{T - u - v}$ has at least one perfect matching, denoted by $\mathcal{M}_1$. Then $\mathcal{M} =$

$\mathcal{M}_1 \cup \{uv\}$ is a perfect matching of $\bar{T}$. The lemma thus holds.

Remark 2 Let T be a tree with $2n$ vertices. Note that $M(\overline{K_{1,2n-1}}) = M(\overline{P_2}) = 0$. Hence, by Lemma 2.2, $M(\bar{T}) = 0$ if and only if $T = K_{1,2n-1}$ or $T = P_2$.

The following corollary is immediate from Lemma 2.1 and Lemma 2.2.

Corollary 2.3 Suppose that T is a forest with n vertices, which is not a tree. Then $M(\bar{T}) > 0$.

Lemma 2.4 If T is a tree with n vertices and edge-independence number $\nu(T) = p$, then T has at most $n - p$ vertices of degree one. In particular, if T has exactly $n - p$ vertices of degree one, then every vertex of degree at least two in T is adjacent to at least one vertex of degree one.

Proof Suppose $\mathcal{M} = \{v_1v_2, v_3v_4, \dots, v_{2p-1}v_{2p}\}$ is a maximum matching of T. Obviously, both vertices v_{2i-1} and v_{2i} for $1 \le i \le p$ are not pendant vertices of T. Hence T has at most $p + (n - 2p) = n - p$ vertices of degree one. Hence the first statement holds. In particular, if T has exactly $n - p$ vertices of degree one then there exists exactly one vertex of degree one for every pair of vertices v_{2i-1} and v_{2i} for $1 \le i \le p$ and every vertex in $V(T)\backslash\{v_1, v_2, \dots, v_{2p-1}, v_{2p}\}$ has degree one. This implies the second statement in the lemma.

There is a well-known formula (see Lovász [24], exercise 5.18, p. 254) for the number of matchings with r edges of the complement of a graph G with n vertices:

Lemma 2.5 (Lovász [24]) Let G be a simple graph with n vertices and $\bar{G}$ the complement of G. The number of matchings with r edges of G, denoted by $m(G, r)$, satisfies

$$m(G, r) = \sum_{i=0}^{r} (-1)^i \binom{n-2i}{2r-2i} (2r - 2i - 1)!!\, m(\bar{G}, i),$$

where $s!! = s \times (s - 2)!!$, and $(-1)!! = 0!! = 1$.

The following corollary is immediate from Lemma 2.5.

Corollary 2.6 Suppose that T is a tree with n vertices. Then

$$M(\bar{T}) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \left(2\left\lfloor \frac{n+1}{2} \right\rfloor - 2i - 1\right)!!\, m(T, i),$$

where $s!! = s \times (s - 2)!!$, and $(-1)!! = 0!! = 1$.

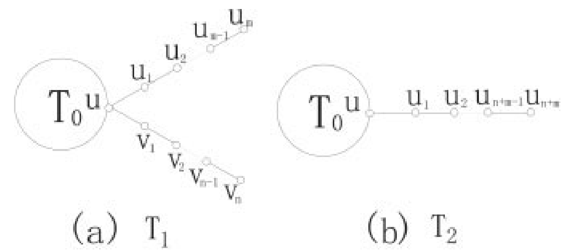

FIGURE 1. Two trees T_1 and T_2 in Definition 3.1.

Next there are five useful elementary results, often probably known for some time, but all stated and proved by Godsil [25].

Lemma 2.7 If $G_1, G_2, \dots, G_t$ are the components of a graph G, then the characteristic polynomial of G is given by

$$\phi(G) = \phi(G_1)\phi(G_2)\cdots\phi(G_t).$$

Lemma 2.8 Let T by a tree and $e = uv$ an edge of T. Then $\phi(T) = \phi(T - e) - \phi(T - u - v)$.

Lemma 2.9 Let T be a tree and u a vertex. Then

$$\phi(T) = x\phi(T - u) - \sum_{\substack{v \in V(T) \\ uv \in E(T)}} \phi(T - u - v),$$

where the sum ranges over all vertices v adjacent to u in T.

Lemma 2.10 Let T be a forest with n vertices. Then

$$\phi(T) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i m(T, i) x^{n-2i},$$

where $m(T, i)$ is the number of matchings with i edges of T for $0 \le i \le \lfloor n/2 \rfloor$, and $m(T, 0) = 1$.

Lemma 2.11 (Godsil [25], Exercise 3 in Chapter 1) Let m and n be two positive integers. Then

$$\phi(P_{m+n}) = \phi(P_m)\phi(P_n) - \phi(P_{m-1})\phi(P_{n-1}).$$

3. Six Types of Transformations of Trees

Definition 3.1 Let T_1 be a tree with $n + m + k$ vertices, shown in Figure 1(a), where T_0 is a tree with k vertices ($k \ge 2$) and u a vertex of T_0. Suppose T_2 is a tree with $n + m + k$ vertices obtained from T_0 by attaching a path P_{m+n+1} to u in T_0 [see Fig. 1(b)].

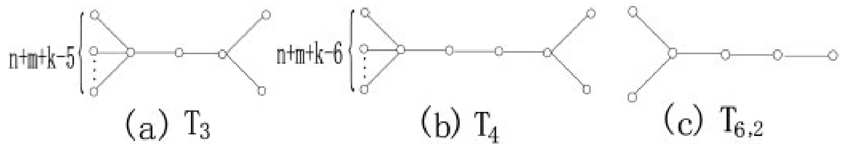

FIGURE 2. Three trees T_3, T_4, and $T_{6,2}$ in Remark 3.

We designate the transformation from T_1 to T_2 as of type I and denote it by $\mathcal{Y}_1 : T_1 \to T_2$ or $\mathcal{Y}_1(T_1) = T_2$.

Theorem 3.2 Let T_0 be a tree with k vertices ($k \geq 2$), and let T_1 and T_2 be the trees defined in Definition 3.1, where $n \geq 1$, $m \geq 1$. Then $M(\overline{T_1}) \leq M(\overline{T_2})$. In particular, if $m + n + k$ is odd then $M(\overline{T_1}) < M(\overline{T_2})$.

Proof By Lemmas 2.7–2.9, we have

$$\phi(T_1) = x\phi(T_0 - u)\phi(P_m)\phi(P_n) - \phi(T_0 - u)\phi(P_m)\phi(P_{n-1}) - \phi(T_0 - u)\phi(P_{m-1})\phi(P_n) - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \phi(T_0 - u - v)\phi(P_m)\phi(P_n),$$

$$\phi(T_2) = x\phi(T_0 - u)\phi(P_{m+n}) - \phi(T_0 - u)\phi(P_{m+n-1}) - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \phi(T_0 - u - v)\phi(P_{m+n}),$$

where the above sums range over all vertices of T_0 adjacent to u. Hence we have

$$\phi(T_1) - \phi(T_2) = x\phi(T_0 - u)[\phi(P_m)\phi(P_n) - \phi(P_{n+m})] - \phi(T_0 - u)[\phi(P_m)\phi(P_{n-1}) - \phi(P_{m+n-1}) + \phi(P_{m-1})\phi(P_n)] - [\phi(P_m)\phi(P_n) - \phi(P_{m+n})] \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \phi(T_0 - u - v).$$

Note that, by Lemma 2.11, $\phi(P_{m+n}) = \phi(P_m)\phi(P_n) - \phi(P_{m-1})\phi(P_{n-1})$. By a routine calculation, we have

$$\phi(T_1) - \phi(T_2) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \phi(T_0 - u - v)\phi(P_{m-1})\phi(P_{n-1}),$$

where the above sum ranges over all vertices of T_0 adjacent to u. Let T_v^* be the forest $(T_0 - u - v) \cup P_{m-1} \cup P_{n-1}$, which has $m + n + k - 4$ vertices, for an arbitrary vertex v adjacent to u in T_0. Hence, by Corollary 2.6 and Lemma 2.10, it is not difficult to see that if $m + n + k$ is even, we have

$$M(\overline{T_1}) - M(\overline{T_2}) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \sum_{i=0}^{m+n+k-4/2} (-1)^i (m + n + k - 4 - 2i - 1)!! m(T_v^*, i) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} M[\overline{T_v^*}].$$

Similarly, we can show that if $m + n + k$ is odd, we have

$$M(\overline{T_1}) - M(\overline{T_2}) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} M[\overline{T_v^*}].$$

It is obvious that $M(\overline{T_1}) \leq M(\overline{T_2})$. Moreover, by Lemma 2.1 and Corollary 2.3, if $m + n + k$ is odd, then $M(\overline{T_v^*}) \neq 0$ for an arbitrary vertex v adjacent to u in T_0, which implies that $M(\overline{T_1}) < M(\overline{T_2})$. The theorem thus holds.

Remark 3 For the trees T_1 and T_2 shown in Figure 1(a) and (b), $M(\overline{T_1}) = M(\overline{T_2})$ if and only if $m + n + k$ is even and T_1 must be one of the three trees T_3, T_4, and $T_{6,2}$ pictured in Figure 2(a)–(c).

Proof In the proof of Theorem 3.2, we have shown that

$$M(\overline{T_1}) - M(\overline{T_2}) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} M(\overline{T_v^*}).$$

Hence $M(\bar{T}_1) = M(\bar{T}_2)$ if and only if

$$\sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} M(\overline{T_v^*}) = \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} M[\overline{(T_0 - u - v) \cup P_{m-1} \cup P_{n-1}}] = 0.$$

Note that by Remark 2 and Corollary 2.3, $M[\overline{(T_0 - u - v) \cup P_{m-1} \cup P_{n-1}}] = 0$ if and only if $m + n + k$ is even, and there exists exactly one vertex v in T_0 adjacent to u, and one of the following two statements holds: (i) $m = n = 1$ and $T_0 - u - v$ is a star with $m + n + k - 4 = k - 2$ vertices and k is even; (ii) $T_0 = P_2$ and $P_{m-1} \cup P_{n-1} = P_2$.

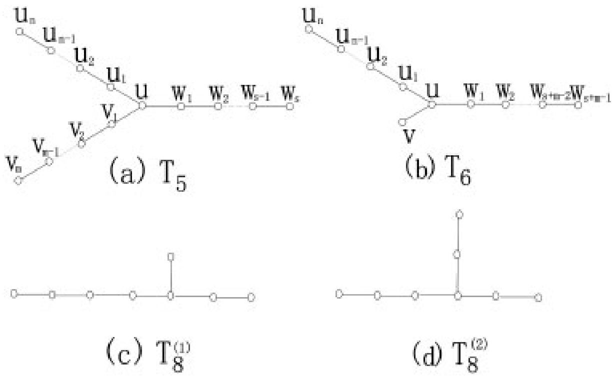

FIGURE 3. Four trees T_5, T_6, $T_8^{(1)}$, and $T_8^{(2)}$ in Theorem 3.3, where $s \geq m \geq 2, n \geq 1$.

Note that T_1 must be the tree T_3 or T_4 shown in Figure 2(a) and (b) if $m = n - 1$ and $T_0 - u - v$ is a star with $m + n + k - 4$ vertices, and otherwise T_1 must be the tree $T_{6,2}$ shown in Figure 2(c). Hence the statement in Remark 3 has been proved.

Theorem 3.3 Let T_5 and T_6 be two trees with $m + n + s + 1$ vertices, shown in Figure 3(a) and (b), respectively, where $s \geq m \geq 2$, $n \geq 1$. Then

$$M(\bar{T}_5) \leq M(\bar{T}_6)$$

with equality if and only if T_5 is one of the two trees $T_8^{(1)}$ and $T_8^{(2)}$ shown in Figure 3(c) and (d).

Proof By the same method as in the proof of Theorem 3.2, we can show that

$$\phi(T_5) - \phi(T_6) = -\phi(P_{m-2})\phi(P_{n-1})\phi(P_{s-2}).$$

Similarly to the proof of Theorem 3.2, we have

$$M(\overline{T_5}) - M(\overline{T_6}) = -M(\overline{P_{m-2} \cup P_{n-1} \cup P_{s-2}}).$$

Hence $M(\overline{T_5}) \leq M(\overline{T_6})$. In particular, by Remark 2 and Corollary 2.3,

$$M(\overline{P_{m-2} \cup P_{n-1} \cup P_{s-2}}) = 0$$

if and only if $P_{m-2} \cup P_{n-1} \cup P_{s-2} = P_2$. Note that $P_{m-2} \cup P_{n-1} \cup P_{s-2} = P_2$ if and only if $m = 2$, $n = 1$ and $s = 4$ or $m = 2$, $n = 3$ and $s = 2$. Hence it is not difficult to see that the equality in the theorem holds if and only if T_5 is the tree $T_8^{(1)}$ or $T_8^{(2)}$ illustrated in Figure 3(c) and (d). The theorem thus follows.

Definition 3.4 We designate the transformation from T_5 to T_6 in Figure 3(a) and (b) as of type II and denote it by $\mathcal{Y}_2 : T_5 \mapsto T_6$ or $\mathcal{Y}_2(T_5) = T_6$.

Theorem 3.5 Let T_7 and T_8 be the two trees with $m + n + 2$ vertices shown in Figure 4(a) and (b), respectively, where $m \geq n \geq 2$. Then

$$M(\overline{T_7}) \leq M(\overline{T_8})$$

with equality if and only if $m - n = 2$.

Proof Similar to the proof of Theorem 3.2, we have

$$\phi(T_7) - \phi(T_8) = -\phi(P_{m-n})$$

and

$$M(\overline{T_7}) - M(\overline{T_8}) = -M(\overline{P_{m-n}}).$$

Hence $M(\overline{T_7}) \leq M(\overline{T_8})$. In particular, by Corollary 2.3, $M(\overline{T_7}) = M(\overline{T_8})$ if and only if $P_{m-n} = P_2$, which implies $m - n = 2$. The theorem has thus been proved.

Definition 3.6 We designate the transformation from T_7 to T_8 in Figure 4(a) and (b) as of type III and denote it by $\mathcal{Y}_3 : T_7 \Rightarrow T_8$ or $\mathcal{Y}_3(T_7) = T_8$.

Definition 3.7 Suppose that T_1' and T_2' are two trees with m ($m > 1$) vertices and with n ($n > 1$) vertices, respectively. Take one vertex u of T_1' and one v of T_2'. Construct two trees T_9 and T_{10} with $m + n$ vertices as follows. The vertex set $V(T_9)$ of T_9 is $V(T_1') \cup V(T_2')$ and the edge set of T_9 is $E(T_1') \cup E(T_2') \cup \{uv\}$ [see Fig. 5(a)]. T_{10} is the tree obtained from T_1' and T_2' by identifying the vertex u of T_1' and the vertex v of T_2' and adding a pendant edge $uw = vw$ to this new vertex u $(=v)$ [see Fig. 5(b)]. We designate the transformation from T_9 to T_{10} as of type IV and denote it by $\mathcal{Y}_4 : T_9 \rightarrow T_{10}$ or $\mathcal{Y}_4(T_9) = T_{10}$.

Theorem 3.8 Suppose that T_9 and T_{10} are two trees with n vertices defined in Definition 3.7. Then

$$M(\overline{T_9}) \geq M(\overline{T_{10}}).$$

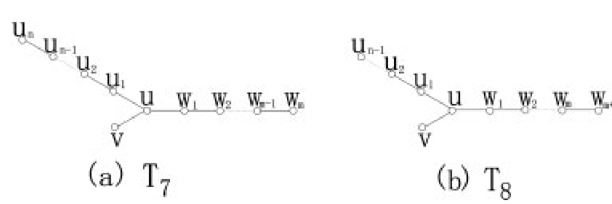

FIGURE 4. Two trees T_7 and T_8 and in Theorem 3.5, where $m \geq n \geq 2$.

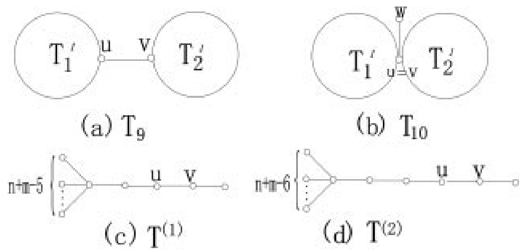

FIGURE 5. Two trees T_9 and T_{10} in Definition 3.7 and two trees $T^{(1)}$ and $T^{(2)}$ in Theorem 3.8, where $m = |V(T_1')| > 1$ and $n = |V(T_2')| > 1$.

with equality if and only if $m + n$ is even and T_9 is one of the two trees $T^{(1)}$ and $T^{(2)}$ shown in Figure 5(c) and (d), where $m > 1$ and $n > 1$.

Proof By Lemmas 2.7–2.9, we have

$$\phi(T_9) = \phi(T_1')\phi(T_2') - \phi(T_1' - u)\phi(T_2' - v)$$

and

$$\phi(T_{10}) = x\phi(T_{10} - w) - \phi(T_1' - u)\phi(T_2' - v).$$

Hence

$$\phi(T_9) - \phi(T_{10}) = \phi(T_1')\phi(T_2') - x\phi(T_{10} - w).$$

Note that, by Lemma 2.9, we have

$$\phi(T_1') = x\phi(T_1' - u) - \sum_{i=1}^{s} \phi(T_1' - u - u_i)$$

and

$$\phi(T_2') = x\phi(T_2' - v) - \sum_{i=1}^{t} \phi(T_2' - v - v_i),$$

where the first sum ranges over all vertices u_i ($1 \leq i \leq s$) of T_1' adjacent to u and the second sum ranges over all vertices v_j ($1 \leq j \leq t$) of T_2' adjacent to v. Hence we have

$$\phi(T_1')\phi(T_2') = x^2\phi(T_1' - u)\phi(T_2' - v) - x \sum_{i=1}^{t} \phi(T_1' - u)\phi(T_2' - v - v_i) - x \sum_{i=1}^{s} \phi(T_2' - v)\phi(T_1' - u - u_i) + \sum_{\substack{1\leq i\leq s\\ 1\leq j\leq t}} \phi(T_1' - u - u_i)\phi(T_2' - v - v_j).$$

Note that

$$x\phi(T_{10} - w) = x^2\phi(T_1' - u)\phi(T_2' - v) - x \sum_{i=1}^{t} \phi(T_1' - u)\phi(T_2' - v - v_i) - x \sum_{i=1}^{s} \phi(T_2' - v)\phi(T_1' - u - u_i).$$

Hence we have

$$\phi(T_9) - \phi(T_{10}) = \sum_{\substack{1\leq i\leq s\\ 1\leq j\leq t}} \phi(T_1' - u - u_i)\phi(T_2' - v - v_j).$$

As in the proof of Theorem 3.2, we can show that

$$M(\overline{T_9}) - M(\overline{T_{10}}) = \sum_{\substack{1\leq i\leq s\\ 1\leq j\leq t}} M[\overline{(T_1' - u - u_i) \cup (T_2' - v - v_i)}],$$

which implies that

$$M(\overline{T_9}) \geq M(\overline{T_{10}}).$$

Moreover, the equality holds if and only if $m + n$ is even and

$$M[\overline{(T_1' - u - u_i) \cup (T_2' - v - v_i)}] = 0$$

for arbitrary u_i and v_j, where $1 \leq i \leq s$, $1 \leq j \leq t$. By Remark 2 and Corollary 2.3, $M[\overline{(T_1' - u - u_i) \cup (T_2' - v - v_i)}] = 0$ if and only if $(T_1' - u - u_i) \cup (T_2' - v - v_i)$ is a star or P_2 for arbitrary u_i and v_j, which implies T_9 must be one of the two trees $T^{(1)}$ and $T^{(2)}$ shown in Figure 5(c) and (d). The theorem has been proved.

Remark 4 Note that neither tree $T^{(1)}$ in Figure 5(c) nor tree $T^{(2)}$ in Figure 5(d) can be transformed

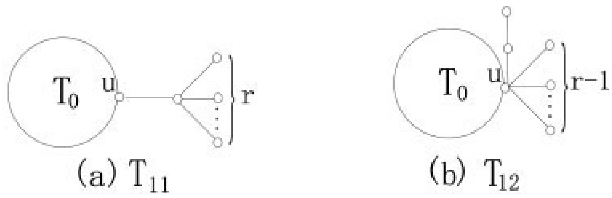

FIGURE 6. Two trees T_{11} and T_{12} in Definition 3.9, where $|V(T_0)| \geq 2$ and $r \geq 1$.

into T^p_{m+n} by a single transformation IV. Hence if T_{10} in Theorem 3.8 is T^p_{m+n} then $M(\overline{T_9}) > M(\overline{T_{10}}) = M(\overline{T^p_{m+n}})$. In particular, $M(\overline{T^p_n}) > M(\overline{T^{p-1}_n})$ for $n \geq 5$.

In fact, by the definition of the transformation IV, the first statement in Remark 4 holds. Note that

$$\phi(T^p_n) - \phi(T^{p-1}_n) = x^{n-2p}(x^2-1)^{p-3} \times [(n-p-1)x^2 - n + 2p - 1].$$

As in the proof of Theorem 3.8, it is not difficult to see that if $n \geq 5$ then $M(\overline{T^p_n}) > M(\overline{T^{p-1}_n})$.

Definition 3.9 Suppose that T_{11} is a tree with n vertices and with the edge-independence number p shown in Figure 6(a), which has exactly $n-p$ pendant vertices, where $|V(T_0)| \geq 2$ and $r \geq 2$. Let T_{12} be the tree with n vertices shown in Figure 6(b), which is obtained from T_{11}. We designate the transformation from T_{11} to T_{12} as of type V and denote it by $\mathcal{Y}_5 : T_{11} \rightsquigarrow T_{12}$ or $\mathcal{Y}_5(T_{11}) = T_{12}$.

Theorem 3.10 Suppose that T_{11} is a tree with n vertices and with the edge-independence number p shown in Figure 6(a), which has exactly $n-p$ pendant vertices, where $|V(T_0)| \geq 2$ and $r \geq 2$. T_{12} is a tree with n vertices obtained from T_{11} by the transformation (V) in Definition 3.9. Then

$$M(\overline{T_{11}}) > M(\overline{T_{12}}).$$

Proof By Lemmas 2.7–2.9, we can show that

$$\phi(T_{11}) - \phi(T_{12}) = -(r-1)x^{r-2}\phi(T_0 - u) + (r-1)x^{r-1}\sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}} \phi(T_0 - u - v),$$

where the sum ranges over all vertices of T_0 incident with u. By Lemma 2.4, every vertex of degree at least two in T_{11} is adjacent to at least one pendant vertex. Hence vertex u of T_0 is adjacent to at least one pendant vertex v' in T_0. Hence $\phi(T_0 - u) = x\phi(T_0 - u - v')$, which implies that

$$\phi(T_{11}) - \phi(T_{12}) = (r-1)x^{r-1}\sum_{\substack{v\in V(T_0), v\neq v'\\ uv\in E(T_0)}} \phi(T_0 - u - v).$$

As in the proof of Theorem 3.2, we can show that

$$M(\overline{T_{11}}) - M(\overline{T_{12}}) = (r-1)\sum_{\substack{v\in V(T_0), v\neq v'\\ uv\in E(T_0)}} M[\overline{(r-1)P_1 \cup (T_0 - u - v)}],$$

which shows that $M(\overline{T_{11}}) \geq M(\overline{T_{12}})$. Moreover,

$M(\overline{T_{11}}) = M(\overline{T_{12}})$ if and only if

$$M[\overline{(r-1)P_1 \cup (T_0 - u - v)}] = 0$$

for every vertex v ($\neq v'$) of T_0 incident with u. Note that $|V(T_0)| \geq 2$ and $r \geq 2$. Since $r \geq 2$, $(r-1)P_1 \cup (T_0 - u - v)$ is an isolated vertex or a forest that is not a tree. It is clear that by Corollary 2.3 we have $M[\overline{(r-1)P_1 \cup (T_0 - u - v)}] \neq 0$. Hence $M(\overline{T_{11}}) > M(\overline{T_{12}})$. The theorem thus follows.

Definition 3.11 Suppose that T_{13} is a tree with n vertices and with the edge-independence number p shown in Figure 7(a), which has exactly $n-p$ pendant vertices, where $|V(T_0)| \geq 2$, $s \geq 1$, and $t \geq 1$. Let T_{14} be the tree with n vertices shown in Figure 7(b), which is obtained from T_{13}. We designate the transformation from T_{13} to T_{14} as of type VI and denote it by $T_{13} \rightarrow T_{14}$ or $\mathcal{Y}_6(T_{13}) = T_{14}$.

Theorem 3.12 Suppose that T_{13} is a tree with n vertices and with the edge-independence number p shown in Figure 7(a), which has exactly $n-p$ pendant vertices, where $|V(T_0)| \geq 2$, $s \geq 1$, and $t \geq 1$. Let T_{14} be the tree with n vertices, shown in Figure 7(b), which is obtained from T_{13}. Then

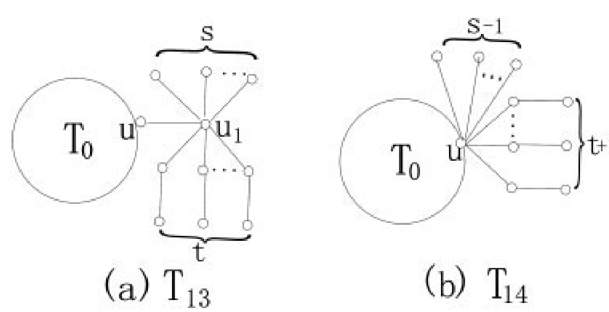

FIGURE 7. Two trees T_{13} and T_{14} in Definition 3.11, where $|V(T_0)| \geq 2$ and $s \geq 1$, $t \geq 1$.

$$M(\overline{T_{13}}) \geq M(\overline{T_{14}})$$

with equality if and only if $T_{13} = T_{14} = T_6^3$.

Proof By Lemmas 2.7–2.9, we can show that if $s > 2$, then

$$\phi(T_{13}) - \phi(T_{14}) = -x^{s-2}(x^2-1)^{t-1}[(s+t-1)x^2 - (s-1)]\phi(T_0-u) + x^{s-1}(x^2-1)^{t-1}[(s+t-1)x^2 - (s-1)] \sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}} \phi(T_0-u-v),$$

where the sum ranges over every vertex v of T_0 adjacent to u. By the same reason as in the proof of Theorem 3.10, there exists at least one pendant vertex v' of T_0 adjacent to u. Hence $\phi(T_0 - u) = x\phi(T_0 - u - v')$. So we have

$$\phi(T_{13}) - \phi(T_{14}) = x^{s-1}(x^2-1)^{t-1}[(s+t-1)x^2 - (s-1)] \sum_{\substack{v\in V(T_0),v\neq v'\\ uv\in E(T_0)}} \phi(T_0-u-v) = (s+t-1) \times [x^{s-1}(x^2-1)^t] \sum_{\substack{v\in V(T_0),v\neq v'\\ uv\in E(T_0)}} \phi(T_0-u-v) + tx^{s-1}(x^2-1)^{t-1} \sum_{\substack{v\in V(T_0),v\neq v'\\ uv\in E(T_0)}} \phi(T_0-u-v).$$

Similarly, we can show that if $s = 1$, the above equalities hold. Hence, by the same method as that in Theorem 3.2, we can show that if $s \geq 1$, then

$$M(\overline{T_{13}}) - M(\overline{T_{14}}) = (s+t-1) \sum_{\substack{v\in V(T_0),v\neq v'\\ uv\in E(T_0)}} \times M[\overline{(s-1)P_1 \cup tP_2 \cup (T_0-u-v)}] + t \sum_{\substack{v\in V(T_0),v\neq v'\\ uv\in E(T_0)}} \times M[\overline{(s-1)P_1 \cup (t-1)P_2 \cup (T_0-u-v)}] \geq 0,$$

which implies that $M(\overline{T_{13}}) \geq M(\overline{T_{14}})$. If $s \geq 2$ then by Corollary 2.3, we have

$$M[\overline{(sP_1) \cup (tP_2) \cup (T_0-u-v)}] > 0.$$

Hence $M(\overline{T_{13}}) = M(\overline{T_{14}})$ if and only if $s = 1$ and

$$M[\overline{tP_2 \cup (T_0-u-v)}] = 0, \quad M[\overline{(t-1)P_2 \cup (T_0-u-4)}] = 0 \quad (1)$$

for arbitrary vertex v ($\neq v'$) of T_0 adjacent to u. Clearly, (1) holds if and only if $t = 1$ and $V(T_0) = \{u, v\}$; that is, $T_{13} = T_{14} = T_6^3$. The theorem thus holds.

4. Main Results

We will first consider the first two largest values of $M(\bar{T})$ for all trees T with N vertices.

Theorem 4.1 Suppose that T is a tree with N vertices and N is odd. If $T \neq P_N$ and $T \neq T_{N,2}$, then

$$M(\bar{T}) < M(\overline{T_{N,2}}) < M(\overline{P_N}),$$

where $M(\bar{T})$ denotes the number of matchings with $(N-1)/2$ edges in the complement of tree T.

Proof We first prove that if $T \neq P_N$ then $M(\bar{T}) < M(\overline{P_N})$. For an arbitrary tree T with N vertices, by repeated applications of the transformation I in Definition 3.1 we can transform T into P_N; that is, there exist trees $T^{(i)}$ for $0 \leq i \leq \ell$ such that

$$T = T^{(0)} \to T^{(1)} \to T^{(2)} \to \ldots \to T^{(\ell-1)} \to T^{(\ell)} = P_N, \quad (2)$$

where $T^{(\ell-1)} \neq P_N$. Since N is odd, by Theorem 3.2 we have

$$M(\bar{T}) < M(\overline{T^{(1)}}) < \ldots < M(\overline{T^{(\ell)}}) = M(\overline{P_N}).$$

Now we need to prove that if $T \neq P_N$ and $T \neq T_{N,2}$ then $M(\bar{T}) < M(\overline{T_{N,2}})$. In (2), if $T^{(\ell-1)} = T_{N,2}$ then $M(\bar{T}) < M(\overline{T^{(\ell-1)}}) = M(\overline{T_{N,2}})$. If $T^{(\ell-1)} \neq T_{N,2}$ then $T^{(\ell-1)}$ has the form of T_5 in Figure 3(a) or Theorem 3.3. Note that T_5 can be transformed into $T_{N,2}$ by repeated applications of the transformations II and III in Definitions 3.4 and 3.6. Moreover, by Theorems 3.3 and 3.5, $M(\bar{T}) < M(\bar{T}_5) < M(\overline{T_{N,2}})$. The theorem thus follows.

Theorem 4.2 Suppose T is a tree with N vertices ($N \geq 8$) and N is even. Then

$$M(\bar{T}) \leq M(\overline{P_N})$$

with equality if and only if $T = P_N$, where $M(\bar{T})$ denotes the number of perfect matchings of the complement $\bar{T}$ of T.

Proof It is clear that $T_{n,2}$ can be transformed into P_N by the transformation I in Definition 3.1. Note that $N \geq 8$, by Remark 3, $M(\overline{T_{N,2}}) < M(\overline{P_N})$. Suppose $T \neq P_N$. It suffices to show that $M(\bar{T}) <$

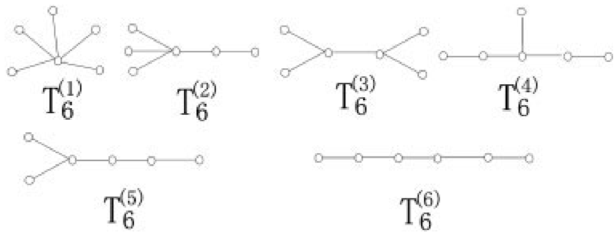

FIGURE 8. Six trees $T_6^{(i)}$ with six vertices for $i = 1, 2, 3, 4, 5, 6$.

$M(\overline{P_N})$. As in the proof of Theorem 4.1, there exist trees $T^{(i)}$ for $0 \le i \le \ell$ such that

$$T = T^{(0)} \rightarrow T^{(1)} \rightarrow T^{(2)} \rightarrow \ldots \rightarrow T^{(\ell-1)} \rightarrow T^{(\ell)} = P_N,$$

where $T^{(\ell-1)} \ne P_N$. By Theorem 3.2, we have

$$M(\bar{T}) \le M(\overline{T^{(1)}}) \le \ldots \le M(\overline{T^{(\ell-1)}}) \le M(\overline{T^{(\ell)}}) = M(\overline{P_N}).$$

If $T^{(\ell-1)} = T_{N,2}$, then $M(\bar{T}) \le M(\overline{T^{(\ell-1)}}) = M(\overline{T_{N,2}}) < M(\overline{P_N})$. If $T^{(\ell-1)} \ne T_{N,2}$ then $T^{(\ell-1)}$ has the form of T_5 in Figure 3(a) or Theorem 3.3, where $N = m + n + s + 1$, $s \ge m \ge 2$, $n \ge 1$. Note that T_5 can be transformed into $T_{N,2}$ by repeated applications of the transformations II and III in Definitions 3.4 and 3.6. Moreover, by Theorems 3.3 and 3.5, $M(\bar{T}) \le M(\overline{T^{(\ell-1)}}) = M(\bar{T}_5) \le M(\overline{T_{N,2}})$. Hence we have proved that $M(\bar{T}) \le M(\overline{T_{N,2}}) < M(\overline{P_N})$. The theorem thus follows.

Remark 4 There are exactly six trees with six vertices, denoted by $T_6^{(i)}$ for $i = 1, 2, 3, 4, 5, 6$ (see Fig. 8). In terms of Corollary 2.6, by a routine calculation, we have $M(\overline{T_6^{(1)}}) = 0$, $M(\overline{T_6^{(2)}}) = 3$, $M(\overline{T_6^{(3)}}) = M(\overline{T_6^{(4)}}) = 4$, $M(\overline{T_6^{(5)}}) = M(\overline{T_6^{(6)}}) = 5$. Hence $M(\overline{T_6^{(1)}}) < M(\overline{T_6^{(2)}}) < M(\overline{T_6^{(3)}}) = M(\overline{T_6^{(4)}}) < M(\overline{T_6^{(5)}}) = M(\overline{T_6^{(6)}}) = 5$.

Theorem 4.3 Suppose T is a tree with N vertices, and N is an even integer greater than 8. If $T \ne P_N$, then

$$M(\bar{T}) \le M(\overline{T_{N,2}})$$

with equality if and only if $T = T_{N,2}$, where $M(\bar{T})$ is the number of perfect matchings of the complement of tree T.

Proof It suffices to prove that if $T \ne P_N$ and $T \ne T_{N,2}$ then $M(\bar{T}) < M(\overline{T_{N,2}})$. In the proof of Theorem 4.2, we have shown that there exist trees $T^{(i)}$ for $0 \le i \le \ell$ such that

$$T = T^{(0)} \rightarrow T^{(1)} \rightarrow T^{(2)} \rightarrow \ldots \rightarrow T^{(\ell-1)} \rightarrow T^{(\ell)} = P_N,$$

where $T^{(\ell-1)} \ne P_N$, $M(\overline{T^{(i-1)}}) \le M(\overline{T^{(i)}})$ for $1 \le i \le l$. Obviously, $T^{(\ell-1)} = T_{N,2}$ or $T^{(\ell-1)}$ has the form of T_5 in Figure 3(a) or Theorem 3.3, where $N = m + n + s + 1$, $s \ge m \ge 2$, $n \ge 1$. We now divide the proof into the following cases:

Case 1:

$$T^{(\ell-1)} = T_{N,2}.$$

By the definition of the transformation I, if $T^{(\ell-1)} = T_{N,2}$, then $T^{(\ell-2)}$ must have the form of T_{15} or T_{16} shown in Figure 9(a) and (b), where $N = m_1 + n_1 + 3 = r_1 + s_1 + t_1 + 4$. Note that N is an even integer greater than 8 (hence $N \ge 10$). By Theorem 3.2 and Remark 3, it is not difficult to see that $M(\overline{T_{15}}) < M(\overline{T_{N,2}})$ and $M(\overline{T_{16}}) < M(\overline{T_{N,2}})$. Hence $M(\bar{T}) \le M(\overline{T^{(\ell-2)}}) = M(\overline{T_{15}}) < M(\overline{T_{N,2}})$ or $M(\bar{T}) \le M(\overline{T^{(\ell-2)}}) = M(\overline{T_{16}}) < M(\overline{T_{N,2}})$. Hence $M(\bar{T}) < M(\overline{T_{N,2}})$.

Case 2: $T^{(\ell-1)}$ has the form of T_5 in Figure 3(a) or Theorem 3.3, where $N = m + n + s + 1$, $s \ge m \ge 2$, $n \ge 1$.

If $n = 1$, then $T^{(\ell-1)}$ has the form of T_6 in Figure 3(b), where $N = m + n + s + 1$, $s \ge m \ge 2$. By repeated applications of transformation III, T_6 can be transformed into $T_{N,2}$. Note that $N \ge 10$. Hence, by Theorem 3.5, we have $M(\overline{T_6}) < M(\overline{T_{N,2}})$. So we have $M(\bar{T}) \le M(\overline{T^{(\ell-1)}}) = M(\overline{T_6}) < M(\overline{T_{N,2}})$.

If $n \ge 2$, then $T^{(\ell-1)}$ can be transformed into the form of T_6 in Figure 3(b) by the transformation II in Definition 3.4. Moreover, since $N \ge 10$, by Theorem 3.3, we have $M(\overline{M^{(\ell-1)}}) = M(\overline{T_5}) < M(\overline{T_6})$. But we

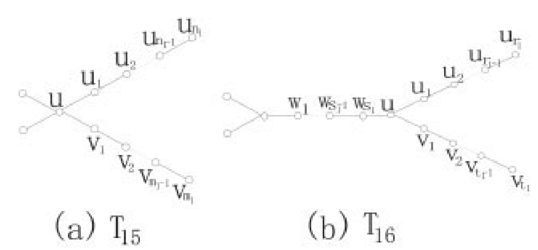

FIGURE 9. Two trees T_{15} and T_{16}, where $N = m_1 + n_1 + 3 = r_1 + s_1 + t_1 + 4$.

have shown that if $N \geq 10$, we have $M(\overline{T_6}) < M(\overline{T_{N,2}})$. Also, $M(\bar{T}) \leq M(\overline{T^{(\ell-1)}}) = M(\overline{T_5}) < M(\overline{T_6}) < M(\overline{T_{N,2}})$.

Hence we have proved the theorem.

Remark 5 Cvetković et al. [26] (Table 2, page 276) show that there exist exactly 23 trees with eight vertices. In terms of Corollary 2.6, by a routine calculation, we can show that if T is a tree with eight vertices and $T \neq P_8$, then

$$M(\bar{T}) \leq M(\overline{T_{8,2}})$$

with equality if and only if $T = T_{8,2}$, $T_8^{(1)}$, or $T_8^{(3)}$. In this case, $T_8^{(1)}$ is the tree with eight vertices shown in Figure 3(c), which is a special case of T_5 in Figure 3(a) for $m = 1$, $n = 2$, and $s = 4$, and $T_8^{(3)}$ is the tree with eight vertices obtained from the path P_4 by attaching two pendant edges to each of two pendant vertices of P_4.

Now, we consider the lower bound of $M(\bar{T})$ for all trees $T \in \mathcal{T}_{N,p}$ with N vertices ($\mathcal{T}_{N,p}$ is the set of trees with N vertices and with the edge-independence number at least p). It is not difficult to see the following lemma holds.

Lemma 4.4 Let T_9 and T_{10} be the two trees defined in Figure 5(a) and (b). Suppose $\mathcal{M}$ is a maximum matching of T_9 in Figure 5(a). Then $\nu(T_9) = \nu(T_{10})$ if $uv \in \mathcal{M}$ or both u and v are not saturated by $\mathcal{M}$ and $\nu(T_9) - \nu(T_{10}) \leq 1$ otherwise.

Hence, by Lemmas 2.4 and 4.4 and with repeated applications of the transformation (IV), the following two lemmas hold.

Lemma 4.5 For an arbitrary tree T with n vertices and with edge-independence number $\nu(T) = p$, if the number of pendant vertices of T is less than $n - p$, then, by repeated applications of the transformation IV in Definition 3.7, T can be transformed into a tree T' with n vertices and with $\nu(T') = p$, the number of pendant vertices of which is exactly $n - p$.

Lemma 4.6 For an arbitrary tree T with n vertices and with $\nu(T) > p$, repeated applications of the transformation IV in Definition 3.7 transform T into a tree T'' with n vertices with $\nu(T') = p$, the number of pendant vertices of which is exactly $n - p$.

Theorem 4.7 Let N and p be two positive integers and $1 \leq p \leq (N/2)$. For an arbitrary tree $T \in \mathcal{T}_{N,p}$,

$$M(\bar{T}) \geq M(\overline{T_N^p})$$

with equality if and only if $T = T_N^p$, where $M(\bar{T})$ is the number of matchings with $\lfloor N/2 \rfloor$ edges in $\bar{T}$ and T_N^p denotes the trees with N vertices obtained from the star $K_{1,N-p}$ by attaching one pendant edge to each of $p - 1$ pendant vertices in $K_{1,N-p}$.

Proof Let $T \in \mathcal{T}_{N,p}$ and $T \neq P_N^p$. It suffices to show that $M(\bar{T}) > M(\overline{T_N^p})$. If $N \leq 6$, it is not difficult to show the statement in Theorem 4.7 holds. So we assume $N \geq 7$ and distinguish the following cases:

Case 1: $\nu(T) = p$ (i.e., the edge-independence number of T is p) and T has exactly $N - p$ pendant vertices. It is not difficult to see that, with repeated applications of transformations V and VI in Definitions 3.9 and 3.11, T can be transformed into T_N^p. By Theorems 3.10 and 3.12, we have $M(\bar{T}) > M(\overline{T_N^p})$.

Case 2: $\nu(T) = p$ and the number of pendant vertices of T is less than $N - p$. Then, by Lemma 4.5, with repeated applications of the transformation IV in Definition 3.7, T can be transformed into one tree T' with N vertices and with $\nu(T') = p$, the number of pendant vertices of which is exactly $N - p$. If $T' \neq T_N^p$, then by Theorem 3.8 we have $M(\bar{T}) \geq M(\overline{T'})$. Note that by Case 1, $M(\overline{T'}) > M(\overline{T_N^p})$. Then $M(\bar{T}) > M(\overline{T_N^p})$. If $T' = T_N^p$, then by Remark 4 we have $M(\bar{T}) > M(\overline{T'})$. Also, by Case 1, we have $M(\bar{T}) > M(\overline{T'}) > M(\overline{T_N^p})$. Hence $M(\bar{T}) > M(\overline{T_N^p})$.

Case 3: $\nu(T) > p$. By Lemma 4.6, with repeated applications of the transformation IV in Definition 3.7, T can be transformed into a tree T'' with N vertices with $\nu(T'') = p$, the number of pendant vertices of which is exactly $N - p$. Moreover, as in Case 2, we can show that $M(\bar{T}) \geq M(\overline{T''}) > M(\overline{T_N^p})$.

The theorem has thus been proved.

If we consider two special cases of Theorem 4.7: $p = 1$ or $p = 2$, then the following corollary is immediate.

Corollary 4.8 Let T be an arbitrary tree with N vertices. If $T \neq K_{1,N-1}$ ($=T_N^1$) and $T \neq T_N^2$, then

$$M(\bar{T}) > M(\overline{T_N^2}) > M(\overline{K_{1,N}}).$$

5. Conclusion

We have considered the combinatorial optimization problem introduced in Section 1 in the case where $f(G)$ denotes the number of matchings with $[n/2]$ edges in G and $\mathcal{E} = \{E(T) | T \in \mathcal{T}_n\}$ or $\mathcal{E} = \{E(T) | T \in \mathcal{T}_{n,p}\}$. We have obtained the first two maximal values in $\{f[K_n - E(T)] | T \in \mathcal{T}_n\}$ and the first minimal value in $\{f[K_n - E(T)] | T \in \mathcal{T}_{n,p}\}$. The key point of our results is that the matchings poly-

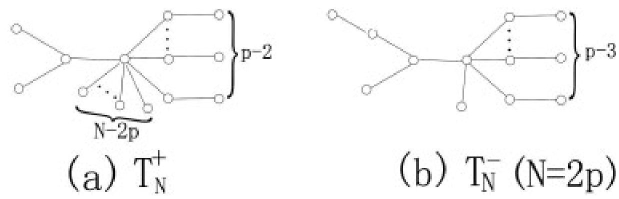

FIGURE 10. Two trees T_N^+ and T_N^-.

nomial of a graph G is determined completely by the matchings polynomial of the complement of G (see Lemma 2.5). Note that the matchings polynomial of a tree T equals the characteristic polynomial of T, which has been extensively considered. Hence, it is hopeful that there will be further results for the complements of trees. In contrast, if we change both f and $\mathcal{E}$, many combinatorial optimization problem can be considered. Finally, we pose the following conjecture.

Conjecture Let N and p be two positive integers and $1 \leq p \leq (N/2)$. For an arbitrary tree $T \in \mathcal{T}_{N,p}$, if $T \neq T_N^p$, then

$$M(\bar{T}) \geq M(\overline{T_N^*})$$

with equality if and only if $T = T_N^* = T_N^+$ if $p \neq (N/2)$ and $T = T_N^* = T_N^-$ otherwise, where T_N^+ and T_N^- are the two trees shown in Figure 10(a) and (b), and $M(\bar{T})$ is the number of matchings with $\lfloor N/2 \rfloor$ edges in $\bar{T}$.

Remark 6 Cvetković et al. [26] (Table 2, page 276) show that the above conjecture holds for $N \leq 10$.

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