# On the Monomer–Dimer Problem of Some Graphs

*By Weigen Yan and Yeong-Nan Yeh*

The pure-dimer problem was solved in exact closed form for many lattice graphs. Although some numerical solutions of the monomer–dimer problem were obtained, no exact solutions of the monomer–dimer problem were available (except in one dimension). Let $G$ be an arbitrary graph with $N$ vertices. Construct a new graph $R(G)$ from $G$ by adding a new verex $e^*$ corresponding to each edge $e = (a, b)$ of $G$ and by joining each new vertex $e^*$ to the vertices $a$ and $b$. If the suitable activities of vertices and edges in $R(G)$ are selected, then the monomer–dimer problem can be solved exactly for the graph $R(G)$, which generalizes the result obtained by Yan and Yeh. As applications, if we select suitable activities for the vertices and edges of $R(\tilde{L}(n, d))$, we obtain the exact formulae for the MD partition function, MD free energy, and MD entropy of $R(\tilde{L}(n, d))$ for the $d$-dimensional lattice $\tilde{L}(n, d)$ with periodic boundaries.

## 1. Introduction

Consider a graph $G = (V(G), E(G))$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_N\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_\epsilon\}$. Suppose that $f : V(G) \to \mathcal{C}$ and $g : E(G) \to \mathcal{C}$ are two activity functions defined on $V(G)$ and $E(G)$, where $\mathcal{C}$ is the complex number field. Monomers can be placed on vertices of $G$, and dimers can be placed on $G$ so as to occupy two vertices connected by an edge. A monomer–dimer arrangement on $G$, denoted by *MD*, is an arrangement of monomers and dimers such that each vertex in $G$ may either be vacant

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(i.e., occupied by a monomer) or may be singly occupied by a dimer. A pure-dimer arrangement on $G$, denote by $PD$, is an arrangement of dimers such that all vertices are singly occupied. Obviously, no such arrangement is possible if the number of vertices is odd. Let $\mathcal{MD}(G)$ and $\mathcal{PD}(G)$ denote the sets of monomer–dimer arrangements and pure-dimer arrangements of $G$, respectively. Given a monomer–dimer arrangement $MD$ of $G$, let $M$ and $D$ denoted the sets of monomers and dimers in $MD$, respectively. Define the weight of an MD of $G$, denoted by $W(MD)$, as follows:

$$W(MD) = \left(\prod_{v\in M} f(v)\right)\left(\prod_{e\in D} g(e)\right).$$

We want to derive close expressions for the MD partition function, denoted by $Z(G; f, g)$, and PD partition function of $G$, denoted by $Z^*(G; g)$, defined as follows:

$$Z(G; f, g) = \sum_{MD\in\mathcal{MD}(G)} W(MD),\ Z^*(G; g) = \sum_{PD\in\mathcal{PD}(G)} W(PD).$$

Obviously, $Z(G; 1, 1)$ and $Z^*(G; 1)$ are exactly the numbers of monomer–dimer arrangements and pure-dimer arrangements of $G$, respectively. $Z(G;\ 1,\ 1)$ (resp. $Z^*(G;\ 1)$) is also equivalent to the numbers of matchings (resp. perfect matchings) of $G$ which was used by mathematicians and mathematical chemists. For a large graph $G$, $Z(G; f, g)$ and $Z^*(G; g)$ are expected to grow exponentially in $N$. Our goal is to compute the MD free energy $\phi(G;\ f,\ g)$ and PD free energy $\phi^*(G;\ g)$:

$$\phi(G; f, g) = \lim_{N\to\infty} \frac{1}{N} \log Z(G; f, g), \quad \phi^*(G; g) = \lim_{N\to\infty} \frac{1}{N} \log Z^*(G; g).$$

Setting $f = g = 1$, the numerical values

$$\chi^*(G) = \phi^*(G; 1, 1) \quad \text{and} \quad \chi^*(G) = \phi^*(G; 1, 1)$$

are the MD entropy and PD entropy of $G$. The MD partition function is called the matching polynomial in combinatorial theory and mathematical chemistry.

In 1961, Kasteleyn [12] found exact formulae of the PD partition function and PD entropy of an $m \times n$ ($mn$ is even) quadratic lattice $G_{mn}$ when the activities of horizontal and vertical edges are $z_1$ and $z_2$, respectively. Temperley and Fisher [18] used a different method and arrived at the same results at almost exactly the same time. They proved that

$$Z^*(G_{mn}; z_1, z_2) = 2^{mn/2} \prod_{k=1}^{m}\prod_{l=1}^{n} \left(z_1^2 \cos^2 \frac{k\pi}{n+1} + z_2^2 \cos^2 \frac{l\pi}{m+1}\right)^{\frac{1}{4}},$$

$$\chi^*(G_{mn}) = \frac{1}{8\pi^2} \int_0^{2\pi}\int_0^{2\pi} \log[4 - 2\cos x - 2\cos y]\, dx\, dy = \frac{c}{\pi} \approx 0.2916,$$

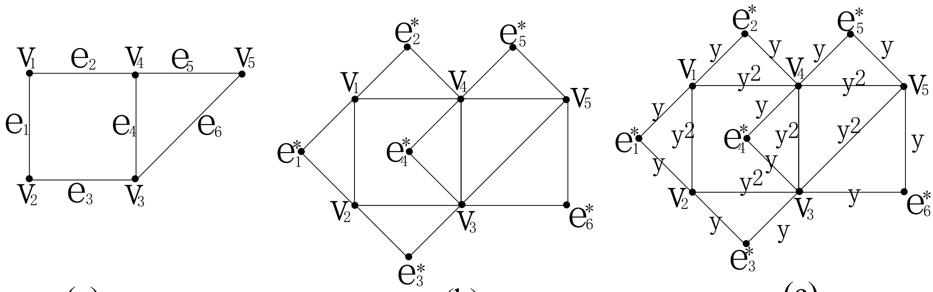


Figure 1. (a) A graph $G$. (b) The corresponding graph $R(G)$. (c) The graph $R(G)$ and its activities of edges.

where $c = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)^2} \approx 0.9160$ is Catalan's constant. Their key idea is to express the PD partition function as a Pfaffian, which in turn can be evaluated as the square root of an associated determinant. In fact, this technique is more general and allows the PD partition function of any planar graph (or indeed, of any family of graphs with fixed genus) to be computed efficiently [13]. Some related work see, for example, [6, 14, 19, 22, 23].

Unfortunately, the MD partition function and the MD entropy are difficult to compute. Jerrum [11] proved that the computation for the MD partition function is *NP*-complete. Some properties of the MD partition function were considered in Heilmann and Lieb [9, 10]. Yan et al. [21] obtained a method to compute the MD partition function, but it is not a polynomial algorithm. Although some numerical solutions of the monomer–dimer problem were obtained (see for example, [1–5, 8, 15, 17), no exact solutions of the monomer–dimer problem were available (except in one dimension) so far.

Let $G$ be an arbitrary graph with $N$ vertices. Construct a new graph $R(G)$ from $G$ by adding a new vertex $e^*$ corresponding to each edge $e = (a, b)$ of $G$ and by joining each new vertex $e^*$ to the vertices $a$ and $b$ (see Figures 1(a) and (b)). If the suitable activities of vertices and edges in $R(G)$ are selected, then the monomer–dimer problem can be solved exactly for the graph $R(G)$. As applications, for the $d$-dimensional lattice $\tilde{L}(n, d)$ of size $n_1 \times n_2 \times \cdots \times n_d$ with periodic boundaries (where $n_1 = n_2 = \cdots = n_d = n$), if suitable activities of vertices and edges in $R(\tilde{L}(n, d))$ are selected, then the exact formulae for the MD partition function, MD free energy, and MD entropy of $R(\tilde{L}(n, d))$ can be obtained.

## 2. A bijection

Consider a graph $G = (V(G), E(G))$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_N\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_\epsilon\}$. By the definition of $R(G)$, its vertex set

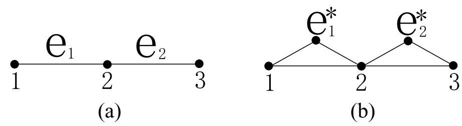


Figure 2. (a) A graph $G$. (b) The corresponding graph $R(G)$.

$V(R(G)) = V(G) \cup \{e_1^*, e_2^*, \ldots, e_\epsilon^*\}$ and its edge set $E(R(G)) = E(G) \cup \{(a_i, e_i^*), (b_i, e_i^*) \mid e_i = (a_i, b_i) \in E(G), 1 \le i \le \epsilon\}$, where $\cup_{i=1}^{\epsilon}\{a_i, b_i\} = V(G)$. In other words, $R(G)$ has two types of vertices: $v$-type vertices, i.e., $v_1, v_2, \ldots, v_N$, and $e$-type vertices, i.e., $e_1^*, e_2^*, \ldots, e_\epsilon^*$, and two types of edges: $(v, v)$-type edges, i.e., $e_1, e_2, \ldots, e_\epsilon$, and $(v, e)$-type edges, i.e., $(a_1, e_1^*)$, $(b_1, e_1^*), \ldots, (a_\epsilon, e_\epsilon^*)$, $(b_\epsilon, e_\epsilon^*)$. Let $v_i$ be an arbitrary $v$-type vertex of $R(G)$. Define $E_i$ to be the set of the $(v, e)$-type edges of $R(G)$ incident with vertex $v_i$. Set $X_i = \{v_i\} \cup E_i$ and $X(G) = X_1 \times X_2 \times \cdots \times X_N$. For the graphs $G$ and $R(G)$ shown Figure 2, $E_1 = \{(1, e_1^*)\}$, $E_2 = \{(2, e_1^*), (2, e_2^*)\}$, $E_3 = \{(3, e_2^*)\}$, $X_1 = \{1, (1, e_1^*)\}$, $X_2 = \{2, (2, e_1^*), (2, e_2^*)\}$, $X_3 = \{3, (3, e_2^*)\}$.

THEOREM 1 [20]. *Suppose $G = (V(G), E(G))$ is a graph with $N$ vertices and $R(G)$ is the graph defined above. Let $\mathcal{MD}(R(G))$ be the set of monomer–dimer arrangements of $R(G)$ and $X(G)$ be the set defined above. Then there exists a bijection between $\mathcal{MD}(R(G))$ and $X(G)$.*

To make the paper self-contained, we need to restate the bijection in Theorem 1 as follows. We only need to construct a bijection $\mu : \mathcal{MD}(R(G)) \longrightarrow X(G)$. Given a monomer–dimer arrangement $MD$ of $R(G)$, we need to define $\mu(MD) = (x_1, x_2, \ldots, x_N) \in X(G)$. Denote the sets of monomers and dimers in $MD$ by $M$ and $D$, respectively. We construct $\mu(MD) = (x_1, x_2, \ldots, x_N)$ from $MD$ by the following procedures:

(i) If $v_i \in M$, then let $x_i = v_i$.
(ii) If $e = (v_j, v_k) \in D$, then let $x_j = (v_j, e^*)$ and $x_k = (v_k, e^*)$.
(iii) If $(v_s, e^*) \in D$, then let $x_s = (v_s, e^*)$.

It is not difficult to see $\mu(MD) \in X(G)$. On the other hand, we can construct an $MD \in \mathcal{MD}(R(G))$ such that $\mu(MD) = Y = (x_1, x_2, \ldots, x_N)$ for any $Y = (x_1, x_2, \ldots, x_N) \in X(G)$. This implies that $\mu$ is bijective. The following example illustrates this bijection.

EXAMPLE. Let $G$ and $R(G)$ the graphs shown in Figures 2(a) and (b). By the definition of $\mathcal{MD}(R(G))$, $\mathcal{MD}(R(G))$ denotes the set of monomer–dimer arrangements of $R(G)$. Obviously, $\mathcal{MD}(R(G)) = \{M_i \mid 0 \le i \le 11\}$, where

$M_0 = \{1, 2, 3, e_1^*, e_2^*\}$, $M_1 = \{3, (1, 2), e_1^*, e_2^*\}$, $M_2 = \{1, (2, 3), e_1^*, e_2^*\}$, $M_3 = \{2, 3, (1, e_1^*), e_2^*\}$, $M_4 = \{1, 3, (2, e_1^*), e_2^*\}$, $M_5 = \{1, 3, (2, e_2^*), e_1^*\}$, $M_6 = \{1, 2, (3, e_2^*), e_1^*\}$, $M_7 = \{(1, e_1^*), (2, 3), e_2^*\}$, $M_8 = \{2, (1, e_1^*), (3, e_2^*)\}$, $M_9 = \{3, (1, e_1^*), (2, e_2^*)\}$, $M_{10} = \{(1, 2), (3, e_2^*), e_1^*\}$, $M_{11} = \{1, (2, e_1^*), (3, e_2^*)\}$. By the definitions of $E_i$ and $X_i$, $E_1 = \{(1, e_1^*)\}$, $E_2 = \{(2, e_1^*), (2, e_2^*)\}$, $E_3 = \{(3, e_2^*)\}$; $X_1 = \{1, (1, e_1^*)\}$, $X_2 = \{2, (2, e_1^*), (2, e_2^*)\}$, $X_3 = \{3, (3, e_2^*)\}$. Hence $X(G) = \{W_i \mid 0 \le i \le 11\}$, where $Y_0 = (1, 2, 3)$, $Y_1 = ((1, e_1^*), (2, e_1^*), 3)$, $Y_2 = (1, (2, e_2^*), (3, e_2^*))$, $Y_3 = ((1, e_1^*), 2, 3)$, $Y_4 = (1, (2, e_1^*), 3)$, $Y_5 = (1, (2, e_2^*), 3)$, $Y_6 = (1, 2, (3, e_2^*))$, $Y_7 = ((1, e_1^*), (2, e_2^*), (3, e_2^*))$, $Y_8 = ((1, e_1^*), 2, (3, e_2^*))$, $Y_9 = ((1, e_1^*), (2, e_2^*), 3)$, $Y_{10} = ((1, e_1^*), (2, e_1^*), (3, e_2^*))$, $Y_{11} = (1, (2, e_1^*), (3, e_2^*))$. Obviously, the mapping $\mu : M_i \longmapsto \mu(M_i) = Y_i$ between $\mathcal{MD}(R(G))$ and $X(G)$ is bijective.

## 3. The MD partition function of *R*(*G*)

Suppose that $G = (V(G), E(G))$ is a graph with $N$ vertices and $f : V(G) \to \mathcal{C}$ and $g : E(G) \to \mathcal{C}$ are two activity functions defined on $V(G)$ and $E(G)$. Let $R(G)$ be the graph defined above. Note that $V(R(G)) = V(G) \cup \{e_1^*, e_2^*, \ldots, e_\epsilon^*\}$ and $E(R(G)) = E(G) \cup E(VG)$, and $E(VG) = \{(a_i, e_i^*), (b_i, e_i^*) \mid e_i = (a_i, b_i) \in E(G), 1 \le i \le \epsilon\}$. Let

$$F_1 : V(R(G)) \to \mathcal{C} \quad \text{and} \quad F_2 : E(R(G)) \to \mathcal{C}$$

such that $F_1(e_i^*) = 1$ for $1 \le i \le \epsilon$, $F_1(v_j) = f(v_j)$ for $1 \le j \le N$, and $F_2(e_i) = g(e_i)$ for $1 \le i \le \epsilon$ and $F_2((v_i, e_k^*)) = \sqrt{g(e_k)}$ for $e_k = (v_i, v_j)$, $1 \le k \le \epsilon$. In other words, the activities of $v$-type vertices $v_i$ and $e$-type vertices $e^*$ of $R(G)$ are $f(v_i)$ and one, respectively. And the activities of $(v, v)$-type edges $e = (v_i, v_j)$ and $(v, e)$-type edges $(v_i, e_k^*)$ of $R(G)$ are $g(e)$ and $\sqrt{g(e_k)}$, respectively.

For an $x_i \in X_i = \{v_i\} \cup E_i$ and a $Y = (x_1, x_2, \ldots, x_N) \in X(G) = X_1 \times X_2 \times \cdots \times X_N$ defined in Section 2, define the weight $W(x_i)$ of $x_i$ to be $f(v_i)$ if $x_i = v_i$ and $\sqrt{g(e_k)}$ if $x_i = (v_i, e_k^*)$, and define the weight of $Y$, denoted by $W(Y)$, to be the product of weights of all $x_i$'s. It is not difficult to see the summation of weights of all elements $Y$ in $X(G)$ equals

$$\prod_{i=1}^{N} \left( f(v_i) + \sum_{e \in N_G(v_i)} \sqrt{g(e)} \right),$$

where $N_G(v_i)$ denotes the set of edges in $G$ incident with vertex $v_i$.

For an $MD \in \mathcal{MD}(R(G))$, let $M = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \cup \{e_{j_1}^*, e_{j_2}^*, \ldots, e_{j_l}^*\}$ and $D = \{(v_{c_1}, v_{d_1}), (v_{c_2}, v_{d_2}), \ldots, (v_{c_s}, v_{d_s})\} \cup \{(v_{p_1}, e_{q_1}^*), (v_{p_2}, e_{q_2}^*), \ldots, (v_{p_t}, e_{q_t}^*)\}$. Obviously, $k + t + 2s = N$ and $k + l + 2s + 2t = N + \epsilon$. By the definition of the weight of $MD$,

$$W(MD) = \left(\prod_{a=1}^{k} F_1(v_{i_a})\right)\left(\prod_{i=1}^{l} F_1\big(e^*_{j_i}\big)\right)\left(\prod_{i=1}^{s} F_2((v_{c_i}, v_{d_i}))\right) \times \left(\prod_{i=1}^{t} F_2\big((v_{p_i}, e^*_{q_i})\big)\right)$$

$$= \left(\prod_{a=1}^{k} f(v_{i_a})\right)\left(\prod_{i=1}^{s} g((v_{c_i}, v_{d_i}))\right)\left(\prod_{i=1}^{t} \sqrt{g\big(e^*_{q_i}\big)}\right). \tag{1}$$

On the other hand, by the definition of the bijection $\mu$, we have $\mu(MD) = (x_1, x_2, \ldots, x_N)$, where $x_{i_a} = v_{i_a}$ for $1 \le a \le k$, $x_{c_i} = (v_{c_i}, e^*_{(c_i,d_i)})$ and $x_{d_i} = (v_{d_i}, e^*_{(c_i,d_i)})$ for $1 \le i \le s$, and $x_{p_i} = (v_{p_i}, e^*_{q_i})$ for $1 \le i \le t$. Hence the weight of $\mu(MD)$ satisfies:

$$W(\mu(MD)) = \prod_{i=1}^{N} W(x_i) = \left(\prod_{a=1}^{k} W(x_{i_a})\right)\left[\prod_{i=1}^{s} \big(W(x_{c_i})W(x_{d_i})\big)\right]\left(\prod_{i=1}^{t} W(x_{p_i})\right)$$

$$= \left(\prod_{a=1}^{k} f(v_{i_a})\right)\left(\prod_{i=1}^{s} g((v_{c_i}, v_{d_i}))\right)\left(\prod_{i=1}^{t} \sqrt{g\big(e^*_{q_i}\big)}\right). \tag{2}$$

Hence by (1) and (2) we have shown that for any an $MD \in \mathcal{MD}(R(G))$

$$W(MD) = W(\mu(MD)). \tag{3}$$

Therefore, a direct result from (3) and Theorem 1 is the following

THEOREM 2. *Suppose $G = (V(G), E(G))$ is a graph with $N$ vertices and $f : V(G) \to \mathcal{C}$ and $g : E(G) \to \mathcal{C}$ are two activity functions defined on $V(G)$ and $E(G)$. Let $R(G)$, $F_1 : V(R(G)) \to \mathcal{C}$ and $F_2 : E(R(G)) \to \mathcal{C}$ are defined as above. Then the MD partition function of $R(G)$*

$$Z(R(G); F_1, F_2) = \prod_{i=1}^{N}\left(f(v_i) + \sum_{e \in N_G(v_i)} \sqrt{g(e)}\right),$$

*where $N_G(v_i)$ denotes the set of edges in $G$ incident with site $v_i$.*

From the theorem above, the number of monomer–dimer arrangements of $R(G)$ is exactly $\prod_{i=1}^{N}(d_i + 1)$, where $d_i$ denotes the degree of $v_i$ in $G$ (i.e., the number of edges in $G$ incident with vertex $v_i$) [20].

For the graph $G$ and $R(G)$ illustrated in Figures 1(a) and (b), we assume that the activities of the old vertices $v_i$'s and the new vertices $e^*_i$'s in $R(G)$ are $x$ and 1, and the activities of the old edges $e_i$'s and the new edges $(v_1, e^*_1)$, $(v_2, e^*_1), \ldots, (v_5, e^*_6)$ are $y^2$ and $y$ (see Figure 1(c)). Then the MD partition function of $R(G)$ equals $(x + 2y)^3(x + 3y)^2$ and the number of monomer–dimer arrangements of $R(G)$ is $3^3 4^2 = 432$.

## 4. Exact formulae for some graphs

We consider the $d$-dimensional rectangular lattices with fixed boundary conditions and with periodic boundary conditions, denoted by $L(n, d)$ and $\tilde{L}(n, d)$, respectively. The vertices of $L(n, d)$ are the $n^d$ integer lattice points in $[1, n]^d$, and two points $x, y$ are connected by an edge iff they are unit distance apart. $\tilde{L}(n, d)$ can be obtained by augmenting $L(n, d)$ with an edge between $(x_1, \ldots, x_{i-1}, n, x_{i+1}, \ldots, x_d)$ and $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d)$ for each $i$. Let the activities of the vertices in $L(n, d)$ and $\tilde{L}(n, d)$ are $z_0$, and the activities of edges of $d$ orientations in $L(n, d)$ and $\tilde{L}(n, d)$ are $z_1, z_2, \ldots, z_d$, respectively.

Note that the lattice graph $R(L(n, d))$ (resp. $R(\tilde{L}(n, d))$) is obtained from $L(n, d)$ (resp. $\tilde{L}(n, d)$) by adding a new vertex $e^*$ corresponding to each edge $e = (a, b)$ of $L(n, d)$ (resp. $\tilde{L}(n, d)$) and by joining each new vertex $e^*$ to the vertices $a$ and $b$ corresponding to it. In other words, $R(L(n, d))$ (resp. $R(\tilde{L}(n, d))$) is obtained from $L(n, d)$ (resp. $\tilde{L}(n, d)$) by replacing each edge $e = (a, b)$ in $L(n, d)$ (resp. $\tilde{L}(n, d)$) by a triangle $ae^*b$ (that is, for each edge $e = (a, b)$, add a new vertex $e^*$ and two new edges $(a, e^*)$ and $(b, e^*)$). For an arbitrary vertex $v$ in $R(L(n, d))$ (resp. $\tilde{L}(n, d)$), define the activity of $v$ to be one if $v$ is a new adding vertex and $z_0$ otherwise. For an arbitrary edge $e'$ in $R(L(n, d))$ (resp. $R(\tilde{L}(n, d))$), define the activity of $e'$ to be $z_j$ if $e'$ is also an edge of the $j$th orientation in $L(n, d)$ (resp. $\tilde{L}(n, d)$) and the activity of $e'$ to be $\sqrt{z_j}$ if $e'$ is an edge with form $(v, e^*)$ and $e$ is an edge of the $j$th orientation in $L(n, d)$ (resp. $\tilde{L}(n, d)$). Figure 3 illustrated $R(L(7, 2))$ and the activities of a unit of $R(L(7, 2))$. For a vertex $v$ of $\tilde{L}(n, d)$, there exists exactly two edges of the $j$th orientation for $1 \leq j \leq d$ and the activities of these two edges equal $z_j$. Hence $\sum_{e \in N_{\tilde{L}(n,d)}(v)} \sqrt{g(e)} = 2 \sum_{j=1}^{d} \sqrt{z_j}$, where $N_{\tilde{L}(n,d)}(v)$ is the set of edges

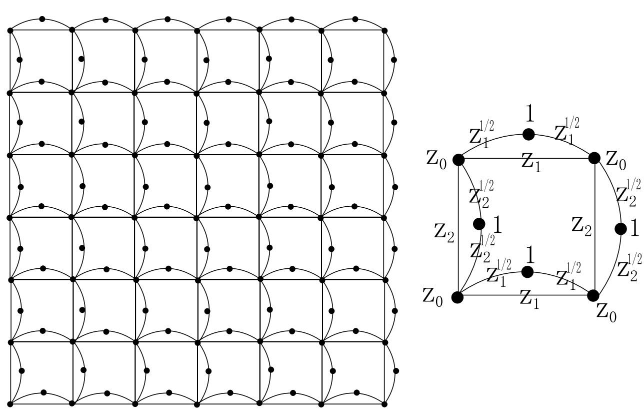


Figure 3. The lattice graph $R(L(7, 2))$ of the two-dimensional rectangular lattice $L(7, 2)$ with fixed boundary conditions and its activities on a unit.

incident with vertex $v$ in $\tilde{L}(n, d)$. Therefore, the following result is immediate from Theorem 2.

COROLLARY 1. *Let $L(n, d)$ and $\tilde{L}(n, d)$ be the d-dimensional rectangular lattices with fixed boundary conditions and with periodic boundary conditions and the activities on vertices and edges be defined as above. Then*

*(1) the MD partition function* $Z(R(\tilde{L}(n, d); \{z_i\}_0^d) = (z_0 + 2\sum_{j=1}^d \sqrt{z_j})^{n^d}$,
*(2) the MD free energies* $\phi(R(\tilde{L}(n, d); \{z_i\}_0^d) = \phi(R(L(n, d); \{z_i\}_0^d) = \frac{1}{d+1} \log(z_0 + 2\sum_{j=1}^d \sqrt{z_j})$,
*(3) the MD entropies* $\chi(R(\tilde{L}(n, d); \{z_i\}_0^d) = \chi(R(n, d)) = \frac{\log(2d+1)}{d+1}$.

Although a closed formula for the MD partition function of $R(L(n, d))$ can be given, we omit it in Corollary 1 because it has a complicated form. For example, the MD partition function $Z(R(L(n, 2); z_0, z_1, z_2)$ of $R(L(n, 2))$ equals $(z_0 + \sqrt{z_1} + \sqrt{z_2})^4 (z_0 + 2\sqrt{z_1} + \sqrt{z_2})^{2(n-2)} (z_0 + \sqrt{z_1} + 2\sqrt{z_2})^{2(n-2)} (z_0 + 2\sqrt{z_1} + 2\sqrt{z_2})^{(n-2)^2}$.

Suppose $G(N, d)$ be an arbitrary bulk regular lattice graph with $N$ vertices and each vertex is incident with $d$ edges. If we assume the activities of the new vertices and the old vertices in $R(G(N, d))$ are one and $z_0$, and the activities of the new edges and the old edges in $R(G(N, d))$ are $\sqrt{z_1}$ and $z_1$, respectively. Similarly, we have the following.

COROLLARY 2. *Let G be a d-regular graph with N vertices ($N \to \infty$) and the activities of vertices and edges in $R(G)$ be defined as above. Then*

*(1) the MD partition function* $Z(R(G(N, d); z_0, z_1) = (z_0 + d\sqrt{z_1})^N$,
*(2) the MD free energy* $\phi(R(G(N, d); z_0, z_1) = \frac{2}{d+2} \log(z_0 + d\sqrt{z_1})$,
*(3) the MD entropy* $\chi(R(G(N, d); z_0, z_1) = \frac{2}{d+2} \log(d + 1)$.

## 5. Concluding remarks

The monomer–dimer problem is a fundamental problem in lattice statistics. From time to time the monomer–dimer system is used as a model of a physical system, but primarily it is interesting as the phototypical lattice statistical mechanics problem. It gained prominence in 1937 through the paper of Fowler and Rushbrooke [7]. Although the pure-dimer problem had been solved for many lattice graphs, no exact solutions of the monomer–dimer problem have been found so far (except that some numerical solutions of the monomer–dimer problem had been reported). In this paper, the monomer–dimer problem has been solved in exact closed form for infinite many lattice graphs. Particularly, for the $d$-dimensional lattice $\tilde{L}(n, d)$ with periodic boundaries, if suitable activities of vertices and edges in $R(\tilde{L}(n, d))$ are selected, then the exact formulae for the MD partition function, MD free energy, and MD entropy of $R(\tilde{L}(n, d))$ have been obtained.

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