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Dimer problem on the cylinder and torus

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ABSTRACT

We obtain explicit expressions of the number of close-packed dimers and entropy for three types of lattices (the so-called 8.8.6, 8.8.4, and hexagonal lattices) with cylindrical boundary condition and the entropy of the 8.8.6 lattice with toroidal boundary condition. Our results and the one on 8.8.4 and hexagonal lattices with toroidal boundary condition by Salinas and Nagle [S.R. Salinas, J.F. Nagle, Theory of the phase transition in the layered hydrogen-bonded $SnCl^2 \cdot 2H_2O$ crystal, Phys. Rev. B 9 (1974) 4920–4931] and Wu [F.Y. Wu, Dimers on two-dimensional lattices, Inter. J. Modern Phys. B 20 (2006) 5357–5371] imply that the 8.8.6 (or 8.8.4) lattices with cylindrical and toroidal boundary conditions have the same entropy whereas the hexagonal lattices have not. Based on these facts we propose the following problem: under which conditions do the lattices with cylindrical and toroidal boundary conditions have the same entropy?

1. Introduction

In 1961, Kasteleyn [14] found a formula for the number of close-packed dimers (perfect matchings) of an $m \times n$ quadratic lattice graph. Temperley and Fisher [34] used a different method and arrived at the same result at almost exactly the same time. Both lines of calculation showed that the logarithm of the number of close-packed dimers, divided by $\frac{mn}{2}$, converges to $2c/\pi \approx 0.5831$ as $m, n \to \infty$, where c is Catalan's constant. This limit is called the entropy of the quadratic lattice graph and the corresponding problem was called the dimer problem by the statistical physicists. In 1992, Elkies et al. [6] studied the enumeration of close-packed dimers of regions called Aztec diamonds, and showed that the entropy equals $\frac{\log 2}{2} \approx 0.35$. Different methods for counting close-packed dimers of Aztec diamonds were considered by many authors (see for example Refs. [19,28,42]). The problem involving enumeration of close-packed dimers of another type of quadratic lattices with different boundary conditions was studied by Sachs and Zeritz [31] and a different entropy was obtained. These facts showed that the entropy of the quadratic lattice is strongly dependent on their boundary conditions. It should be pointed out that the dimer model on the hexagonal (Kasteleyn or brick) lattice has a "frozen" ground state, which sort of resembles the ground state of the ferromagnetic six-vertex model. It has been shown that the entropy of the six-vertex model does depend on the boundary conditions [18]. See also the works of chemists cited in [10].

Cohn, Kenyon, and Propp [4] demonstrated that the behavior of random perfect matchings (close-packed dimers) of large regions R was determined by a variational (or entropy maximization) principle, as was conjectured in Section 8 of Ref. [6], and they gave an exact formula for the entropy of simply-connected regions of arbitrary shape. Particularly, they showed that computation of the entropy is intimately linked with an understanding of long-range variations in the local statistics of random domino tilings. Kenyon, Okounkov, and Sheffield [17] considered the problem of enumerating close-packed dimers

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of the doubly period bipartite graph on a torus, which generalized the results in Ref. [4]. They proved that the number of close-packed dimers of the doubly period plane bipartite graph G can be expressed in terms of four determinants and they expressed the entropy of G as a double integral.

The exact solution of the dimer problem was obtained for many lattices such as the quadratic lattice, 8.8.4 lattice, hexagonal lattice, triangular lattice, kagome lattice, 3-12-12 lattice, union Jack lattice, and etc. with toroidal boundary condition [8,14,32,37]. The exact solution of the dimer problem has been extended to the cylindrical condition [22,26]. Wu and Wang [36] obtained the exact result on the enumeration of close-packed dimers on a finite kagome lattice with general asymmetric dimer weights under the cylindrical boundary condition. The result by Wu and Wang implies that the kagome lattices with the cylindrical and toroidal boundary conditions have the same entropy. This phenomenon also took place for some other lattices with the cylindrical and toroidal boundary conditions [9].

Some related work about the dimer problem can be found in, for example, Refs. [7,8,10,14–16,23,27,32,34,39,40] by physicists and chemists and Refs. [2–4,17,29,33,41–43] by mathematicians.

In this paper, we consider three types of lattices—8.8.6, 8.8.4, and hexagonal lattices. We obtain explicit expressions of the number of close-packed dimers and entropy for these three types of lattices with cylindrical boundary condition. Based on the result by Kenyon, Okounkov, and Sheffield [17], we compute the entropy of the 8.8.6 lattice with toroidal boundary condition. Combining our results and the one on 8.8.4 and hexagonal lattices with toroidal boundary condition [32,37], we can see that the 8.8.6 (or 8.8.4) lattices with cylindrical and toroidal boundary conditions have the same entropy whereas the effect of the boundary for the hexagonal lattices is not trivial (that is, the hexagonal lattices with cylindrical and toroidal boundary conditions have different entropies). Based on these facts we would propose the following problem: under which conditions do the lattices with cylindrical and toroidal boundary conditions have the same entropy?

2. Pfaffians

The Pfaffian method enumerating close-packed dimers of plane graphs was independently discovered by Fisher [8], Kasteleyn [14], and Temperley [34]. Given a plane graph G, the method produces a skew symmetric matrix A such that the number of close-packed dimers of G can be expressed by the Pfaffian of the matrix A. Alternatively, the Pfaffian can be replaced by the square root of the determinant of A. By using this method, Fisher [8], Kasteleyn [14], and Temperley [34] solved independently a famous problem on enumerating close-packed dimers of an $m \times n$ quadratic lattice graph in statistical physics–Dimer problem. Given a simple graph $G = (V(G), E(G))$ with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, let G^e be an arbitrary orientation. The skew adjacency matrix of G^e, denoted by $A(G^e)$, is defined as follows:

$$A(G^e) = (a_{ij})_{n\times n},$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an arc of } G^e, \\ -1 & \text{if } (v_j, v_i) \text{ is an arc of } G^e, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $A(G^e)$ is a skew symmetric matrix.

Let D be an orientation of a graph G. A cycle C of even length in D is said to be oddly oriented in D if for either choice of the two directions of traversal around C, the number of edges of C directed in the direction of the traversal is odd (note that this definition is independent of the choice of traversal, since C has an even length). A cycle C in D is said to be nice if the subgraph D–C (obtained from D by deleting all vertices of C) has a close-packed dimer. We say that D is a Pfaffian orientation of G if every nice cycle of even length of G is oddly oriented in D. It is well known that if a bipartite graph G contains no subdivision of $K_{3,3}$ then G has a Pfaffian orientation (see Little [20]). McCuaig [24], and McCuaig, Robertson et al. [25], and Robertson, Seymour et al. [30] found a polynomial-time algorithm to show whether a bipartite graph has a Pfaffian orientation. Stembridge [33] proved that the number (or generating function) of nonintersecting r-tuples of paths from a set of r vertices to a specified region in an acyclic digraph D can, under favorable circumstances, be expressed as a Pfaffian. For a recent survey of Pfaffian orientations of graphs, please see Thomas [35]. Throughout this paper, we denote by $M(G)$ the number of close-packed dimers of a graph G.

Lemma 2.1 (*Lovász et al. [21]*). *If G^e is a Pfaffian orientation of a graph G, then*

$$M(G) = \sqrt{\det(A(G^e))},$$

where $A(G^e)$ is the skew adjacency matrix of G^e.

Lemma 2.2 (*Lovász et al. [21]*). *If a plane graph G has an orientation G^e such that every boundary face – except possibly the infinite face – has an odd number of edges oriented clockwise, then in every cycle the number of edges oriented clockwise is of opposite parity to the number of vertices of G^e inside the cycle. Consequently, G^e is a Pfaffian orientation.*

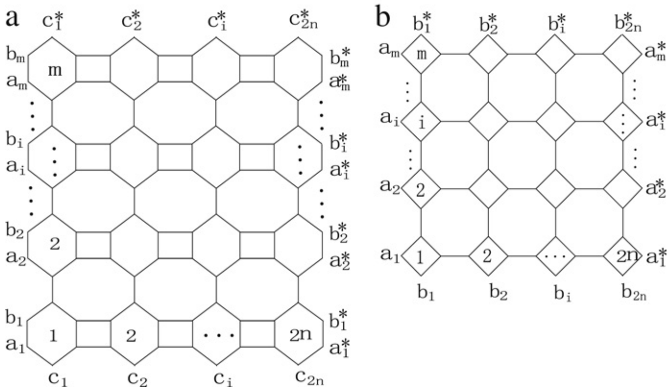

Fig. 1. (a) The 8.8.6 lattice $G_1^*(m, 2n)$. (b) The 8.8.4 lattice $G_2^*(m, 2n)$.

3. Three types of cylinders

Two bulk lattices, denoted by $G_1^*(m, 2n)$ and $G_2^*(m, 2n)$, are illustrated in Fig. 1(a) and Fig. 1(b), respectively, where $G_1^*(m, 2n)$ is a finite subgraph of an edge-to-edge tiling of the plane with two types of vertices—8.8.6 and 8.8.4 vertices, and $G_2^*(m, 2n)$ is a finite subgraph of 8.8.4 tiling in the Euclidean plane which has been used to describe phase transitions in the layered hydrogen-bonded $SnCl^2 \cdot 2H_2O$ crystal [32] in physical systems [1,27,32]. The 8.8.6 lattice $G_1^*(m, 2n)$, whose fundamental part is a hexagon, is composed of $2mn$ hexagons. Similarly, The 8.8.4 lattice $G_2^*(m, 2n)$, whose fundamental part is a quadrangle, is composed of $2mn$ quadrangles. Each of such bulk graphs is called "an $(m, 2n)$-bipartite graph with free boundary condition" (see Ref. [23]).

If we add edges (a_i, a_i^*), (b_i, b_i^*) for $1 \le i \le m$ and (c_j, c_j^*) for $1 \le j \le 2n$ in $G_1^*(m, 2n)$, we obtain an $(m, 2n)$-bipartite graph with toroidal boundary condition, denoted by $G_1^t(m, 2n)$. Similarly, if we add edges (a_i, a_i^*) for $1 \le i \le m$ and (b_j, b_j^*) for $1 \le j \le 2n$ in $G_2^*(m, 2n)$, then an $(m, 2n)$-bipartite graph with toroidal boundary condition, denoted by $G_2^t(m, 2n)$, is obtained. For some related work on the plane bipartite graphs with the toroidal boundary condition, see Kenyon, Okounkov, and Sheffield [17] and Cohn, Kenyon, and Propp [4]. Salinas and Nagle [32] and Wu [37] showed that the entropy of $G_2^t(m, 2n)$, that is $\lim_{n,m\to\infty} \frac{2}{8mn} \log[M(G_2^t(m, 2n))]$, equals

$$\frac{1}{2\pi}\int_0^{\pi/2} \log\left[\frac{5+\sqrt{25-16\cos^2\theta}}{2}\right] \mathrm{d}_\theta \approx 0.3770. \tag{1}$$

The hexagonal lattices with toroidal and cylindrical boundary conditions, denoted by $H^t(n, m)$ and $H^c(n, m)$, are illustrated in Fig. 2(a) and Fig. 2(b), where $(a_1, b_1), (a_2, b_2), \ldots, (a_{m+1}, b_{m+1}), (a_1, c_1^*), (c_1, c_2^*), (c_2, c_3^*), \ldots, (c_{n-1}, c_n^*), (c_n, b_{m+1})$ are edges in $H^t(n, m)$, and $(a_1, b_1), (a_2, b_2), \ldots, (a_{m+1}, b_{m+1})$ are edges in $H^c(n, m)$. Wu [37–39] showed that the entropy of $H^t(n, m)$, i.e.,

$$\lim_{n,m\to\infty} \frac{2}{(n+1)(2m+2)} \log[M(H^t(n, m))] = \frac{2}{\pi}\int_0^{\pi/3} \log(2\cos\theta)\mathrm{d}_\theta \approx 0.3230. \tag{2}$$

In this section, we enumerate close-packed dimers of the 8.8.6, 8.8.4, and hexagonal lattices $G_1(m, 2n)$, $G_2(m, 2n)$, and $H^c(n, m)$ with cylindrical boundary condition, where $G_1(m, 2n)$ (resp. $G_2(m, 2n)$) is obtained from $G_1^*(m, 2n)$ (resp. $G_2^*(m, 2n)$) by adding extra edges (a_i, a_i^*), (b_i, b_i^*) for $1 \le i \le m$ (resp. (a_i, a_i^*) for $1 \le i \le m$) between each pair of opposite vertices of both sides of them. We call each of $G_1(m, 2n)$ and $G_2(m, 2n)$ "an $(m, 2n)$-bipartite graph with cylindrical boundary condition" (simply cylinder). We also obtain the exact solutions for the entropies of the 8.8.6 lattice $G_1(m, 2n)$, 8.8.4 lattice $G_2(m, 2n)$, and hexagonal lattice $H^c(n, m)$ with cylindrical boundary condition.

3.1. The cylinder $G_1(m, 2n)$

Let $G_1(m, 2n)^e$ be the orientation of $G_1(m, 2n)$ illustrated in Fig. 3(a). For $G_1(m, 2n)^e$, all hexagons in the first column have the same orientation, all hexagons in the second column have the inverse of the orientation of hexagons in the first column, and so on. Obviously, $G_1(m, 2n)^e$ satisfies the conditions in Lemma 2.2 and hence is a Pfaffian orientation.

Theorem 3.1. *For the cylinder $G_1(m, 2n)$, the number of close-packed dimers of $G_1(m, 2n)$ can be expressed by*

$$M(G_1(m, 2n)) = \frac{1}{2^n}\prod_{j=0}^{n-1}\frac{1}{\sqrt{4+\beta_j^2}}\left[\left(\sqrt{4+\beta_j^2}+\beta_j\right)^{2m+1}+\left(\sqrt{4+\beta_j^2}-\beta_j\right)^{2m+1}\right], \tag{3}$$

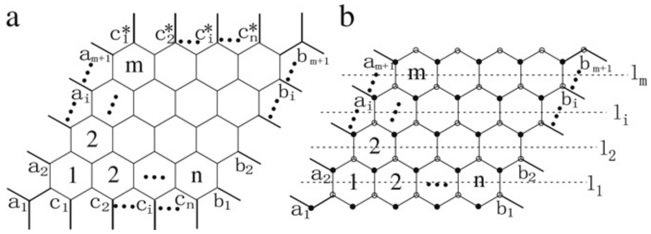

Fig. 2. (a) The hexagonal lattice $H^t(n, m)$ with toroidal boundary condition. (b) The hexagonal lattice $G^c(n, m)$ with cylindrical boundary condition.

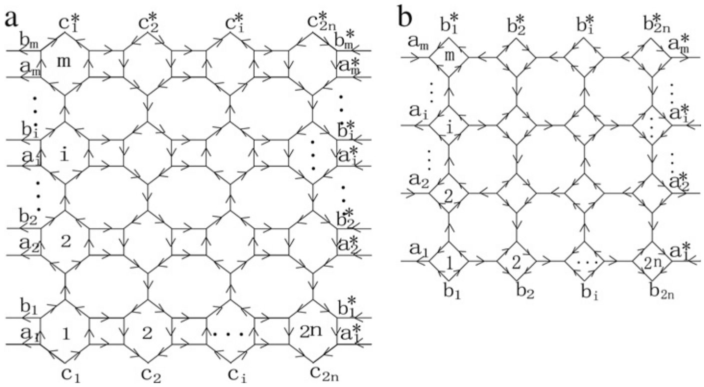

Fig. 3. (a) The orientation $G_1(m, 2n)^e$ of $G_1(m, 2n)$. (b) The orientation $G_2(m, 2n)^e$ of $G_2(m, 2n)$.

and the entropy of $G_1(m, 2n)$*, i.e.,* $\lim_{m,n\to\infty} \frac{2}{12mn} \log M(G_1(m, 2n))$*, equals*

$$\frac{2}{3\pi}\int_0^{\frac{\pi}{2}} \log\left(\cos x + \sqrt{4+\cos^2 x}\right) \mathrm{d}_x \approx 0.3344,$$

where $\beta_j = \cos\frac{2j\pi}{2n}$ *if n is odd and* $\beta_j = \cos\frac{(2j+1)\pi}{2n}$ *otherwise.*

In order to prove Theorem 3.1, we must introduce some lemmas as follows.
For $j = 0, 1, \ldots, n-1$, define:

$$\alpha_j = \alpha_j(n) = \begin{cases} \cos\dfrac{2j\pi}{2n} + \mathrm{i}\sin\dfrac{2j\pi}{2n} & \text{if } n = 1 \pmod 2; \\ \cos\dfrac{(2j+1)\pi}{2n} + \mathrm{i}\sin\dfrac{(2j+1)\pi}{2n} & \text{otherwise.} \end{cases} \tag{4}$$

Obviously, $\alpha_j(n)^{2n} = 1$ if n is odd and $\alpha_j(n)^{2n} = -1$ otherwise.

Lemma 3.2 ([5]). *Let*

$$B = \begin{pmatrix} B_0 & B_1 & \cdots & B_{n-1} \\ B_{n-1} & B_0 & \cdots & B_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_0 \end{pmatrix}$$

be a block circulant matrix over the complex number field, where all B_t*'s are* $r \times r$ *matrices,* $t = 0, 1, \ldots, n-1$*. Then B satisfies the following factorization equality*

$$U^*BU = \mathrm{diag}(J_0, J_1, \ldots, J_{n-1}),$$

where $U = (u_{pq}I_r)_{0\le p,q\le n-1}$ *is a fixed block matrix,* $u_{pq} = \frac{1}{\sqrt{n}}\omega^{pq}$*,* I_r *is a unit matrix of order r, and* ω *is the nth root of unity, meanwhile*

$$J_t = B_0 + B_1\omega^t + B_2\omega^{2t} + \cdots + B_{n-1}\omega^{(n-1)t}.$$

A direct consequence of Lemma 3.2 is the following:

Corollary 3.3. *Let*

$$B = \begin{pmatrix} A & R & 0 & \cdots & 0 & R^{\mathrm{T}} \\ R^{\mathrm{T}} & A & R & \cdots & 0 & 0 \\ 0 & R^{\mathrm{T}} & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A & R \\ R & 0 & 0 & \cdots & R^{\mathrm{T}} & A \end{pmatrix}_{n\times n}$$

be a block circulant matrix over the real number field, where both A and R are $r \times r$ matrices. Then there exists an invertible matrix U of order nr such that

$$U^{-1}BU = \mathrm{diag}(J_0, J_1, \ldots, J_{n-1}),$$

where $J_t = A + \omega^t R + \omega^{-t} R^{\mathrm{T}}$, $t = 0, 1, \ldots, n-1$, and ω is the nth root of unity.

Lemma 3.4. *Let*

$$B = \begin{pmatrix} A & R & 0 & \cdots & 0 & -R^{\mathrm{T}} \\ R^{\mathrm{T}} & A & R & \cdots & 0 & 0 \\ 0 & R^{\mathrm{T}} & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A & R \\ -R & 0 & 0 & \cdots & R^{\mathrm{T}} & A \end{pmatrix}_{n\times n}$$

be a block circulant matrix over the real number field, where both A and R are $r \times r$ matrices. Then there exists an invertible matrix U of order nr such that

$$U^{-1}BU = \mathrm{diag}(J_0, J_1, \ldots, J_{n-1}),$$

where $J_t = A + \omega_t R + \omega_t^{-1} R^{\mathrm{T}}$, and $\omega_t = \cos\frac{(2t+1)\pi}{n} + \mathrm{i}\sin\frac{(2t+1)\pi}{n}$, $t = 0, 1, \ldots, n-1$.

Proof. Note that the set of roots of equation $x^n = -1$ is exactly $\{\omega_0, \omega_1, \ldots, \omega_{n-1}\}$. Let $U = (U_{ij})_{n\times n}$ and $U^* = (U^*_{ij})_{n\times n}$ be two $n \times n$ block matrices such that $U_{ij} = \frac{\omega_j^i}{\sqrt{n}} I_r$ and $U^*_{ij} = \frac{\omega_i^{-j}}{\sqrt{n}} I_r$ for $0 \le i, j \le n-1$, where I_r is the identity matrix of order r. It is not difficult to show that $U^*U = I_{nr}$. Hence $U^* = U^{-1}$. It suffices to prove that $U^*BU = \mathrm{diag}(J_0, J_1, \ldots, J_{n-1})$. For convenience, let $B = (B_{st})_{n\times n}$ and $U^*BU = (X_{st})_{n\times n}$, where B_{st} and X_{st} are $r \times r$ matrices. Note that, for $0 \le i, j \le n-1$,

$$\begin{aligned} X_{ij} &= \sum_{0\le s,t\le n-1} U^*_{is} B_{st} U_{tj} = \frac{1}{n} \sum_{0\le s,t\le n-1} \omega_i^{-s} B_{st} \omega_j^t \\ &= \frac{1}{n}\left[\sum_{k=0}^{n-1} (\omega_i^{-1}\omega_j)^k A + \sum_{k=0}^{n-1} \omega_j (\omega_i^{-1}\omega_j)^k R + \sum_{k=0}^{n-1} \omega_j^{-1} (\omega_i^{-1}\omega_j)^k R^{\mathrm{T}}\right] \\ &= \begin{cases} J_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence we have finished the proof of the lemma. □

Proof of Theorem 3.1. By a suitable labelling of vertices of $G_1(m, 2n)^e$, the skew adjacency matrix of $G_1(m, 2n)^e$, denoted by $A_1(G^e)$, has the following form:

$$A_1(G^e) = \begin{pmatrix} A & R & 0 & \cdots & 0 & R^{\mathrm{T}} \\ -R^{\mathrm{T}} & -A & R & \cdots & 0 & 0 \\ 0 & -R^{\mathrm{T}} & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A & R \\ -R & 0 & 0 & \cdots & -R^{\mathrm{T}} & -A \end{pmatrix}_{2n\times 2n},$$

where A denotes the skew adjacency matrix of the part in first column of $G_1(m, 2n)^e$, and R is the adjacency relation between the two parts in the first and second columns of $G_1(m, 2n)^e$.

Set

$$M_1 = \begin{pmatrix} A & R & 0 & \cdots & 0 & R^{\mathrm{T}} \\ R^{\mathrm{T}} & A & R & \cdots & 0 & 0 \\ 0 & R^{\mathrm{T}} & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A & R \\ R & 0 & 0 & \cdots & R^{\mathrm{T}} & A \end{pmatrix}_{2n\times 2n}$$

and

$$M_2 = \begin{pmatrix} A & R & 0 & \cdots & 0 & -R^{\mathrm{T}} \\ R^{\mathrm{T}} & A & R & \cdots & 0 & 0 \\ 0 & R^{\mathrm{T}} & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A & R \\ -R & 0 & 0 & \cdots & R^{\mathrm{T}} & A \end{pmatrix}_{2n\times 2n}.$$

Multiplying by -1 to rows 2 and 3 of the block matrix $A_1(G^e)$, then to columns 3 and 4, then to rows 6 and 7, then to columns 7 and 8, and so on and so forth, finally we obtain the matrix M_1 when $n = 1 \pmod 2$, or the matrix M_2 when $n = 0 \pmod 2$. Clearly we have

$$\det(A_1(G^e)) = \begin{cases} |\det(M_1)| & \text{if } n = 1 \pmod 2, \\ |\det(M_2)| & \text{if } n = 0 \pmod 2. \end{cases}$$

Note that, by Corollary 3.3 and Lemma 3.4, we have

$$\det(M_1) = \prod_{j=0}^{2n-1} \det(A + \omega^j R + \omega^{-j} R^{\mathrm{T}}) = \left(\prod_{j=0}^{n-1} \det(A + \omega^j R + \omega^{-j} R^{\mathrm{T}}) \right)^2$$

and

$$\det(M_2) = \prod_{j=0}^{2n-1} \det(A + \omega_j R + \omega_j^{-1} R^{\mathrm{T}}) = \left(\prod_{j=0}^{n-1} \det(A + \omega_j R + \omega_j^{-1} R^{\mathrm{T}}) \right)^2,$$

where $\omega = \cos\frac{2\pi}{2n} + \mathrm{i}\sin\frac{2\pi}{2n}$ (*i.e.*, ω is the $2n$th root of unity) and $\omega_j = \cos\frac{(2j+1)\pi}{2n} + \mathrm{i}\sin\frac{(2j+1)\pi}{2n}$.

Now we construct n weighted digraphs D_j's with $6m$ vertices from $G_1(m, 2n)^e$ which are illustrated in Fig. 3(a), where α_j's satisfy (1), and each arc (v_s, v_t) in D_j whose weight is neither α_j nor α_j^{-1} must be regarded as two arcs (v_s, v_t) and (v_t, v_s) with weights 1 and -1, respectively.

By the definitions of α_j's and D_j's, it is not difficult to see that the adjacency matrix A_j of weighted digraph D_j defined above equals exactly $A + \omega^j R + \omega^{-j} R^{\mathrm{T}}$ if $n = 1 \pmod 2$ and $A + \omega_j R + \omega_j^{-1} R^{\mathrm{T}}$ otherwise. Thus, by Lemma 2.1, we have

$$M(G_1(m, 2n)) = \sqrt{\det(A(G_1^e))} = \prod_{j=0}^{n-1} |\det(A_j)|, \tag{5}$$

where A_j is the adjacency matrix of D_j.

Let D_j' be the digraph obtained from D_j by deleting vertex c_1^* (see Fig. 4(b)). For $j = 0, 1, \ldots, n-1$, set

$$L_m(j) = \det(A_j), \qquad L_m'(j) = \det(A_j'),$$

where A_j (resp. A_j') is the adjacency matrix of the digraph D_j (resp. D_j'). It is not difficult to prove that $\{L_m(j)\}_{m\geq 0}$ and $\{L_m'(j)\}_{m\geq 0}$ satisfy the following recurrences:

$$\begin{cases} L_m(j) = (4 + 4\beta_j^2) L_{m-1}(j) + 4\beta_j L_{m-1}'(j) & \text{for } m \geq 1, \\ L_m'(j) = 4\beta_j L_{m-1}(j) + 4L_{m-1}'(j) & \text{for } m \geq 1, \\ L_0(j) = 1, \qquad L_0'(j) = 0. \end{cases}$$

Then we have

$$\begin{cases} L_m(j) = (8 + 4\beta_j^2) L_{m-1}(j) - 16 L_{m-2}(j) & \text{for } m \geq 2, \\ L_0(j) = 1, \qquad L_1(j) = 4 + 4\beta_j^2, \end{cases}$$

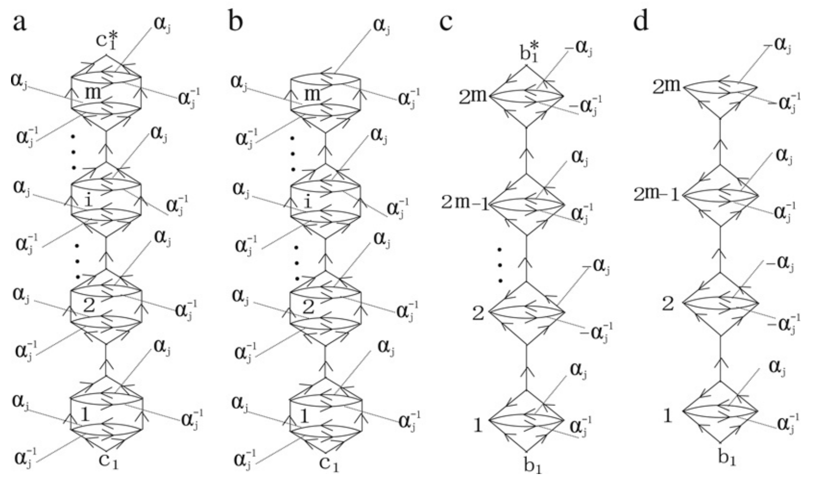

Fig. 4. (a) The digraphs D_j's. (b) The digraph D_j' obtained from D_j by deleting vertex c_1^* shown in Fig. 4(a). (c) The digraphs C_j's. (d) The digraph C_j' obtained from D_j by deleting vertex b_1^* shown in Fig. 4(c).

which implies the following:

$$L_m(j) = \frac{1}{2\sqrt{4+\beta_j^2}}\left[\left(\sqrt{4+\beta_j^2}+\beta_j\right)^{2m+1} + \left(\sqrt{4+\beta_j^2}-\beta_j\right)^{2m+1}\right]. \tag{6}$$

Hence the equality (3) follows from (5) and (6). So the entropy of $G_1(m, 2n)$

$$\begin{aligned}
&\lim_{m,n\to\infty} \frac{2}{12mn} \log M(G_1(m, 2n)) \\
&= \lim_{m,n\to\infty} \frac{1}{6mn}\left\{-n\log 2 - \frac{1}{2}\sum_{j=0}^{n-1}\log(4+\beta_j^2) + \sum_{j=0}^{n-1}\log\left[\left(\sqrt{4+\beta_j^2}+\beta_j\right)^{2m+1} + \left(\sqrt{4+\beta_j^2}-\beta_j\right)^{2m+1}\right]\right\} \\
&= \frac{2}{3\pi}\int_0^{\pi/2} \log\left(\cos x + \sqrt{4+\cos^2 x}\right) \mathrm{d}_x \approx 0.3344
\end{aligned}$$

and this completes the proof. □

In order to prove the following corollary, we need to introduce a formula of the entropy for an (n, n)-bipartite graph with toroidal boundary condition obtained by Kenyon, Okounkov, and Sheffield [17]. Let G be a Z^2-period bipartite graph which is embedded in the plane so that translations in the plane act by color-preserving isomorphisms of G—isomorphisms which map black vertices to black vertices and white to white. Let G_n be the quotient of G by the action of nZ^2. Then G_n is a bipartite graph with the doubly period condition. Let $P(z, w)$ be the characteristic polynomial of G (see the definition in page 1029 in Ref. [17]). Authors of Ref. [17] showed that the entropy of G_n

$$\lim_{n\to\infty} \frac{2}{n^2|G_1|}\log M(G_n) = \frac{2}{|G_1|(2\pi \mathrm{i})^2}\int_D \log|P(z, w)|\frac{\mathrm{d}_z}{z}\frac{\mathrm{d}_w}{w}, \tag{7}$$

where $D = \{(z, w) \in C^2 : |z| = |w| = 1\}$ and $\mathrm{i}^2 = -1$.

Corollary 3.5. *Both $G_1(m, 2n)$ and $G_1^t(m, 2n)$ have the same entropy, that is,*

$$\lim_{m,n\to\infty} \frac{2}{12mn}\log(M(G_1(m, 2n))) = \lim_{m,n\to\infty} \frac{2}{12mn}\log(M(G_1^t(m, 2n))) \approx 0.3344.$$

Proof. Note that by the definition in Ref. [17] the fundamental domain of $G_1^t(m, 2n)$ is composed of two hexagons (see Fig. 5). Otherwise, if we use a hexagon as the fundamental domain, then it does not satisfy the condition "color-preserving isomorphisms". It is not difficult to show that the characteristic polynomial of $G_1^t(m, 2n)$

$$P(z, w) = 10 - 4(z + z^{-1}) - (w + w^{-1}).$$

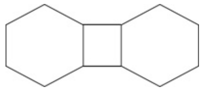

Fig. 5. The fundamental domain of $G_1^t(m, 2n)$.

Hence, by (7), we have

$$\lim_{m,n\to\infty} \frac{2}{12mn} \log(M(G_1^t(m, 2n))) = \frac{2}{12(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log(10 - 8\cos x - 2\cos y) \mathrm{d}_x \mathrm{d}_y$$
$$= \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log(10 - 8\cos x - 2\cos y) \mathrm{d}_x \mathrm{d}_y.$$

It is not difficult to see that

$$\frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log(10 - 8\cos x - 2\cos y) \mathrm{d}_x \mathrm{d}_y = \frac{2}{3\pi} \int_0^{\pi/2} \log\left(\cos x + \sqrt{4 + \cos^2 x}\right) \mathrm{d}_x.$$

The corollary has thus been proved. □

3.2. The cylinder $G_2(m, 2n)$

Let $G_2(m, 2n)^e$ be the orientation of $G_2(m, 2n)$ illustrated in Fig. 3(b). For $G_2(m, 2n)^e$, all quadrangles in the first column have the same orientation, all quadrangles in the second column have the inverse of the orientation of quadrangles in the first column, and so on. Obviously, $G_2(m, 2n)^e$ satisfies the conditions in Lemma 2.2 and hence is a Pfaffian orientation.

Theorem 3.6. *For the cylinder $G_2(m, 2n)$, the number of close-packed dimers of $G_2(m, 2n)$ can be expressed by*

$$M(G_2(m, 2n)) = \prod_{j=0}^{n-1} \left[\frac{\sqrt{9 + 16\beta_j^2} + 3}{2\sqrt{9 + 16\beta_j^2}} \left(\frac{5 + \sqrt{9 + 16\beta_j^2}}{2} \right)^m + \frac{\sqrt{9 + 16\beta_j^2} - 3}{2\sqrt{9 + 16\beta_j^2}} \left(\frac{5 - \sqrt{9 + 16\beta_j^2}}{2} \right)^m \right], \tag{8}$$

and the entropy, i.e., $\lim_{m,n\to\infty} \frac{2}{8mn} \log M(G_2(m, 2n))$, *equals*

$$\frac{1}{2\pi} \int_0^{\pi/2} \log\left[\frac{5 + \sqrt{25 - 16\cos^2\theta}}{2} \right] \mathrm{d}_\theta \approx 0.3770,$$

where $\beta_j = \cos\frac{j\pi}{n}$ *if n is odd and* $\beta_j = \cos\frac{(2j+1)\pi}{2n}$ *otherwise.*

Proof. We can prove easily the statement in the theorem on the entropy from (8). Hence it suffices to prove that (8) holds. For the orientation $G_2(m, 2n)^e$ of the cylinder $G_2(m, 2n)$ shown in Fig. 3(b). Let C_j denote the digraph illustrated in Fig. 4(c) for $0 \le j \le n - 1$. Similarly to the proof of Theorem 3.1, we can prove that

$$M(G_2(m, 2n)) = \prod_{j=0}^{n-1} |\det(F_j)|, \tag{9}$$

where F_j is the adjacency matrix of C_j.

Let C'_j be the digraph obtained from C_j by deleting vertex b_1^* (see Fig. 4(d)) and F'_j the adjacency matrix of C'_j.

For $j = 0, 1, \ldots, n - 1$, set

$$P_m(j) = \det(F_j), \qquad P'_m(j) = \det(F'_j),$$

where F_j (resp. F'_j) is the adjacency matrix of the digraph C_j (resp. C'_j) illustrated in Fig. 4(c) (resp. Fig. 4(d)). It is not difficult to prove that $\{P_m(j)\}_{m\ge0}$ and $\{P'_m(j)\}_{m\ge0}$ satisfy the following recurrences:

$$\begin{cases} P_{2m+1}(j) = 4P_{2m}(j) + 2\beta_j P'_{2m}(j), & P'_{2m+1}(j) = -2\beta_j P_{2m}(j) - P'_{2m}(j) \qquad m \ge 0, \\ P_{2m}(j) = 4P_{2m-1}(j) - 2\beta_j P'_{2m-1}(j), & P'_{2m}(j) = 2\beta_j P_{2m-1}(j) - P'_{2m-1}(j) \qquad m \ge 1, \\ P_0(j) = 1, \quad P'_0(j) = 0, \quad P_1(j) = 4, \quad P'_1(j) = -2\beta_j. \end{cases}$$

Then we have

$$\begin{pmatrix} P_{2m+1}(j) \\ P'_{2m+1}(j) \end{pmatrix} = \begin{pmatrix} 16 + 4\beta_j^2 & -10\beta_j \\ -10\beta_j & 4\beta_j^2 + 1 \end{pmatrix} \begin{pmatrix} P_{2m-1}(j) \\ P'_{2m-1}(j) \end{pmatrix} \tag{10}$$

and

$$\begin{pmatrix} P_{2m}(j) \\ P'_{2m}(j) \end{pmatrix} = \begin{pmatrix} 16+4\beta_j^2 & 10\beta_j \\ 10\beta_j & 4\beta_j^2+1 \end{pmatrix} \begin{pmatrix} P_{2(m-1)}(j) \\ P'_{2(m-1)}(j) \end{pmatrix}. \tag{11}$$

Let $a_m = P_{2m+1}(j)$ and $b_m = P_{2m}(j)$ for $j \geq 0$. Hence we have

$$\begin{cases} a_m = (18\beta_j^2+17)a_{m-1} - 16(1-\beta_j^2)^2 a_{m-2} & \text{for } m \geq 2, \\ b_m = (18\beta_j^2+17)b_{m-1} - 16(1-\beta_j^2)^2 b_{m-2} & \text{for } m \geq 2, \\ a_0 = 4, \quad a_1 = 64+36\beta_j^2, \quad b_0 = 1, \quad b_1 = 16+4\beta_j^2. \end{cases} \tag{12}$$

By solving the recurrences in (12), then we have

$$a_m = \prod_{j=0}^{n-1} \left[\frac{\sqrt{9+16\beta_j^2}+3}{2\sqrt{9+16\beta_j^2}} \left(\frac{5+\sqrt{9+16\beta_j^2}}{2} \right)^{2m+1} + \frac{\sqrt{9+16\beta_j^2}-3}{2\sqrt{9+16\beta_j^2}} \left(\frac{5-\sqrt{9+16\beta_j^2}}{2} \right)^{2m+1} \right],$$

and

$$b_m = \prod_{j=0}^{n-1} \left[\frac{\sqrt{9+16\beta_j^2}+3}{2\sqrt{9+16\beta_j^2}} \left(\frac{5+\sqrt{9+16\beta_j^2}}{2} \right)^{2m} + \frac{\sqrt{9+16\beta_j^2}-3}{2\sqrt{9+16\beta_j^2}} \left(\frac{5-\sqrt{9+16\beta_j^2}}{2} \right)^{2m} \right].$$

Hence (8) has been proved and the theorem follows. □

From (1) and Theorem 3.6, the 8.8.4 lattices with cylindrical and toroidal boundary conditions have the same entropy which is approximately 0.3770.

3.3. The cylinder $H^c(n, m)$

Note that the hexagonal lattice $H^c(n, m)$ with cylindrical boundary condition is a finite bipartite graph whose vertex set can be colored by two colors black and white (see Fig. 2(b)). One can see that no edge intersected by the cut segment l_i ($i = 1, 2, \ldots, m$) illustrated in Fig. 2(b) can belong to one close-packed dimers, since the the upper (or bottom) vertices of edges intersected by the cut segment l_i have the same color. Thus the number of close-packed dimers of $H^c(n, m)$ equals the number of close-packed dimers of $m + 1$ cycles with $2n + 2$ vertices. Hence we have the following:

Theorem 3.7. *The number of close-packed dimers of $H^c(n, m)$ can be expressed by*

$$M(H^c(n, m)) = 2^{m+1}$$

and the entropy equals zero.

From (2) and the above theorem, we have the following:

Remark 3.8. The hexagonal lattices $H^t(n, m)$ and $H^c(n, m)$ with toroidal and cylindrical boundary conditions have different entropies. That is, for the hexagonal lattices, the entropy is dependent on boundary conditions.

4. Concluding remarks

In statistical mechanics some examples implied that the thermodynamic limit of the free energy (including the entropy) is independent of boundary conditions [9]. Kasteleyn [14] discussed the related problem of the $m \times n$ quadratic lattices with the free and toroidal boundary conditions. The results by Wu [37,22] and by Wu and Wang [36] also imply that the kagome lattices with toroidal and cylindrical boundary conditions have the same entropy. In this paper, we computed the entropies of the 8.8.6, 8.8.4, and hexagonal lattices with cylindrical boundary condition and the entropy of the 8.8.6 lattice with toroidal boundary condition. We showed that the 8.8.6 lattices with the cylindrical and toroidal boundary conditions have the same entropy. Comparing with the result by Salinas and Nagle [32] and Wu [37] we can see that the 8.8.4 lattices have the same property. But, for the hexagonal lattices, the entropy is dependent on the boundary conditions. Based on these results, it is natural to pose the following

Problem 4.1. Let $G^c(m, n)$ and $G^t(m, n)$ be the lattices with the cylindrical and toroidal boundary conditions, where their fundamental domain is a plane bipartite graph G with close-packed dimers. Under which conditions do $G^c(m, n)$ and $G^t(m, n)$ have the same entropy?

In Refs. [11–13] the finite-size corrections for the dimer model on the square and triangular lattices have been studied. It is an interesting problem to study the finite-size corrections of the dimer model on 8.8.6 and 8.8.4 lattices and compare the results with the present paper.

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References

[1] G.R. Allen, Dimer models for the antiferroelectric transition in copper formate tetrahydrate, J. Chem. Phys. 60 (1974) 3299–3309.
[2] M. Ciucu, Enumeration of perfect matchings in graphs with reflective symmetry, J. Combin. Theory Ser. A 77 (1997) 67–97.
[3] H. Cohn, N. Elkies, J. Propp, Local statistics for random domino tilings of the Aztec diamond, Duke Math. J. 85 (1996) 117–166.
[4] H. Cohn, R. Kenyon, J. Propp, A variational principle for domino tilings, J. Amer. Math. Soc. 14 (2001) 297–346.
[5] P.J. Davis, Circulant Matrices, Wiley, New York, 1979.
[6] N. Elkies, G. Kuperberg, M. Larsen, J. Propp, Alternating sign matrices and domino tilings, J. Alg. Combin. 1 (1992) 111–132 and 219–234.
[7] V. Elser, Solution of the dimer problem on a hexagonal lattice with boundary, J. Phys. A 17 (1984) 1509–1513.
[8] M.E. Fisher, Statistical mechanics of dimers on a plane lattice, Phys. Rev. 124 (1961) 1664–1672.
[9] M.E. Fisher, J.L. Lebowitz, Asymptotic free energy of a system with periodic boundary conditions, Commun. Math. Phys. 19 (1970) 251–272.
[10] I. Gutman, S.J. Cyvin, Kekulé Structures in Benzenoid Hydrocarbons, Springer, Berlin, 1988.
[11] N.Sh. Izmailian, K.B. Oganesyan, C.-K. Hu, Exact finite-size corrections of the free energy for the square lattice dimer model under different boundary conditions, Phys. Rev. E 67 (2003) 066114.
[12] N.Sh. Izmailian, K.B. Oganesyan, M.-C. Wu, C.-K. Hu, Finite-size corrections and scaling for the triangular lattice dimer model with periodic boundary conditions, Phys. Rev. E 73 (2006) 016128.
[13] N.Sh. Izmailian, V.B. Priezzhev, P. Ruelle, C.-K. Hu, Logarithmic conformal field theory and boundary effects in the Dimer model, Phys. Rev. Lett. 95 (2005) 260602.
[14] P.W. Kasteleyn, The statistics of dimers on a lattice I: The number of dimer arrangements on a quadratic lattice, Physica 27 (1961) 1209–1225.
[15] P.W. Kasteleyn, Dimer statistics and phase transitions, J. Math. Phys. 4 (1963) 287–293.
[16] P.W. Kasteleyn, Graph theory and crystal physics, in: F. Harary (Ed.), Graph Theory and Theoretical Physics, Academic Press, 1967, pp. 43–110.
[17] R. Kenyon, A. Okounkov, S. Sheffield, Dimers and amoebae, Ann. Math. 163 (2006) 1019–1056.
[18] V. Korepin, P. Zinn-Justin, Thermodynamic limit of the six-vertex model with domain wall boundary conditions, J. Phys. A 33 (2000) 7053.
[19] E.H. Kuo, Applications of graphical condensation for enumerating matchings and tilings, Theoret. Comput. Sci. 319 (2004) 29–57.
[20] C.H.C. Little, A characterization of convertible (0, 1)-matrices, J. Combin. Theory Ser. B 18 (1975) 187–208.
[21] L. Lovász, M.D. Plummer, Matching Theory, Elsevier, Amsterdam, New York, 1986.
[22] W.T. Lu, F.Y. Wu, Physica A 258 (1998) 157–170.
[23] W.T. Lu, F.Y. Wu, Dimer statistics on the Mobius strip and the Klein bottle, Phys. Lett. A 259 (1999) 108–114.
[24] W. McCuaig, Polyá's permanent problem, Electron. J. Combin. 11 (2004) R79.
[25] W. McCuaig, N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations, and even directed circuits (Extended abstract), in: Proc. Symp. on the Theory of Computing, STOC'97, 1997.
[26] M. McCoy, T.T. Wu, The Two-dimensional Ising Model, Harvard University Press, Cambridge, 1973.
[27] J.F. Nagle, C.S.O. Yokoi, S.M. Bhattacharjee, Dimer models on anisotropic lattices, in: C. Domb, J.L. Lebowitz (Eds.), Phase Transitions and Critical Phenomena, vol. 13, Academic Press, New York, 1989.
[28] J. Propp, Generalized domino-shuffling, Theoret. Comput. Sci. 303 (2003) 267–301.
[29] J. Propp, Enumerations of matchings: Problems and progress, in: New Perspectives in Geometric Combinatorics, in: MSRI Publications, vol. 38, Cambridge University Press, Cambridge, UK, 1999, pp. 255–291.
[30] N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, Ann. Math. 150 (1999) 929–975.
[31] H. Sachs, H. Zeritz, Remark on the dimer problem, Discrete Appl. Math. 51 (1994) 171.
[32] S.R. Salinas, J.F. Nagle, Theory of the phase transition in the layered hydrogen-bonded $SnCl^2 \cdot 2H_2O$ crystal, Phys. Rev. B 9 (1974) 4920–4931.
[33] J.R. Stembridge, Nonintersecting paths, pfaffians, and plane partitions, Adv. Math. 38 (1990) 96–131.
[34] H.N.V. Temperley, M.E. Fisher, Dimer problem in statistical mechanics—An exact result, Philos. Magazine 6 (1961) 1061–1063.
[35] R. Thomas, A Survey of Pfaffian Orientations of Graphs, ICM, 2006.
[36] F.Y. Wu, F. Wang, Dimers on the kagome lattice I: Finite lattices, Physica A 387 (2008) 4148–4156.
[37] F.Y. Wu, Dimers on two-dimensional lattices, Internat. J. Modern Phys. B 20 (2006) 5357–5371.
[38] F.Y. Wu, Exactly soluble model of the ferroelectric phase transition in two dimensions, Phys. Rev. Lett. 18 (1967) 605.
[39] F.Y. Wu, Remarks on the modified potassium dihydrogen phosphate model of a ferroelectric, Phys. Rev. 168 (1968) 539–543.
[40] F.Y. Wu, X.N. Wu, H.W.J. Blóte, Critical frontier of the antiferromagnetic Ising model in a magnetic field: The honeycomb lattice, Phys. Rev. Lett. 62 (1989) 2773–2776.
[41] W.G. Yan, F.J. Zhang, Enumeration of perfect matchings of graphs with reflective symmetry by Pfaffians, Adv. Appl. Math. 32 (2004) 655–668.
[42] W.G. Yan, F.J. Zhang, Graphical condensation for enumerating perfect matchings, J. Combin. Theory Ser. A 110 (2005) 113–125.
[43] W.G. Yan, F.J. Zhang, Enumeration of perfect matchings of a type of Cartesian products of graphs, Disrete Appl. Math. 154 (2006) 145–157.