Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **4** (2009), No. 3, pp. 299-332

GENERALIZATIONS OF CHUNG-FELLER THEOREMS

BY

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Abstract

In this paper, we develop a method to find Chung-Feller extensions for three kinds of different rooted lattice paths and prove Chung-Feller theorems for such lattice paths. In particular, we compute a generating function $S(z)$ of a sequence formed by rooted lattice paths. We give combinatorial interpretations to the function of Chung-Feller type $\frac{S(z)-yS(yz)}{1-y}$ for the generating function $S(z)$. Using our method, we first prove Chung-Feller theorems of up-down type for three kinds of rooted lattice paths. Our results are generalizations of the classical Chung-Feller theorem of up-down type for Dyck paths. We also find Motzkin paths have Chung-Feller properties of up-down type. Then we prove Chung-Feller theorems of left-right type for two among three kinds of rooted lattice paths. Chung-Feller theorem of left-right type for Motzkin paths is a special case of our theorems. We also show that Dyck paths have Chung-Feller phenomenons of left-right type. By the main theorems in this paper, many new Chung-Feller theorems for rooted lattice paths are derived.

1. Introduction

Let $\mathbb{Z}$ denote the set of integers and $[m] := \{1, 2, \ldots, m\}$. An m-Dyck path D in the plane $\mathbb{Z}\times\mathbb{Z}$ is a sequence of vectors $(x_1, y_1)(x_2, y_2)\cdots(x_{2m}, y_{2m})$ in the set $\{(1,-1), (1,1)\}$ such that $\sum_{i=1}^{2m} y_i = 0$. We call vectors $(1,1)$ and $(1,-1)$ up- and down-steps respectively. Then the path D has $2m$ steps.

Received and Communicated by T. P. Liu April 17, 2009.

Key words and phrases: Chung-Feller Theorem, Dyck path, Motzkin path, Rooted lattice path.

[1]Partially supported by NSC 96-2115-M-001-005.

Clearly, $\sum_{i=1}^{2m} x_i = 2m$. We say that the *semilength* of the path D is m. Let $a_0 = 0$, $b_0 = 0$, $a_i = \sum_{j=1}^{i} x_j$ and $b_i = \sum_{j=1}^{i} y_j$ for every $i \in [2m]$. Then the m-Dyck path D is a sequence of points $(a_0, b_0)(a_1, b_1) \cdots (a_{2m}, b_{2m})$ in the plane $\mathbb{Z} \times \mathbb{Z}$. For every $i \in [2m]$, we say that a step (x_i, y_i) in the path D is nonpositive if $b_i \leq 0$. A Catalan path of semilength m is an m-Dyck path which has no nonpositive up-steps. The number of such paths is the m-th Catalan number $c_m = \frac{1}{m+1}\binom{2m}{m}$. The generating function $C(z) := \sum_{m\geq 0} c_m z^m$ satisfies the functional equation $C(z) = 1 + zC(z)^2$ and $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ explicitly. We state the classical Chung-Feller theorem [2] as follows.

For every $0 \leq r \leq m$, the number of m-Dyck paths with r nonpositive up-steps is equal to c_m and independent on r.

Since the line $y = 0$ partitions a Dyck path into two parts of up and down, we say that the classical Chung-Feller theorem is of *up-down type*. The classical Chung-Feller Theorem was proved by MacMahon [8]. Chung and Feller reproved this theorem by using analytic method in [2]. Narayana [10] showed the Chung-Feller Theorem by combinatorial methods. Mohanty's book [9] devotes an entire section to exploring the Chung-Feller theorem. Eu, Liu and Yeh [3] proved the Chung-Feller Theorem by using the Taylor expansions of generating functions and gave a refinement of this theorem. In [4], Eu, Fu and Yeh gave a strengthening of the Chung-Feller Theorem and a weighted version for Schröder paths. Chen [1] revisited the Chung-Feller Theorem by establishing a bijection.

An m-Motzkin path M is a sequence of vectors $(x_1, y_1)(x_2, y_2) \cdots (x_{m+1}, y_{m+1})$ in the set $\{(1, -1), (1, 1), (1, 0)\}$ such that $\sum_{i=1}^{m+1} y_i = 1$. We call the vector $(1, 0)$ level-step. Thus, the path M has $m+1$ steps. We also call $m+1$ the length of M since $\sum_{i=1}^{m+1} x_i = m + 1$. Let $a_0 = 0$, $b_0 = 0$, We view the m-Motzkin path M as a sequence of points $(a_0, b_0)(a_1, b_1) \cdots (a_{m+1}, b_{m+1})$ in the plane $\mathbb{Z} \times \mathbb{Z}$, where $a_i = \sum_{j=1}^{i} x_j$ and $b_i = \sum_{j=1}^{i} y_j$ for every $i \in [m+1]$. A minimum point (a_i, b_i) is a point in the path M such that $b_i \leq b_j$ for all $j \neq i$. A rightmost minimum point (a_i, b_i) is a minimum point such that $i > j$ if (a_j, b_j) also is a minimum point and $j \neq i$. We suppose that the rightmost minimum point of the Motzkin path M is (a_i, b_i) for some $i \in \{0, 1, \ldots, m\}$. We say that a step (x_j, y_j) in the path M is left if $j \leq i$. An m-Motzkin path M is called a positive-Motzkin path of length $m + 1$ if it has no left steps.

The number of positive-Motzkin paths of lenth $m+1$ is the m-th Motzkin number, denoted by e_m. The generating function $M(z) := \sum_{m\geq 0} e_m z^m$ satisfies $M(z) = 1+zM(z)+z^2M(z)^2$ and explicitly $M(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$. Shapiro [12] found the following Chung-Feller theorem for Motzkin paths.

For every $0 \leq r \leq m$, the number of m-Motzkin paths M with r left steps is equal to e_m and independent on r.

The Chung-Feller theorem for Motzkin paths was investigated in [3]. For an m-Motzkin path, its rightmost minimum point partitions itself into two parts of left and right. Hence, We say that the Chung-Feller theorem for Motakin paths is of *left-right type.*

1.1. Chung-Feller theorems of up-down type and left-right type

There are the following two interesting problems.

- Is there a Chung-Feller theorem of left-right type for Dyck paths?
- Is there a Chung-Feller theorem of up-down type for Motzkin paths?

Let us check the following two examples.

Example 1.1. Given an m-Dyck path $D = (x_1, y_1)(x_2, y_2)\cdots(x_{2m}, y_{2m})$, we note that the m-Dyck path can be viewed a sequence of points $(a_0, b_0)(a_1, b_1)\cdots(a_{2m}, b_{2m})$ in the plane $\mathbb{Z}\times\mathbb{Z}$ where $a_0 = 0$, $b_0 = 0$, $a_i = \sum_{j=1}^{i} x_j$ and $b_i = \sum_{j=1}^{i} y_j$ for every $i \in [2m]$. We can define the rightmost minimum

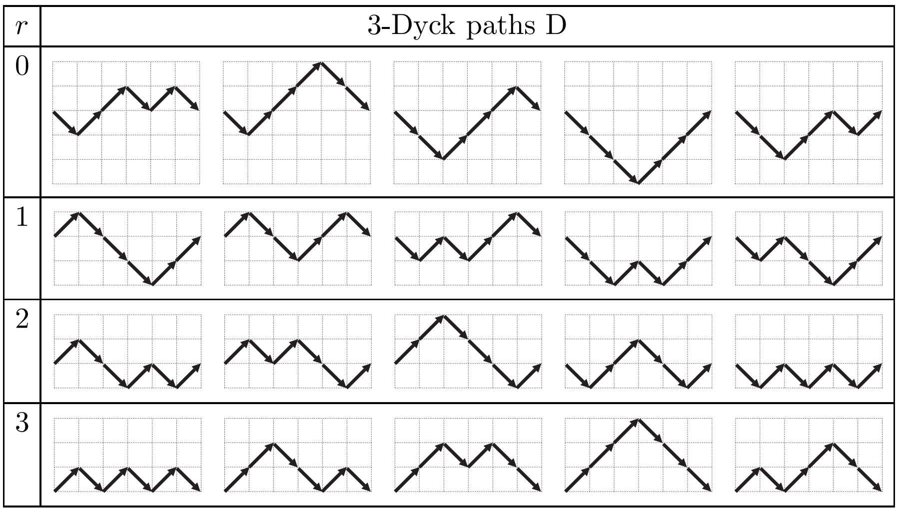

Figure 1. All the 3-Dyck paths.

point and left steps of a Dyck path as those for Motzkin paths. We draw all the 3-Dyck paths with r left up-steps in Figure 1.

We observe that the number of 3-Dyck paths D with r left up-steps is 5 and independent on r for $r = 0, 1, 2, 3$.

Example 1.2. Given an m-Motzkin path $M = (x_1, y_1)(x_2, y_2) \cdots (x_{m+1}, y_{m+1})$, we can define nonpositive steps of a Motzkin path as those for Dyck paths. We draw all the 3-Motzkin paths with r nonpositive steps as follows.

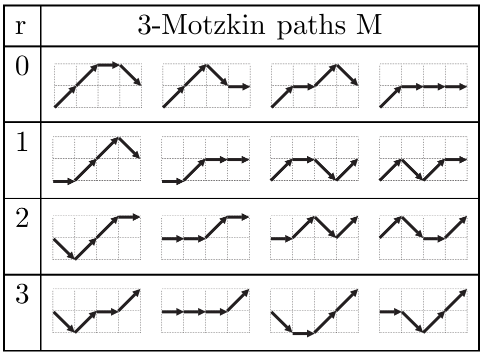

Figure 2. All the 3-Motzkin paths

We observe that the number of 3-Motzkin paths M with r nonpositive steps is 4 and independent on r for $r = 0, 1, 2, 3$.

In this paper, we focus on three kinds of different rooted lattice paths. In the Section 3, we prove the Chung-Feller theorems of not only up-down type but also left-right type for the first kind of rooted lattice paths. The results about schröder paths in [4] is a special case of our theorems. As applications of our main theorems in this section, we reprove the classical Chung-Feller theorem of up-down type for Dyck paths. We also find a Chung-Feller theorem of left-right type for Dyck paths. In the Section 4, we prove Chung-Feller theorems of not only up-down type but also left-right type for the second kind of rooted lattice paths. As applications of our main results in this section, we reprove the Chung-Feller theorem of left-right type for Motzking paths. We also find a Chung-Feller theorem of up-down type for Motzkin paths. Since it is tedious to state Chung-Feller theorems of left-right type for the third kind of lattice paths, we only prove Chung-Feller theorems of up-down type for the third kind of lattice paths in the Section 5.

1.2. Chung-Feller extension

In general, let $\mathscr{S}$ be a set of some combinatorial structures S. We call the set $\mathscr{S}$ a combinatorial model. Let θ be a mapping from the set $\mathscr{S}$ to the set $\mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. Let $\bar{\mathscr{S}}$ be a combinatorial model as well. Let $\bar{\theta}$ and $\bar{\lambda}$ be two mappings from the set $\bar{\mathscr{S}}$ to the set $\mathbb{N}$ such that $0 \le \bar{\lambda}(S) \le \bar{\theta}(S)$ for every $S \in \bar{\mathscr{S}}$.

Definition 1.3. $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ is a Chung-Feller extension for $(\mathscr{S}, \theta)$ if the number of combinatorial structures S in the combinatorial model $\bar{\mathscr{S}}$ such that $\bar{\theta}(S) = m$ and $\bar{\lambda}(S) = r$ is equal to the number of combinatorial structures S in the combinatorial model $\mathscr{S}$ such that $\theta(S) = m$ and independent on r for all $r = 0, 1, \ldots, m$. We say that $\bar{\lambda}$ is a Chung-Feller parameter for $(\bar{\mathscr{S}}, \bar{\theta})$.

We give some examples for Chung-Feller extensions.

Example 1.4. Let $\mathscr{S}$ be the set of all the Catalan paths in the plane $\mathbb{Z} \times \mathbb{Z}$. For every $S \in \mathscr{S}$, let $\theta(S)$ be the semi-length of the catalan path S. Define D_m as the set of all the m-Dyck paths. Let $\bar{\mathscr{S}} = \bigcup_{m \ge 0} D_m$. For every $S \in \bar{\mathscr{S}}$, let $\bar{\theta}(S)$ be the semi-length of a Dyck path S and $\bar{\lambda}(S)$ denote the number of nonpositive up-steps in a Dyck path S. By the classical Chung-Feller theorem of up-down type for Dyck paths, $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ is a Chung-Feller extension of up-down type for $(\mathscr{S}, \theta)$.

Example 1.5. Let $\mathscr{S}$ be the set of all the positive-Motzkin paths in the plane $\mathbb{Z} \times \mathbb{Z}$. For every $S \in \mathscr{S}$, let $\theta(S)$ be the length of S. Define B_m as the set of all the m-Motzkin paths. Let $\bar{\mathscr{S}} = \bigcup_{m \ge 0} B_m$. For every $S \in \bar{\mathscr{S}}$, let $\bar{\theta}(S)$ be the length of a Motzkin path S and $\bar{\lambda}(S)$ denote the number of left steps in a Motzkin path S. By the Chung-Feller theorem of left-right for Motzkin paths, $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ is a Chung-Feller extension of left-right type for $(\mathscr{S}, \theta)$.

Example 1.6. There are m drivers which are labeled by $\{1, 2, \ldots, m\}$ and $m + 1$ parking spaces which are arranged in a cycle and labeled by $\{0, 1, \ldots, m\}$ clockwise. Each driver i has an initial parking preference a_i. We call such a sequence $S = (a_1, \ldots, a_m)$ a preference function of length m. Driver enter the parking area in the order in which they are labeled.

Each driver proceeds to his preferential parking space and parks there if it is free, or moves clockwise to the next unoccupied parking space and parks there. Every preference function S leaves one parking space unoccupied. We denote this unoccupied parking space by $\bar{\lambda}(S)$.

A preference functions S of length m is a parking function of length m if $\bar{\lambda}(S) = 0$. Let $\mathscr{S}_m$ be the set of parking functions of length m and let $\mathscr{S} = \bigcup_{m\geq 0} \mathscr{S}_m$. Let θ be a mapping from $\mathscr{S}$ to $\mathbb{N}$ such that $\theta(S)$ is the length of parking function S.

Let $\bar{\mathscr{S}}_m$ be the set of preference functions of length m and let $\bar{\mathscr{S}} = \bigcup_{m\geq 0} \bar{\mathscr{S}}_m$. For every $S \in \bar{\mathscr{S}}$, let $\bar{\theta}(S)$ be the length of S. Riordan [11] proved that the number of preference functions S in $\bar{\mathscr{S}}$ with $\bar{\theta}(S) = m$ and $\bar{\lambda}(S) = r$ is equal to the number of parking functions S of length m in $\mathscr{S}$ and independent on r. Hence, $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ is a Chung-Feller extension of $(\mathscr{S}, \theta)$.

Example 1.7. Let $\mathcal{S} = \{(1, 1), (1, -1), (5, -1)\}$ be a set of vectors, n and m two positive integers. An $(\mathcal{S}, m, n)$-lattice path is a sequence of vectors $(x_1, y_1)(x_2, y_2)\cdots(x_n, y_n)$ in the set $\mathcal{S}$ such that $\sum_{i=1}^{n} y_i = 0$ and $\sum_{i=1}^{n} x_i = 2m$. We call m the semi-length of this lattice path. An $(\mathcal{S}, m, n)$-nonnegative lattice path is an $(\mathcal{S}, m, n)$-lattice path such that $\sum_{j=1}^{i} y_j \geq 0$ for all $i \in [n]$. Define $\mathscr{S}_{\mathcal{S},m,n}$ as the set of all the $(\mathcal{S}, m, n)$-nonnegative lattice paths. We consider the set $\mathscr{S} = \mathscr{S}_{\mathcal{S}} = \bigcup_{m\geq 0, n\geq 0} \mathscr{S}_{\mathcal{S},m,n}$. For every lattice path $L \in \mathscr{S}$, let $\theta(L)$ be the semi-length of L. There are exactly 6 lattice paths with semi-length 3 in the set $\mathscr{S}$. We draw them as follows.

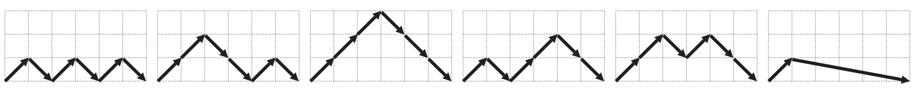

Figure 3. All the lattice paths with semi-length 3 in the set $\mathscr{S}$.

Define $\bar{\mathscr{S}}_{\mathcal{S},m,n}$ as the set of all the $(\mathcal{S}, m, n)$-lattice paths. Let $\bar{\mathscr{S}} = \bar{\mathscr{S}}_{\mathcal{S}} = \bigcup_{m\geq 0, n\geq 0} \bar{\mathscr{S}}_{\mathcal{S},m,n}$. For every lattice path $L \in \bar{\mathscr{S}}$, let $\bar{\theta}(L)$ be the semi-length of L. There are exactly 22 lattice paths with semi-length 3 in the set $\bar{\mathscr{S}}$. We draw them in Figure 4.

Clearly, there are no mappings $\bar{\lambda}$ from $\bar{\mathscr{S}}$ to $\mathbb{N}$ such that $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ is a Chung-Feller extension for $(\mathscr{S}, \theta)$ since 22 can not be divided by 6. What

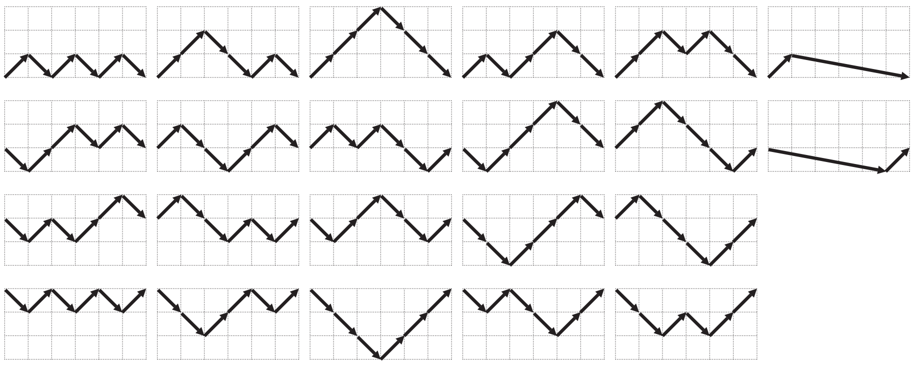

Figure 4. All the lattice paths with semi-length 3 in the set $\bar{\mathscr{S}}$.

are Chung-Feller extensions for $(\mathscr{S}, \theta)$? In the Section 3, we will give Chung-Feller extensions of not only up-down type but also left-right type for $(\mathscr{S}, \theta)$.

We are given a combinatorial model $\mathscr{S}$ and a mapping θ from $\mathscr{S}$ to $\mathbb{N}$. Some natural problems arise.

- Is there a Chung-Feller extension $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ for $(\mathscr{S}, \theta)$?
- How to find it if there is a Chung-Feller extesion $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ for $(\mathscr{S}, \theta)$?
- Suppose $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ is a Chung-Feller extension for $(\mathscr{S}, \theta)$. Fix $\bar{\mathscr{S}}$ and $\bar{\theta}$. How many Chung-Feller parameters are there for $(\bar{\mathscr{S}}, \bar{\theta})$?

Given a combinatorial model $\mathscr{S}$ and a mapping θ from $\mathscr{S}$ to $\mathbb{N}$, we define a generating function $S(z) = \sum_{S \in \mathscr{S}} z^{\theta(S)}$. For any $m \geq 0$, let s_m be the number of combinatorial structures S in $\mathscr{S}$ with $\theta(S) = m$. Then $S(z) = \sum_{m \geq 0} s_m z^m$. We easily obtain a bivariate function $\frac{S(z)-yS(yz)}{1-y}$ from $S(z)$. Given a combinatorial model $\bar{\mathscr{S}}$, two mappings $\bar{\theta}$ and $\bar{\lambda}$ from $\bar{\mathscr{S}}$ to $\mathbb{N}$, we define a generating function $\bar{S}(y, z) = \sum_{S \in \bar{\mathscr{S}}} y^{\bar{\lambda}(S)} z^{\bar{\theta}(S)}$. Let $s_{m,r}$ be the number of combinatorial structures S in $\bar{\mathscr{S}}$ with $\bar{\theta}(S) = m$ and $\bar{\lambda}(S) = r$. Then $\bar{S}(y, z) = \sum_{m \geq 0} \sum_{r=0}^{m} s_{m,r} y^r z^m$. It is easy to see that a necessary condition for $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ to be a Chung-Feller extension for $(\mathscr{S}, \theta)$ is $\bar{S}(y, z) = \frac{S(z)-yS(yz)}{1-y}$. In [5], Liu, Wang and Yeh gave the notion of the function of Chung-Feller type for a generating function $S(z)$.

Definition 1.8. Let $S(z)$ be a generating function of a sequence $(s_0, s_1, \ldots)$. We call the following bivariate function

$$\frac{S(z) - yS(yz)}{1 - y},$$

denoted by $CS(y,z)$, the function of Chung-Feller type for $S(z)$.

Example 1.9. We consider lattice paths in Example 1.7. Let $\mathscr{S}$ and θ be defined as those in Example 1.7. Let s_m be the number of lattice paths of length $2m$ in $\mathscr{S}$. Define a generating function $S(z)=\sum_{m\geq 0} s_m z^m$. Simple computations tell us that $S(z)$ satisfies the functional equation $S(z)=1+(z+z^3)[S(z)]^2$ and $S(z)=\frac{1-\sqrt{1-4z-4z^3}}{2(z+z^3)}$. We easily obtain the bivariate function of Chung-Feller type $CS(y,z)=\frac{S(z)-yS(yz)}{1-y}$ for the generating function $S(z)$.

The function of Chung-Feller type $CS(y,z)$ for $S(z)$ implies us that it is possible to find a Chung-Feller extension $(\bar{\mathscr{S}},\bar{\theta},\bar{\lambda})$ for $(\mathscr{S},\theta)$. The key is to give a combinatorial interpretation for the function $CS(y,z)$. Liu, Wang and Yeh [5] attempted to do this for some functions of Chung-Feller type. In the sections 3,4 and 5, we focus on three different combinatorial models $\mathscr{S}$ formed by combinatorial structures "lattice paths". By the function of Chung-Feller type for a generating function, we develope a method to find Chung-Feller extesions for $(\mathscr{S},\theta)$. Particularly, we consider a mapping θ from $\mathscr{S}$ to $\mathbb{N}$. Define a generating function $S(x)=\sum_{S\in\mathscr{S}} z^{\theta(S)}$. We first find a functional equation which $S(z)$ satisfies. Then we study the function of Chung-Feller type $CS(y,z)$ for $S(z)$ and give it a combinatorial interpretation. Thus, we can find a Chung-Feller extension $(\bar{\mathscr{S}},\bar{\theta},\bar{\lambda})$ for $(\mathscr{S},\theta)$ and prove a Chung-Feller theorem for $(\bar{\mathscr{S}},\bar{\theta},\bar{\lambda})$.

1.3. New Chung-Feller theorems

Narayana [10] related cycle permutations of lattice paths to the Chung-Feller theorem. Mohanty's book [9] devotes an entire section to exploring the Chung-Feller theorem. We are interested in the Theorem 2 in the page 70 of the book. Many Chung-Feller theorems are consequences of this theorem. In Section 6, we investigate relations between our main results and this theorem. We find that the main results of this paper can not be derived directly as special cases of this theorem. Hence, by the main theorems of this paper, many new Chung-Feller theorems for rooted lattice paths are derived. We also mention the notion of incomplete Chung-Feller phenomenents and give some examples.

2. Rooted Lattice Paths

In this section, we introduce the notion of rooted lattice paths. Let $\mathcal{S}$ be a set of vectors in $\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$. Let k, n and m be three integers with $n\geq 1$ and $m\geq 1$. An $(\mathcal{S},m,k,n)$-*lattice path* L is a sequence of vectors $(x_1,y_1)(x_2,y_2)\cdots(x_n,y_n)$ such that :

- $(x_i,y_i)\in\mathcal{S}$;
- $\sum_{i=1}^{n} y_i = k$;
- $\sum_{i=1}^{n} x_i = m$.

$\mathcal{S}$ is called the *step set* and each vector in $\mathcal{S}$ is called a step. Then the lattice path L contains n steps. We say that the path L is of *order* n and *size* m. Let $\mathscr{L}_{\mathcal{S},m,k,n}$ be the set of all the $(\mathcal{S},m,k,n)$-lattice paths and let $\mathscr{L}_{\mathcal{S},k}=\bigcup_{m\geq 0,n\geq 0}\mathscr{L}_{\mathcal{S},m,k,n}$.

Let w and l be two mappings from $\mathcal{S}$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We say that w and l are the weight function and the length function of $\mathcal{S}$, respectively. For any $(x,y)\in\mathcal{S}$, $w(x,y)$ and $l(x,y)$ are called the weight and the length of the step (x,y), respectively. For any $L=(x_1,y_1)(x_2,y_2)\cdots(x_n,y_n)\in\mathscr{L}_{\mathcal{S},m,k,n}$, define the *weight* of the path L, denoted by $w(L)$, as $w(L)=\prod_{j=1}^{n} w(x_j,y_j)$; define the *length* of the path L, denoted by $\theta(L)$, as $\theta(L)=\sum_{j=1}^{n} l(x_j,y_j)$.

We can view the lattice path L as the sequence of points in the plane $\mathbb{Z}\times\mathbb{Z}$

$$(0,0)=(a_0,b_0),(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n),$$

where $a_j=\sum_{i=1}^{j} x_i$ and $b_j=\sum_{i=1}^{j} y_i$ for any $i\in[n]$. For any a step (x_j,y_j) in the path L, we define the height of the step (x_j,y_j), denoted by $h(x_j,y_j)$, as $h(x_j,y_j)=b_j$. Let $NP(L)$ be a subset of $[n]$ such that $NP(L)=\{j\mid b_j\leq 0\}$. Define the *non-positive length* of L, denoted by $npl(L)$, as $npl(L)=\sum_{j\in NP(L)} l(x_j,y_j)$.

A minimum point is a point (a_i,b_i) in the path L such that $b_i\leq b_j$ for all $j\neq i$. A rightmost minimum point is a minimum point (a_i,b_i) such that $j < i$ if (a_j, b_j) also is a minimum point in the path L and $j \neq i$. Finally, suppose (a_i, b_i) is the rightmost minimum point of the path L. We define the *rightmost minimum length* of the path L, denoted by $rml(L)$, as $rml(L) = \sum_{1 \le j \le i} l(x_j, y_j)$.

Example 2.1. Let $\mathcal{S} = \{(1,1), (1,0), (5,-1), (1,-1)\}$, $m = 16$, $k = 0$, $n = 8$, $l(1,1) = 1$, $l(1,-1) = 1$, $l(1,0) = 1$ and $l(5,-1) = 5$. Let $L = (1,0)(1,-1)(1,1)(5,-1)(1,1)(1,0)(1,1)(5,-1)$. We draw the $(\mathcal{S}, 16, 0, 8)$-lattice path L as follows,

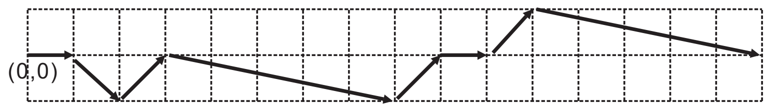

Figure 5. A $(\mathcal{S}, m, k, n)$-lattice path L.

It is easy to see that the length of the path L is 16, i.e., $\theta(L) = 16$. We have $NP(L) = \{1,2,3,4,5,6,8\}$. Hence, the non-positive length of the path L is 15, i.e., $npl(L) = 15$. The points $(2,-1)$ and $(8,-1)$ are the minimum points of the path L. The point $(8,-1)$ is the rightmost minimum point of the path L. Thus, the rightmost minimum length of the path L is 8, i.e., $rml(L) = 8$.

An $(\mathcal{S}, m, k, n)$-*nonnegative path* is an $(\mathcal{S}, m, k, n)$-lattice path which never goes below the line $y = k$. Let $\mathscr{N}_{\mathcal{S},m,k,n}$ be the set of all the $(\mathcal{S}, m, k, n)$-nonnegative paths and $\mathscr{N}_{\mathcal{S},k} = \bigcup_{m \ge 0, n \ge 0} \mathscr{N}_{\mathcal{S},m,k,n}$.

We define rooted lattice paths as follows.

Definition 2.2. Let k be an integer. Let n and m be two positive integers. Let $\mathcal{S}$ be a set of vectors in $\mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$. Let l be the length function of $\mathcal{S}$. A rooted $(\mathcal{S}, m, k, n)$-lattice path is a pair $[L; j]$ such that:

(a) $L = (x_1, y_1) \cdots (x_n, y_n)$ is a $(\mathcal{S}, m, k, n)$-lattice path;

(b) $0 \le j \le l(x_n, y_n) - 1$.

We let the point $(m - j, 0)$ the root of $\bar{L}$. Given a $\bar{L} = [L; j] \in \bar{\mathscr{L}}_{\mathcal{S},k}$, we define the *non-positive root length* of $\bar{L}$, denoted by $nprl(\bar{L})$, as $nprl(\bar{L}) = npl(L) + j$ and the *rightmost minimum root length* of $\bar{L}$, denoted by $rmrl(\bar{L})$, as $rmrl(\bar{L}) = rml(L) + j$.

Example 2.3. We consider the lattice path L given in Example 2.1. We draw a rooted $(\mathcal{S}, 16, 0, 8)$-lattice path $\bar{L} = [L; 2]$ as follows,

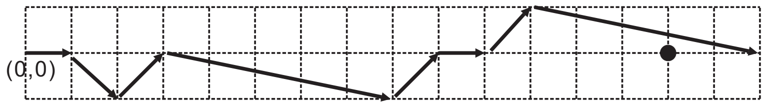

Figure 6. A rooted $(\mathcal{S}, m, k, n)$-lattice path $\bar{L}$.

where the point $(14, 0)$ is the root of L since $m = 16$ and $j = 2$, denoted by the notation "$\bullet$". It is easy to see that $nprl(\bar{L}) = 17$ and $rmrl(\bar{L}) = 10$.

We use $\bar{\mathscr{L}}_{\mathcal{S},m,k,n}$ to denote the set of rooted $(\mathcal{S}, m, k, n)$-lattice paths and $\bar{\mathscr{L}}_{\mathcal{S},k} = \bigcup_{m\geq 0, n\geq 0} \bar{\mathscr{L}}_{\mathcal{S},m,k,n}$.

We define rooted $(\mathcal{S}, m, k, n)$-nonnegative paths as follows.

Definition 2.4. A rooted $(\mathcal{S}, m, k, n)$-nonnegative path is a rooted $(\mathcal{S}, m, k, n)$-lattice path which never goes below the line $y = k$.

Let $\bar{\mathscr{N}}_{\mathcal{S},m,k,n}$ be the set of all the rooted $(\mathcal{S}, m, k, n)$-nonnegative paths and $\bar{\mathscr{N}}_{\mathcal{S},k} = \bigcup_{m\geq 0, n\geq 0} \bar{\mathscr{N}}_{\mathcal{S},m,k,n}$.

3. Chung-Feller Extensions of Two Types for $(\mathscr{N}_{\mathcal{S}_1,0}, \theta)$

Throughout the paper, we always let A and B be two finite subsets of the set $\mathbb{P}$, where $\mathbb{P}$ is the set of positive integers. For any $i \in A$ (resp. $j \in B$), let a_i (resp. b_j) be a real number. In this section, we consider rooted lattice paths with the step set, the weight function and the length function in the following case.

Let $\mathcal{S}_1 = \mathcal{S}_A \cup \mathcal{S}_B \cup \{(1,1)\}$, where $\mathcal{S}_A = \{(2i-1, -1) \mid i \in A\}$ and $\mathcal{S}_B = \{(2i, 0) \mid i \in B\}$.

For any step $(x, y) \in \mathcal{S}_1$, let

$$l_1(x,y) = \begin{cases} i & \text{if } (x,y) = (2i, 0), \\ i-1 & \text{if } (x,y) = (2i-1, -1), \\ 1 & \text{if } (x,y) = (1,1), \end{cases}$$

$$w_1(x,y) = \begin{cases} b_i & \text{if } (x,y) = (2i,0), \\ a_i & \text{if } (x,y) = (2i-1,-1), \\ 1 & \text{if } (x,y) = (1,1). \end{cases}$$

Recall that $\mathscr{N}_{\mathcal{S}_1,0} = \bigcup_{m\geq 0,n\geq 0} \mathscr{N}_{\mathcal{S}_1,m,0,n}$, where $\mathscr{N}_{\mathcal{S}_1,m,0,n}$ denotes the set of all the $(\mathcal{S}_1,m,0,n)$-nonnegative paths. For every $L \in \mathscr{N}_{\mathcal{S}_1,0}$, $\theta(L)$ denote the length of L. Note that θ can be viewed as a mapping from $\mathscr{N}_{\mathcal{S}_1,0}$ to $\mathbb{N}$. Our goal is to find Chung-Feller extensions of not only up-down type but also left-right type for $(\mathscr{N}_{\mathcal{S}_1,0},\theta)$.

3.1. The generating function $S_1(z)$

Define a generating function $S_1(z)$ as follows:

$$S_1(z) = \sum_{L\in\mathscr{N}_{\mathcal{S}_1,0}} w(L)z^{\theta(L)}.$$

Let $s_{1;m}$ be the sum of weights of lattice paths in the set $\mathscr{N}_{\mathcal{S}_1,0}$ with length m for $m \geq 1$ and $s_{1;0} = 1$. It is easy to see that $S_1(z) = \sum_{m\geq 0} s_{1;m}z^m$.

Lemma 3.1. $S_1(z) = 1 + \left(\sum_{i\in B} b_i z^i\right) S_1(z) + \left(\sum_{i\in A} a_i z^i\right) [S_1(z)]^2$

Proof. Given a path $L \in \mathscr{N}_{\mathcal{S}_1,0}$ and $L \neq \emptyset$, we suppose that $(x,y) \in \mathcal{S}_1$ is the first step of L and discuss the following two cases:

Case I. $(x,y) = (2i,0)$ for some $i \in B$. We can decompose the path L into $(x,y)R$, where $R \in \mathscr{N}_{\mathcal{S}_1,0}$. Note that $l_1(x,y) = i$ and $w_1(x,y) = b_i$. This provides the second term $\left(\sum_{i\in B} b_i z^i\right) S_1(z)$.

Case II. $(x,y) = (1,1)$. Let (x',y') be the first step returning to the x-axis in L. We can decompose the path L into $(x,y)R(x',y')Q$, where $R,Q \in \mathscr{N}_{\mathcal{S}_1,0}$ and $(x',y') = (2i-1,-1)$ for some $i \in A$. Note that $l_1(x,y) = 1$ and $w_1(x,y) = 1$, $l_1(x',y') = i-1$ and $w_1(x',y') = a_i$. This provides the third term $\left(\sum_{i\in A} a_i z^i\right) [S_1(z)]^2$. $\square$

3.2. A Chung-Feller extension $(\bar{\mathscr{L}}_{\mathcal{S}_1,1}, \bar{\theta}, nprl)$ of up-down type for $(\mathscr{N}_{\mathcal{S}_1,0}, \theta)$

Recall that $\bar{\mathscr{L}}_{\mathcal{S}_1,1} = \bigcup_{m\geq 0, n\geq 0} \bar{\mathscr{L}}_{\mathcal{S}_1,m,1,n}$, where $\bar{\mathscr{L}}_{\mathcal{S}_1,m,1,n}$ denotes the set of all the rooted $(\mathcal{S}_1, m, 1, n)$-lattice paths. For every $\bar{L} \in \bar{\mathscr{L}}_{\mathcal{S}_1,1}$, let $\bar{\theta} : \bar{\mathscr{L}}_{\mathcal{S}_1,1} \to \mathbb{N}$ be a mapping such that $\bar{\theta}(\bar{L}) = \theta(L) - 1$; $nprl(\bar{L})$ denotes the non-positive root length of $\bar{L}$. In this subsection, we will show that $(\bar{\mathscr{L}}_{\mathcal{S}_1,1}, \bar{\theta}, nprl)$ is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S},0}, \theta)$.

First, we investigate the function of Chung-Feller type $CS_1(y, z)$ for $S_1(z)$. By Lemma 3.1, we have $S_1(yz) = 1 + \left(\sum_{i\in B} b_i y^i z^i\right) S_1(yz) + (\sum_{i\in A} a_i y^i z^i)[S_1(yz)]^2$. Hence,

$$
\begin{aligned}
CS_1(y,z) &= \frac{S_1(z) - yS_1(yz)}{1-y} \\
&= \frac{1 + \left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_1(z) + \left(\sum_{i\in A} a_i z^i \sum_{j=0}^{i-2} y^j\right) [S_1(z)]^2}{1 - \sum_{i\in B} b_i y^i z^i - \left(\sum_{i\in A} a_i y^i z^i\right) S_1(yz) - \left(\sum_{i\in A} a_i y^{i-1} z^i\right) S_1(z)}
\end{aligned}
$$

So, let

$$
P_1(y,z) = 1 + \left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_1(z) + \left(\sum_{i\in A} a_i z^i \sum_{j=0}^{i-2} y^j\right) [S_1(z)]^2
$$

and

$$
G_1(y,z) = \frac{1}{1 - \sum_{i\in B} b_i y^i z^i - \left(\sum_{i\in A} a_i y^i z^i\right) S_1(yz) - \left(\sum_{i\in A} a_i y^{i-1} z^i\right) S_1(z)}.
$$

We need to give combinatorial interpretations for $P_1(y, z)$, $G_1(y, z)$ and $CS_1(y, z)$, respectively.

Recall that $\bar{\mathscr{N}}_{\mathcal{S}_1,0} = \bigcup_{m\geq 0, n\geq 0} \bar{\mathscr{N}}_{\mathcal{S}_1,m,0,n}$, where $\bar{\mathscr{N}}_{\mathcal{S}_1,m,0,n}$ is the set of all the rooted $(\mathcal{S}_1, m, 0, n)$-nonnegative paths. Define a generating function $\tilde{P}_1(y, z)$ as follows:

$$
\tilde{P}_1(y,z) = \sum_{[L;j]\in \bar{\mathscr{N}}_{\mathcal{S}_1,0}} w(L) y^j z^{\theta(L)}.
$$

Lemma 3.2. $\tilde{P}_1(y, z) = 1 + \left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_1(z) + \Big(\sum_{i\in A} a_i z^i$

$\sum_{j=0}^{i-2} y^j\Big)[S_1(z)]^2$.

Proof. Given a rooted lattice path $\bar{L} = [L, j] \in \bar{\mathscr{N}_{\mathcal{S}_1,0}}$ and $\bar{L} \neq \emptyset$, we suppose that the final step of L is (x, y). Note that $0 \le j \le l_1(x, y) - 1$. We discuss the following two cases:

Case I. $(x, y) = (2i, 0)$ for some $i \in B$. We can decompose the path L into $R(x, y)$, where $R \in \mathscr{N}_{\mathcal{S}_1,0}$. Note that $l_1(x, y) = i$, $w_1(x, y) = b_i$ and $j \in \{0, 1, \dots, i-1\}$. This provides the second term $\left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_1(z)$.

Case II. $(x, y) = (2i-1, -1)$ for some $i \in A$ and $i \ge 2$. Let (x', y') be the right-most step leaving the x-axis. We can decompose the path L into $Q(x', y')R(x, y)$, where $R, Q \in \mathscr{N}_{\mathcal{S}_1,0}$ and $(x', y') = (1, 1)$. Note that $l_1(x', y') = 1$, $w_1(x', y') = 1$, $l_1(x, y) = i-1$, $w_1(x, y) = a_i$ and $j \in \{0, 1, \dots, i-2\}$. This provides the third term $\left(\sum_{i\in A} a_i z^i \sum_{j=0}^{i-2} y^j\right)$ $[S_1(z)]^2$. $\square$

Lemma 3.2 tells us that $\tilde{P}_1(y, z) = \sum_{[L;j]\in\bar{\mathscr{N}_{\mathcal{S}_1,0}}} w(L)y^j z^{\theta(L)}$ is a combinatorial interpretation for $P_1(y, z)$.

Now, recall that $\mathscr{L}_{\mathcal{S}_1,0} = \bigcup_{m\ge0,n\ge0} \mathscr{L}_{\mathcal{S}_1,m,0,n}$, where $\mathscr{L}_{\mathcal{S}_1,m,0,n}$ denotes the set of all the $(\mathcal{S}_1, m, 0, n)$-lattice paths. Define a generating function $\tilde{G}_1(y, z)$ as follows:

$$\tilde{G}_1(y, z) = \sum_{L\in\mathscr{L}_{\mathcal{S}_1,0}} w(L)y^{npl(L)} z^{\theta(L)}.$$

Lemma 3.3.

$$\begin{aligned}\tilde{G}_1(y, z) \;=\; & 1 + \left(\sum_{i\in B} b_i y^i z^i\right) \tilde{G}_1(y, z) + \left(\sum_{i\in A} a_i y^i z^i\right) S_1(yz)\tilde{G}_1(y, z) \\ & + \left(\sum_{i\in A} a_i y^{i-1} z^i\right) S_1(z)\tilde{G}_1(y, z),\end{aligned}$$

Equivalently,

$$\tilde{G}_1(y, z) \;=\; \frac{1}{1-\sum_{i\in B} b_i y^i z^i - \left(\sum_{i\in A} a_i y^i z^i\right) S_1(yz) - \left(\sum_{i\in A} a_i y^{i-1} z^i\right) S_1(z)}.$$

Proof. Since the second identity is equivalent to the first identity, we just prove the first identity. Given a path $L \in \mathscr{L}_{\mathcal{S},0}$ and $L \neq \emptyset$, we suppose that (x, y) is the first step of L. We discuss the following three cases:

Case I. $(x, y) = (2i, 0)$ for some $i \in B$. We can decompose the path L into $(x, y)R$, where $R \in \mathscr{L}_{\mathcal{S}_1,0}$. Note that $l_1(x, y) = i$, $w_1(x, y) = b_i$ and $h(x, y) = 0$. This provides the second term $\left(\sum_{i\in B} b_i y^i z^i\right) \tilde{G}_1(y, z)$.

Case II. $(x, y) = (2i - 1, -1)$ for some $i \in A$. Let $(x', y') = (1, 1)$ be the first step returning to the x-axis. We can decompose the path L into $(x, y)Q(x', y')R$. Clearly, $R \in \mathscr{L}_{\mathcal{S}_1,0}$. For any a lattice path $P = (x_1, y_1)\cdots(x_n, y_n)$, we define P' as $P' = (x_n, y_n)(x_{n-1}, y_{n-1})\cdots(x_1, y_1)$. Then $Q' \in \mathscr{N}_{\mathcal{S}_1,0}$. Note that $l_1(x', y') = 1$, $w_1(x', y') = 1$, $l_1(x, y) = i - 1$, $w_1(x, y) = a_i$, $h(x, y) = -1$ and $h(x', y') = 0$. This provides the third term $\left(\sum_{i\in A} a_i y^i z^i\right) \times S_1(yz)\tilde{G}_1(y, z)$.

Case III. $(x, y) = (1, 1)$. Let $(x', y') = (2i-1, -1)$ be the first step returning to the x-axis. We can decompose the path L into $(x, y)Q(x', y')R$, where $R \in \mathscr{L}_{\mathcal{S}_1,0}$ and $Q \in \mathscr{N}_{\mathcal{S}_1,0}$. Note that $l_1(x', y') = i - 1$, $w_1(x', y') = a_i$, $l_1(x, y) = 1$, $w_1(x, y) = 1$, $h(x, y) = 1$ and $h(x', y') = 0$. This provides the fourth term $\left(\sum_{i\in A} a_i y^{i-1} z^i\right) S_1(z)\tilde{G}_1(y, z)$. □

Hence, $\tilde{G}_1(y, z) = \sum_{L\in\mathscr{L}_{\mathcal{S}_1,0}} w(L)y^{npl(L)}z^{\theta(L)}$ is a combinatorial interpretation for $G_1(y, z)$.

Now, we give a combinatorial interpretation for the function $CS_1(y, z)$ of Chung-Feller type for $S_1(z)$. Define a generating function $D_1(y, z)$ as follows:

$$D_1(y, z) = \sum_{\bar{L}\in\bar{\mathscr{L}}_{\mathcal{S}_1,1}} w(L)y^{nprl(\bar{L})}z^{\bar{\theta}(\bar{L})}.$$

Let $\bar{s}_{1;m,r}$ be the sum of weights of rooted lattice paths in the set $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with length $m + 1$ and non-positive root length r for $(m, r) \neq (0, 0)$ and let $\bar{s}_{1;0,0} = 1$. It is easy to see that $D_1(y, z) = \sum_{m\geq 0}\sum_{r=0}^{m} \bar{s}_{1;m,r}y^r z^m$.

Lemma 3.4. $D_1(y, z) = G_1(y, z)P_1(y, z)$.

Proof. Let $\bar{L} \in \bar{\mathscr{L}}_{\mathcal{S}_1,1}$. Let (x, y) be the right-most step $(1, 1)$ leaving x-axis and reaching the line $y = 1$. We can decompose the path $\bar{L}$ into $R(x, y)\bar{Q}$, where $R \in \mathscr{L}_{\mathcal{S}_1,0}$ and $\bar{Q} \in \bar{\mathscr{N}}_{\mathcal{S}_1,0}$. Hence, $D_1(y, z) = G_1(y, z)P_1(y, z)$. □

Hence, $D_1(y,z) = \sum_{m\geq 0}\sum_{r=0}^{m} \bar{s}_{1;m,r} y^r z^m$ is a combinatorial interpretation for $CS_1(y,z)$.

Theorem 3.5. *$(\bar{\mathscr{L}}_{\mathcal{S}_1,1}, \bar{\theta}, nprl)$ is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S},0}, \theta)$.*

Theorem 3.6. *Let $\bar{s}_{1;m,r}$ be the sum of weights of rooted lattice paths in the set $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ which*

(a) *have length $m+1$,*

(b) *have non-positive root length r.*

Let $s_{1;m}$ be the sum of weights of lattice paths in the set $\mathscr{N}_{\mathcal{S}_1,0}$ with length m. Then $\bar{s}_{1;m,r} = s_{1;m}$ and $\bar{s}_{1;m,r}$ is independent on r.

The classical Chung-Feller theorem can be derived as a special case of Theorem 3.6.

Corollary 3.7. (Chung-Feller [2]) *For every $0 \leq r \leq m$, the number of m-Dyck paths with r nonpositive up-steps is equal to c_m and independent on r.*

Proof. Let $\mathcal{S} = \{(1,1),(1,-1)\}$, $w(x,y) = 1$ for any $(x,y) \in \mathcal{S}$, $l(1,1) = 1$ and $l(1,-1) = 0$. Suppose $\bar{L} = [L;j] \in \bar{\mathscr{L}}_{\mathcal{S},1}$. Since $l(1,1) = 1$ and $l(1,-1) = 0$, the final step of $\bar{L}$ must be $(1,1)$ and $j = 0$. Deleting the root and the final step of $\bar{L}$, we obtain a lattice path in $\mathscr{L}_{\mathcal{S},0}$. This implies that the number of rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S},1}$ with length $\bar{\theta}(\bar{L}) = m$ and $nprl(\bar{L}) = r$ is equal to the number of lattice paths L in $\mathscr{L}_{\mathcal{S},0}$ with $\theta(L) = m$ and $npl(L) = r$. It is easy to see that $\mathscr{L}_{\mathcal{S},0}$ is the set of Dyck paths. For every $L \in \mathscr{L}_{\mathcal{S},0}$, the number of nonpositive up-steps and the semilength of the path L are $npl(L)$ and $\theta(L)$ respectively.

By Theorem 3.6, the number of rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S},1}$ with length $\bar{\theta}(\bar{L}) = m$ and $nprl(\bar{L}) = r$ is equal to the number of lattice paths L in $\mathscr{N}_{\mathcal{S},0}$ with $\theta(L) = m$. By Lemma 3.1, we have $S_1(z) = 1 + z[S_1(z)]^2$ since $\mathcal{S} = \{(1,1),(1,-1)\}$, $w(s) = 1$ for any $s \in \mathcal{S}$, $l(1,1) = 1$ and $l(1,-1) = 0$. Hence, the number of the lattice paths L in $\mathscr{N}_{\mathcal{S}_1,0}$ with $\theta(L) = m$ is the m-th Catalan number. Thus, the number of m-Dyck paths D with r nonpositive up-steps is equal to c_m and independent on r. $\square$

3.3. A Chung-Feller extension $(\bar{\mathscr{L}}_{\mathcal{S}_1,1}, \bar{\theta}, rmrl)$ of left-right type for $(\mathscr{N}_{\mathcal{S}_1,0}, \theta)$

For every $\bar{L} \in \bar{\mathscr{L}}_{\mathcal{S}_1,1}$, $rmrl(\bar{L})$ denotes the rightmost minimum root length of $\bar{L}$. We will show that $(\bar{\mathscr{L}}_{\mathcal{S}_1,1}, \bar{\theta}, rmrl)$ is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S}_1,0}, \theta)$. Note that

$$CS_1(y,z) = \frac{S_1(z) - yS_1(yz)}{1-y} = \frac{P_1(y,z)S_1(yz)}{1 - \left[\sum_{i\in A} a_i y^{i-1} z^i\right] [S_1(yz)S_1(z)]},$$

where $P_1(y,z) = 1 + \left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_1(z) + \left(\sum_{i\in A} a_i z^i \sum_{j=0}^{i-2} y^j\right) [S_1(z)]^2$. We give a new combinatorial interpretation for $CS_1(y,z)$.

Let $k \geq 0$. Recall that $\mathscr{N}_{\mathcal{S}_1,-k} = \bigcup_{m\geq 0, n\geq 0} \mathscr{N}_{\mathcal{S}_1,m,-k,n}$, where $\mathscr{N}_{\mathcal{S}_1,m,-k,n}$ denotes the set of all the $(\mathcal{S}_1, m, -k, n)$-nonnegative paths. Define a generating function

$$H_{1;k}(z) = \sum_{L\in \mathscr{N}_{\mathcal{S}_1,-k}} w(L) z^{\theta(L)}.$$

Lemma 3.8. $H_{1;k}(z) = [S_1(z)]^{k+1} \left[\sum_{i\in A} a_i z^{i-1}\right]^k$.

Proof. For any a path $L \in \mathscr{N}_{\mathcal{S}_1,-k}$ and $L \neq \emptyset$, we consider the first step with height $-m$ in L, denoted by (x_m, y_m), where $1 \leq m \leq k$. Thus we can decompose the path L into $L_0(x_1,y_1)L_1(x_2,y_2)\cdots L_{k-1}(x_k,y_k)L_k$, where $L_r \in \mathscr{N}_{\mathcal{S}_1,0}$ for all $0 \leq r \leq k$ and $(x_j, y_j) \in \mathcal{S}_A$ for all j. Thus, $H_{1;k}(z) = [S_1(z)]^{k+1} \left[\sum_{i\in A} a_i z^{i-1}\right]^k$. ☐

Define a generating function $M_1(y,z)$ as follows:

$$M_1(y,z) = \sum_{\bar{L}\in \bar{\mathscr{L}}_{\mathcal{S}_1,1}} w(L) y^{rmrl(\bar{L})} z^{\bar{\theta}(L)}.$$

Let $\bar{g}_{1;m,r}$ be the sum of weights of rooted lattice paths in the set $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with length $m+1$ and rightmost minimum root length r for $(m,r) \neq (0,0)$ and $\bar{g}_{1;0,0} = 1$. It is easy to see that $M_1(y,z) = \sum_{m\geq 0} \sum_{r=0}^{m} \bar{g}_{1;m,r} y^r z^m$.

Lemma 3.9. $M_1(y,z) = \dfrac{P_1(y,z)S_1(yz)}{1 - \left[\sum_{i\in A} a_i y^{i-1} z^i\right] [S_1(yz)S_1(z)]}$.

Proof. Given a rooted lattice path $\bar{L} \in \bar{\mathscr{L}}_{\mathcal{S}_1,1}$ we suppose the rightmost minimum point of L is $(a, -k)$, where $k \geq 0$. Furthermore, suppose the first step at the right of the point $(a, -k)$ in L is (x, y). Then $(x, y) = (1, 1)$. Using this step, we can decompose $\bar{L}$ into $R(1, 1)\bar{T}$, where $R \in \mathscr{N}_{\mathcal{S}_1,-k}$. For the path $\bar{T}$, we denote the rightmost step $(1, 1)$ with height $-m$ in the path $\bar{T}$ as (x_{m+1}, y_{m+1}) , where $-1 \leq m \leq k-1$. Thus we can decompose the path $\bar{T}$ into $L_{k-1}(x_k, y_k)L_{k-2}(x_{k-1}, y_{k-1}) \cdots L_0(x_0, y_0)\bar{Q}$, where $L_j \in \mathscr{N}_{\mathcal{S}_1,0}$ for all $0 \leq j \leq k-1$ and $\bar{Q} \in \bar{\mathscr{N}}_{\mathcal{S}_1,0}$. Hence, by Lemmas 3.1, 3.2 and 3.8, we get $M_1(y, z) = \sum_{k \geq 0} H_{1;k}(yz)[S_1(z)]^k z^k P_1(y, z)$. Hence,

$$\begin{aligned} M_1(y,z) &= P_1(y,z)S_1(yz)\sum_{k\geq 0}[S_1(yz)]^k \left[\sum_{i\in A} a_i y^{i-1}z^{i-1}\right]^k [S_1(z)]^k z^k \\ &= \frac{P_1(y,z)S_1(yz)}{1-\left[\sum_{i\in A} a_i y^{i-1}z^{i}\right]\left[S_1(yz)S_1(z)\right]}, \end{aligned}$$

□

Hence, $M_1(y, z) = \sum_{\bar{L} \in \bar{\mathscr{L}}_{\mathcal{S}_1,1}} w(L)y^{rmrl(\bar{L})}z^{\bar{\theta}(L)}$ is a combinatorial interpretation for $CS_1(y, z)$

Theorem 3.10. *$(\bar{\mathscr{L}}_{\mathcal{S}_1,1}, \bar{\theta}, rmrl)$ is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S}_1,0}, \theta)$.*

Theorem 3.11. *Let $\bar{g}_{1;m,r}$ be the sum of weights of rooted lattice paths in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ which:*

(a) *have length $m + 1$,*

(b) *have rightmost minimum root length r.*

Let $s_{1;m}$ be the sum of weights of lattice paths in $\mathscr{N}_{\mathcal{S}_1,0}$ with length m. Then $\bar{g}_{1;m,r} = s_{1;m}$ and $\bar{g}_{1;m,r}$ is independent on r.

By Theorem 3.11, we derive Chung-Feller Theorem of left-right type for Dyck paths. See also Example 1.1.

Corollary 3.12. *For every $0 \leq r \leq m$, the number of m-Dyck paths with r left up-steps is equal to c_m and independent on r.*

Proof. Let $\mathcal{S} = \{(1, 1), (1, -1)\}$, $w(x, y) = 1$ for any $(x, y) \in \mathcal{S}$, $l(1, 1) = 1$ and $l(1, -1) = 0$. Suppose $\bar{L} = [L; j] \in \bar{\mathscr{L}}_{\mathcal{S},1}$. Since $l(1, 1) = 1$ and $l(1, -1) = 0$, the final step of $\bar{L}$ must be $(1, 1)$ and $j = 0$. Deleting the root and the final step of $\bar{L}$, we obtain a lattice path in $\mathscr{L}_{\mathcal{S},0}$. This implies that the number of rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S},1}$ with length $\bar{\theta}(\bar{L}) = m$ and $rmrl(\bar{L}) = r$ is equal to the number of lattice paths L in $\mathscr{L}_{\mathcal{S},0}$ with $\theta(L) = m$ and $rml(L) = r$. It is easy to see that $\mathscr{L}_{\mathcal{S},0}$ is the set of Dyck paths. For every $L \in \mathscr{L}_{\mathcal{S},0}$, the number of left up-steps and the semilength of the path L are $rml(L)$ and $\theta(L)$ respectively.

By Theorem 3.11, the number of rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S},1}$ with length $\bar{\theta}(\bar{L}) = m$ and $rmrl(\bar{L}) = r$ is equal to the number of lattice paths L in $\mathscr{N}_{\mathcal{S},0}$ with $\theta(L) = m$. Hence, the number of m-Dyck paths with r left up-steps is equal to c_m and independent on r. □

By Theorems 3.6 and 3.11, we can derive many new Chung-Feller Theorems for rooted lattice paths. As example, we give the following corollary. This corollary answers the problem in Example 1.7.

Corollary 3.13. *Let* $\mathcal{S} = \{(1,1),(5,-1),(1,-1)\}$, $w(x,y) = 1$ *for any* $(x,y) \in \mathcal{S}$, $l(1,1) = 1$, $l(5,-1) = 2$ *and* $l(1,-1) = 0$. *Then*

(1) *the number of lattice paths in* $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ *with length* $m+1$ *and non-positive root length* r *equal to the number of lattice paths in* $\mathscr{N}_{\mathcal{S}_1,0}$ *with length* m *and independent on* r;

(2) *the number of lattice paths in* $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ *with length* $m+1$ *and rightmost minimum root length* r *equal to the number of lattice paths in* $\mathscr{N}_{\mathcal{S}_1,0}$ *with length* m *and independent on* r.

Example 3.14. Let $m = 3$. Let the step set $\mathcal{S}$, the weight function w and the length function l be given as those in Corollary 3.13. We draw all the lattice paths L in $\mathscr{N}_{\mathcal{S}_1,0}$ with $\theta(L) = 3$ as follows:

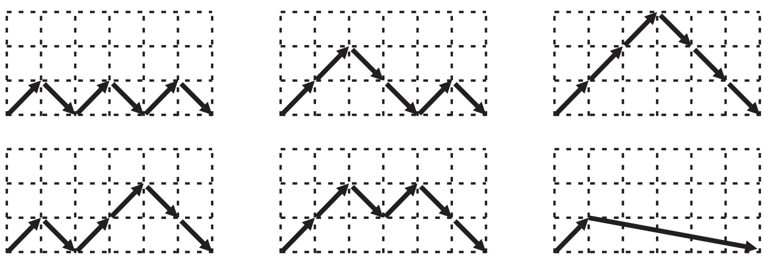

Figure 7. All the lattice paths in $\mathscr{N}_{\mathcal{S}_1,0}$ with $\theta(L) = 3$.

We draw rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = 3$ and $nprl(\bar{L}) = r$ as follows:

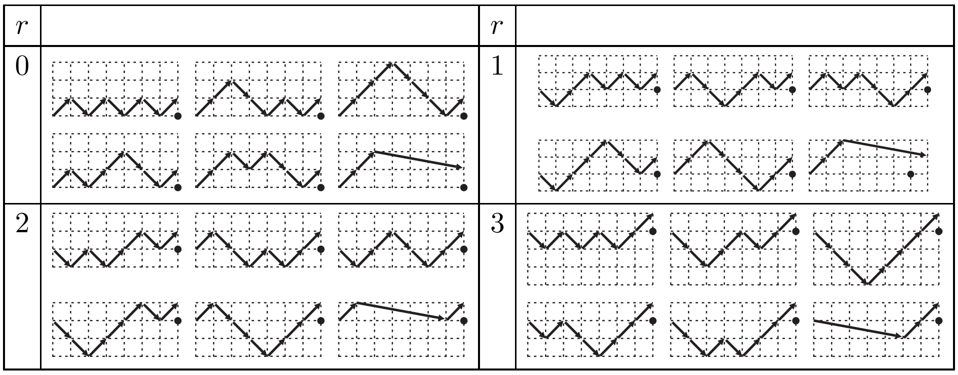

Figure 8. Rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = 3$ and $nprl(\bar{L}) = r$.

where the notation " $\bullet$ " denotes the root of the corresponding path. We draw rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = 3$ and $rmrl(\bar{L}) = r$ as follows:

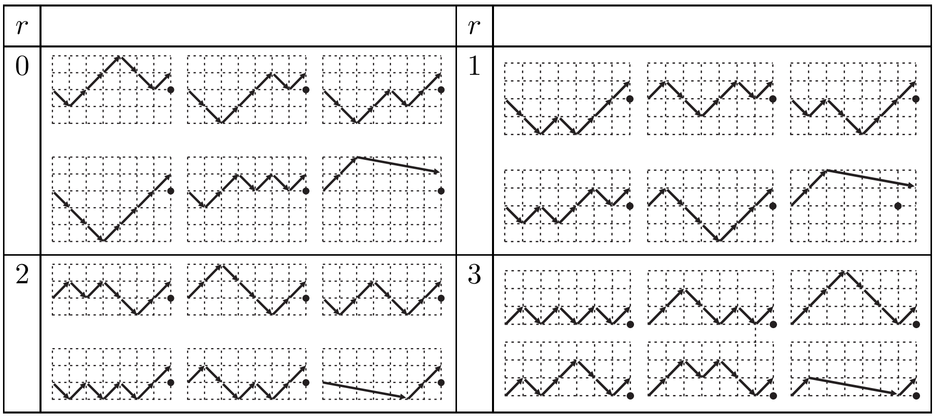

Figure 9. Rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = 3$ and $rmrl(\bar{L}) = r$.

where the notation " $\bullet$ " denotes the root of the corresponding path.

4. Chung-Feller Extensions of Two Types for $(\mathscr{N}_{\mathcal{S}_2,0}, \boldsymbol{\theta})$

In this section, we give another example of Chung-Feller extensions obtained by the function of Chung-Feller type for a generating function. Since we use similar methods to those used in Section 3, we don't give the detail of the proofs for lemmas and theorems in this section and only list them. In this subsection, we consider rooted lattice paths with the step set, the weight function and the length function in the following case.

Let $\mathcal{S}_2 = \mathcal{S}_A \cup \mathcal{S}_B \cup \{(1,1)\}$, where $\mathcal{S}_A = \{(i,-1) \mid i \in A\}$ and $\mathcal{S}_B = \{(i,0) \mid i \in B\}$.

For any step $(x,y) \in \mathcal{S}_2$, let

$$l_2(x,y) = \begin{cases} i & \text{if } (x,y) = (i,0) \text{ and } (i,-1), \\ 1 & \text{if } (x,y) = (1,1), \end{cases}$$

$$w_2(x,y) = \begin{cases} b_i & \text{if } (x,y) = (i,0), \\ a_i & \text{if } (x,y) = (i,-1), \\ 1 & \text{if } (x,y) = (1,1). \end{cases}$$

Recall that $\bar{\mathscr{L}}_{\mathcal{S}_2,1} = \bigcup_{m\geq 0, n\geq 0} \bar{\mathscr{L}}_{\mathcal{S}_2,m,1,n}$, where $\bar{\mathscr{L}}_{\mathcal{S}_2,m,1,n}$ denotes the set of all the rooted $(\mathcal{S}_2,m,1,n)$-lattice paths. For every $\bar{L} \in \bar{\mathscr{L}}_{\mathcal{S}_2,1}$, let $\bar{\theta} : \bar{\mathscr{L}}_{\mathcal{S}_1,1} \to \mathbb{N}$ be a mapping such that $\bar{\theta}(\bar{L}) = \theta(L) - 1$; $nprl(\bar{L})$ denotes the non-positive root length of $\bar{L}$. We first show that $(\bar{\mathscr{L}}_{\mathcal{S}_2,1}, \bar{\theta}, nprl)$ is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S}_2,0}, \theta)$.

4.1. A Chung-Feller extension $(\bar{\mathscr{L}}_{\mathcal{S}_2,1}, \bar{\theta}, nprl)$ of up-down type for $(\mathscr{N}_{\mathcal{S}_2,0}, \theta)$

Define a generating function $S_2(z)$ as $S_2(z) = \sum_{L\in\mathscr{N}_{\mathcal{S}_2,0}} w(L) z^{\theta(L)}$. Let $s_{2;m}$ be the sum of weights of lattice paths in the set $\mathscr{N}_{\mathcal{S}_2,0}$ with length m for $m \geq 1$ and $s_{2;0} = 1$. It is easy to see that $S_2(z) = \sum_{m\geq 0} s_{2;m} z^m$.

Lemma 4.1. $S_2(z) = 1 + \left(\sum_{i\in B} b_i z^i\right) S_2(z) + \left(\sum_{i\in A} a_i z^{i+1}\right) [S_2(z)]^2$.

We consider the function of Chung-Feller type $CS_2(y,z)$ for $S_2(z)$ as follows.

$$\begin{aligned} CS_2(y,z) &= \frac{S_2(z) - yS_2(yz)}{1-y} \\ &= \frac{1 + \left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_2(z) + \left(\sum_{i\in A} a_i z^{i+1} \sum_{j=0}^{i-1} y^j\right) [S_2(z)]^2}{1 - \sum_{i\in B} b_i y^i z^i - \left(\sum_{i\in A} a_i y^{i+1} z^{i+1}\right) S_2(yz) - \left(\sum_{i\in A} a_i y^i z^{i+1}\right) S_2(z)}. \end{aligned}$$

Hence, let

$$P_2(y,z) = 1 + \left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_2(z) + \left(\sum_{i\in A} a_i z^{i+1} \sum_{j=0}^{i-1} y^j\right) [S_2(z)]^2$$

and

$$G_2(y,z) = \frac{1}{1-\sum_{i\in B} b_i y^i z^i - \left(\sum_{i\in A} a_i y^{i+1} z^{i+1}\right) S_2(yz) - \left(\sum_{i\in A} a_i y^i z^{i+1}\right) S_2(z)}$$

We give combinatorial interpretations for $P_2(y,z)$, $G_2(y,z)$ and $CS_2(y,z)$.

Define a generating function $\tilde{P}_2(y,z)$ as $\tilde{P}_2(y,z) = \sum_{[L;j]\in \bar{\mathscr{N}}_{\mathcal{S}_2,0}} w(L) y^j z^{\theta(L)}$.

Lemma 4.2. $\tilde{P}_2(y,z) = 1 + \left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) F_2(z)$
$+ \left(\sum_{i\in A} a_i z^{i+1} \sum_{j=0}^{i-1} y^j\right) [F_2(z)]^2$.

Hence, $\tilde{P}_2(y,z) = \sum_{[L;j]\in \bar{\mathscr{N}}_{\mathcal{S}_2,0}} w(L) y^j z^{\theta(L)}$ is a combinatorial interpretation for $P_2(y,z)$.

Define a generating functions $\tilde{G}_2(y,z)$ as $\tilde{G}_2(y,z) = \sum_{L\in \mathscr{L}_{\mathcal{S}_2,0}} w(L) y^{npl(L)} z^{\theta(L)}$.

Lemma 4.3.

$$\tilde{G}_2(y,z) = 1 + \left(\sum_{i\in B} b_i y^i z^i\right) \tilde{G}_2(y,z) + \left(\sum_{i\in A} a_i y^{i+1} z^{i+1}\right) S_2(yz) \tilde{G}_2(y,z) + \left(\sum_{i\in A} a_i y^i z^{i+1}\right) S_2(z) \tilde{G}_2(y,z).$$

Equivalently,

$$G_2(y,z) = \frac{1}{1-\sum_{i\in B} b_i y^i z^i - (\sum_{i\in A} a_i y^{i+1} z^{i+1}) S_2(yz) - (\sum_{i\in A} a_i y^i z^{i+1}) S_2(z)}.$$

Hence, $\tilde{G}_2(y,z) = \sum_{L\in \mathscr{L}_{\mathcal{S}_2,0}} w(L) y^{npl(L)} z^{\theta(L)}$ is a combinatorial interpretation for $G_2(y,z)$.

Define a generating functions $D_2(y,z)$ as $D_2(y,z) = \sum_{\bar{L}\in \bar{\mathscr{L}}_{\mathcal{S}_2,1}} w(L) \times y^{nprl(\bar{L})} z^{\bar{\theta}(L)}$. Let $\bar{s}_{2;m,r}$ be the sum of weights of rooted lattice paths in the set $\bar{\mathscr{L}}_{\mathcal{S}_2,1}$ with length $m+1$ and non-positive root length r for $(m,r) \neq (0,0)$

and $\bar{s}_{2;0,0} = 1$. It is easy to see that $D_2(y,z) = \sum_{m\geq 0}\sum_{r=0}^{m} \bar{s}_{2;m,r} y^r z^m$.

Lemma 4.4. $D_2(y,z) = G_2(y,z)P_2(y,z)$.

Hence, $D_2(y,z) = \sum_{\bar{L}\in\bar{\mathscr{L}}_{\mathcal{S}_2,1}} w(L) y^{nprl(\bar{L})} z^{\bar{\theta}(L)}$ is a combinatorial interpretation for $CS_2(y,z)$.

Theorem 4.5. *$(\bar{\mathscr{L}}_{\mathcal{S}_2,1}, \bar{\theta}, nprl)$ is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S}_2,0}, \theta)$.*

Theorem 4.6. *Let $\bar{s}_{2;m,r}$ be the sum of weights of rooted lattice paths in the set $\bar{\mathscr{L}}_{\mathcal{S}_2,1}$ which*

(a) *have length $m+1$,*

(b) *have non-positive root length r.*

Let $s_{2;m}$ be the sum of weights of lattice paths in the set $\mathscr{N}_{\mathcal{S}_2,0}$ with length m. Then $\bar{s}_{2;m,r} = s_{2;m}$ and $\bar{s}_{2;m,r}$ is independent on r.

By Theorem 4.6, we derive a Chung-Feller Theorem of up-down type for Motzkin paths. See also Example 1.2.

Corollary 4.7. *For every $0 \leq r \leq m$, the number of m-Motzkin paths with r nonpositive steps is equal to the m-th Motzkin number and independent on r.*

Proof. Let $\mathcal{S} = \{(1,1),(1,-1),(1,0)\}$, $w(x,y) = 1$ and $l(x,y) = 1$ for any $(x,y) \in \mathcal{S}$. Suppose $\bar{L} = [L;j] \in \bar{\mathscr{L}}_{\mathcal{S}_1,1}$. Since $l(x,y) = 1$ for any $(x,y) \in \mathcal{S}$, we have $j = 0$. Deleting the root from the rooted lattice path $\bar{L}$, we obtain a lattice path in $\mathscr{L}_{\mathcal{S},1}$. This implies that the number of rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = m$ and $nprl(\bar{L}) = r$ is equal to the number of lattice paths L in $\mathscr{L}_{\mathcal{S},1}$ with length $\theta(L) = m$ and $npl(L) = r$. It is easy to see that $\mathscr{L}_{\mathcal{S},1}$ is the set of Motzkin paths. For every $L \in \mathscr{L}_{\mathcal{S},1}$, the number of nonpositive steps and the length of the path L are $npl(L)$ and $\theta(L)$ respectively.

By Theorem 4.6, the number of rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = m$ and $nprl(\bar{L}) = r$ is equal to the number of lattice paths L in $\mathscr{N}_{\mathcal{S}_1,0}$ with $\theta(L) = m$. By Lemma 4.1, we have $S_2(z) = 1 + zS_2(z) + z^2[S_2(z)]^2$ since $\mathcal{S} = \{(1,1),(1,-1),(1,0)\}$, $w(x,y) = 1$ and $l(x,y) = 1$ for any $(x,y) \in \mathcal{S}$. Hence, the number of lattice paths L in $\mathscr{N}_{\mathcal{S}_2,0}$ with $\theta(L) = m$ is the m-th Motzkin number. Thus the number of m-Motzkin paths with r nonpositive steps is equal to the m-th Motzkin number and independent on r. $\square$

4.2. A Chung-Feller extension $(\bar{\mathscr{L}}_{\mathcal{S}_2,1},\bar{\theta},rmrl)$ of left-right type for $(\mathscr{N}_{\mathcal{S}_2,0},\theta)$

For every $\bar{L}\in\bar{\mathscr{L}}_{\mathcal{S}_2,1}$, $rmrl(\bar{L})$ denotes the rightmost minimum root length of $\bar{L}$. We will show that $(\bar{\mathscr{L}}_{\mathcal{S}_2,1},\bar{\theta},rmrl)$ also is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S}_2,0},\theta)$.

It is easy to see that

$$CS_2(y,z)=\frac{S_2(z)-yS_2(yz)}{1-y}=\frac{P_2(y,z)S_2(yz)}{1-\left[\sum_{i\in A}a_iy^iz^{i+1}\right]\left[S_2(yz)S_2(z)\right]}.$$

We will give a new combinatorial interpretation for $CS_2(y,z)$.

Let $k\geq 0$. Define a generating function $H_{2,k}(z)=\sum_{L\in\mathscr{N}_{\mathcal{S}_2,-k}}w(L)z^{\theta(L)}$.

Lemma 4.8. $H_{2,k}(z)=[S_2(z)]^{k+1}\left[\sum_{i\in A}a_iz^i\right]^k$.

Define a generating function $M_2(y,z)$ as $M_2(y,z)=\sum_{\bar{L}\in\bar{\mathscr{L}}_{\mathcal{S}_2,1}}w(L)\times y^{rmrl(\bar{L})}z^{\bar{\theta}(L)}$. Let $\bar{g}_{2;m,r}$ be the sum of weights of rooted lattice paths in the set $\bar{\mathscr{L}}_{\mathcal{S}_2,1}$ with length $m+1$ and rightmost minimum root length r for $(m,r)\neq(0,0)$ and $\bar{g}_{2;0,0}=1$. It is easy to see that $M_2(y,z)=\sum_{m\geq 0}\sum_{r=0}^{m}\bar{g}_{2;m,r}y^rz^m$.

Lemma 4.9. $M_2(y,z)=\dfrac{P_2(y,z)S_2(yz)}{1-\left[\sum_{i\in A}a_iy^iz^{i+1}\right]\left[S_2(yz)S_2(z)\right]}$.

Hence, $M_2(y,z)=\sum_{\bar{L}\in\bar{\mathscr{L}}_{\mathcal{S}_2,1}}w(L)y^{rmrl(\bar{L})}z^{\bar{\theta}(L)}$ is a combinatorial interpretation for $CS_2(y,z)$

Theorem 4.10. *$(\bar{\mathscr{L}}_{\mathcal{S}_2,1},\bar{\theta},rmrl)$ is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S}_2,0},\theta)$.*

Theorem 4.11. *Let $\bar{g}_{2;m,r}$ be the sum of weights of rooted lattice paths in $\bar{\mathscr{L}}_{\mathcal{S}_2,1}$ which:*

(a) *have length $m+1$,*
(b) *have rightmost minimum root length r.*

Let $s_{2;m}$ be the sum of weights of lattice paths in $\mathscr{N}_{\mathcal{S}_2,0}$ with length m. Then $\bar{g}_{2;m,r}=s_{2;m}$ and $\bar{g}_{2;m,r}$ is independent on r.

The result which Shapiro [12] found is a corollary of Theorem 4.11.

Corollary 4.12. (Shapiro [12]) *For every $0 \le r \le n$, the number of m-Motzkin paths with r left steps is equal to the m-th Motzkin number and independent on r.*

Proof. Let $\mathcal{S} = \{(1,1),(1,-1),(1,0)\}$, $w(x,y) = 1$ and $l(x,y) = 1$ for any $(x,y) \in \mathcal{S}$. Suppose $\bar{L} = [L;j] \in \bar{\mathscr{L}}_{\mathcal{S}_1,1}$. Since $l(x,y) = 1$ for any $(x,y) \in \mathcal{S}$, we have $j = 0$. Deleting the root from the rooted lattice path $\bar{L}$, we obtain a lattice path in $\mathscr{L}_{\mathcal{S},1}$. This implies that the number of rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = m$ and $rmrl(\bar{L}) = r$ is equal to the number of lattice paths L in $\mathscr{L}_{\mathcal{S},1}$ with $\theta(L) = m$ and $rml(L) = r$. It is easy to see that $\mathscr{L}_{\mathcal{S},1}$ is the set of Motzkin paths. For every $L \in \mathscr{L}_{\mathcal{S},1}$, the number of left steps and the length of the path L are $\theta(L) = m$ and $rml(L) = r$ respectively.

By Theorem 4.11, the number of rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = m$ and $rmrl(\bar{L}) = r$ is equal to the number of lattice paths L in $\mathscr{N}_{\mathcal{S}_1,0}$ with $\theta(L) = m$. Hence, the number of m-Motzkin paths with r left steps is equal to the m-th Motzkin number and independent on r. □

By Theorem 4.6 and 4.11, we can derive many new Chung-Feller Theorems for lattice paths. As example, we give the following corollary.

Corollary 4.13. *Let $\mathcal{S} = \{(1,1),(3,-1),(1,-1)\}$, $w(x,y) = 1$ for any $(x,y) \in \mathcal{S}$, $l(1,1) = 1$, $l(3,-1) = 3$ and $l(1,-1) = 1$. Then*

(1) *the number of lattice paths in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with length $m+1$ and non-positive root length r equal to the number of lattice paths in $\mathscr{N}_{\mathcal{S}_1,0}$ with length m and independent on r;*

(2) *the number of lattice paths in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with length $m+1$ and rightmost minimum root length r equal to the number of lattice paths in $\mathscr{N}_{\mathcal{S}_1,0}$ with length m and independent on r.*

Example 4.14. Let $m = 4$. Let the step set $\mathcal{S}$, the weight function w and the length function l be given as those in Corollary 4.13. We draw all the lattice paths L in $\mathscr{N}_{\mathcal{S}_1,0}$ with $\theta(L) = 4$ as follows:

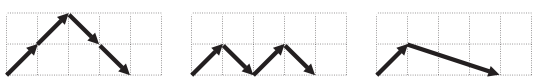

Figure 10. All the lattice paths in $\mathscr{N}_{\mathcal{S}_1,0}$ with $\theta(L) = 4$.

We draw rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = 4$ and $nprl(\bar{L}) = r$ as follows:

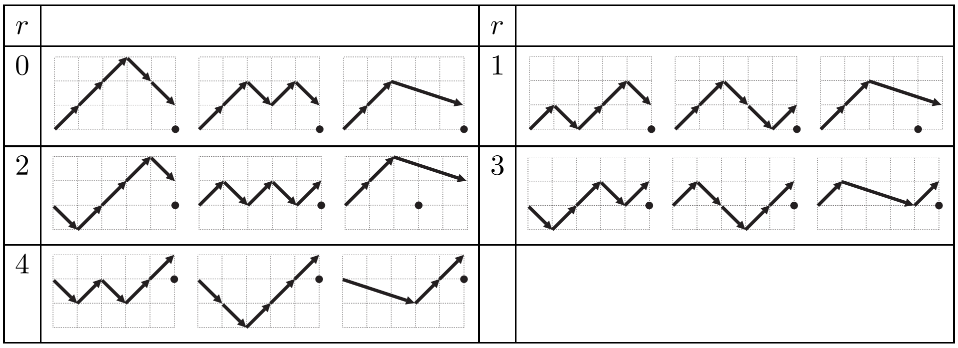

Figure 11. Rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = 4$ and $nprl(\bar{L}) = r$.

where the notation " $\bullet$ " denotes the root of the corresponding path. We draw rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ $\bar{\theta}(\bar{L}) = 4$ and $rmrl(\bar{L}) = r$ as follows:

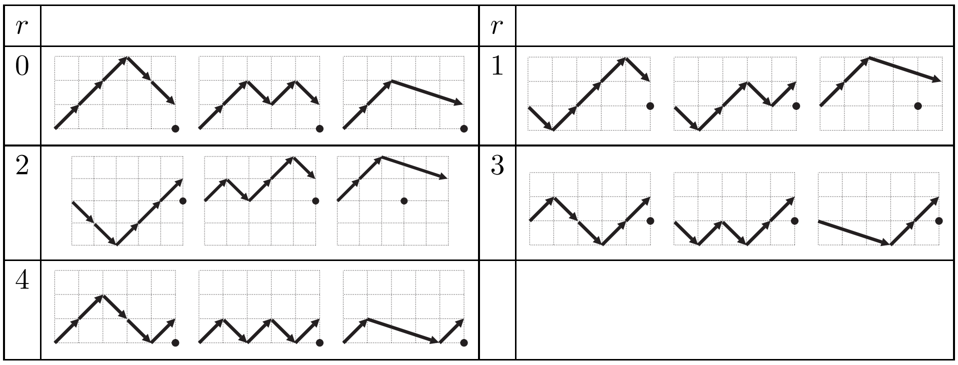

Figure 12. Rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ with $\bar{\theta}(\bar{L}) = 4$ and $rmrl(\bar{L}) = r$.

where the notation " $\bullet$ " denotes the root of the corresponding path.

5. A Chung-Feller extension of up-down type for $(\mathscr{N}_{\mathcal{S}_3,0}, \theta)$

In this section, we give the third example of Chung-Feller extensions obtained by the function of Chung-Feller type for a generating function. Since we use similar methods to those used in Section 3, we don't give the detail of the proofs for lemmas and theorems in this section and only list them. In this subsection, we consider rooted lattice paths with the step set, the weight function and the length function in the following case.

Let $\mathcal{S}_3 = \mathcal{S}_A \cup \mathcal{S}_B \cup \{(1,1)\}$, where $\mathcal{S}_A = \{(1, -2i+1) \mid i \in A\}$ and $\mathcal{S}_B = \{(2i, 0) \mid i \in B\}$.

For any step $(x,y) \in \mathcal{S}_3$, let

$$l_3(x,y) = \begin{cases} i & \text{if } (x,y) = (2i,0), \\ 0 & \text{if } (x,y) = (1,-2i+1), \\ 1 & \text{if } (x,y) = (1,1), \end{cases}$$

$$w_3(x,y) = \begin{cases} b_i & \text{if } (x,y) = (2i,0), \\ a_i & \text{if } (x,y) = (1,-2i+1), \\ 1 & \text{if } (x,y) = (1,1). \end{cases}$$

Define a generating function $S_3(z)$ as $S_3(z) = \sum_{L \in \mathscr{N}_{\mathcal{S}_{3,0}}} w(L) z^{\theta(L)}$. Let $s_{3;m}$ be the sum of the weights of lattice paths in the set $\mathscr{N}_{\mathcal{S}_{3,0}}$ with length m for $m \geq 1$ and $s_{3;0} = 1$. It is easy to see that $S_3(z) = \sum_{n \geq 0} s_{3;m} z^m$.

Lemma 5.1. $S_3(z) = 1 + \left(\sum_{i \in B} b_i z^i\right) S_3(z) + \left(\sum_{i \in A} a_i z^i\right) [S_3(z)]^{i+1}$.

By Lemma 5.1, we have

$$\begin{aligned} CS_3(y,z) &= \frac{S_3(z) - yS_3(yz)}{1-y} \\ &= \frac{1 + \left(\sum_{i \in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_3(z)}{1 - \sum_{i \in B} b_i y^i z^i - \sum_{i \in A} a_i z^i \sum_{j=0}^{i} y^{i-j} [S_3(yz)]^{i-j} [S_3(z)]^j}. \end{aligned}$$

Hence, let

$$P_3(y,z) = 1 + \left(\sum_{i \in B} b_i z^i \sum_{j=0}^{i-1} y^j \right) S_3(z)$$

and

$$G_3(y,z) = \frac{1}{1 - \sum_{i \in B} b_i y^i z^i - \sum_{i \in A} a_i z^i \sum_{j=0}^{i} y^{i-j} [S_3(yz)]^{i-j} [S_3(z)]^j}$$

We give combinatorial interpretations for $P_3(y,z)$, $G_3(y,z)$ and $CS_3(y,z)$.

Define a generating function $\tilde{P}_3(y,z)$ as $\tilde{P}_3(y,z) = \sum_{[L;j]\in\bar{\mathscr{N}}_{\mathcal{S}_3,0}} w(L)y^j z^{\theta(L)}$.

Lemma 5.2. $\tilde{P}_3(y,z) = 1 + \left(\sum_{i\in B} b_i z^i \sum_{j=0}^{i-1} y^j\right) S_3(z)$.

Hence, $\tilde{P}_3(y,z) = \sum_{[L;j]\in\bar{\mathscr{N}}_{\mathcal{S}_3,0}} w(L)y^j z^{\theta(L)}$ is a combinatorial interpretation for $P_3(y,z)$.

Define a generating function $\tilde{G}_3(y,z)$ as $\tilde{G}_3(y,z) = \sum_{L\in\mathscr{L}_{\mathcal{S}_3,0}} w(L)y^{npl(L)} z^{\theta(L)}$.

Lemma 5.3.

$$\begin{aligned}\tilde{G}_3(y,z) &= 1 + \left(\sum_{i\in A} a_i z^i \sum_{j=0}^{i} y^{i-j}[S_3(yz)]^{i-j}[S_3(z)]^j\right)\tilde{G}_3(y,z)\\ &\quad + \left(\sum_{i\in B} b_i y^i z^i\right)\tilde{G}_3(y,z).\end{aligned}$$

Equivalently,

$$\tilde{G}_3(y,z) = \frac{1}{1 - \sum_{i\in B} b_i y^i z^i - \sum_{i\in A} a_i z^i \sum_{j=0}^{i} y^{i-j}[S_3(yz)]^{i-j}[S_3(z)]^j}$$

Hence, $\tilde{G}_3(y,z) = \sum_{L\in\mathscr{L}_{\mathcal{S}_3,0}} w(L)y^{npl(L)} z^{\theta(L)}$ is a combinatorial interpretation for $G_3(y,z)$.

Define a generating function $D_3(y,z)$ as $D_3(y,z) = \sum_{\bar{L}\in\bar{\mathscr{L}}_{\mathcal{S}_3,1}} w(L)y^{nprl(\bar{L})} z^{\theta(L)-1}$. Let $\bar{s}_{3;m,r}$ be the sum of weights of rooted lattice paths in the set $\bar{\mathscr{L}}_{\mathcal{S}_3,1}$ with length $m+1$ and non-positive root length r for $(m,r)\neq(0,0)$ and $\bar{s}_{3;0,0}=1$. It is easy to see that $D_3(y,z) = \sum_{m\geq 0}\sum_{r=0}^{m} \bar{s}_{3;m,r} y^r z^m$.

Lemma 5.4. $D_3(y,z) = G_3(y,z)P_3(y,z)$.

Hence, $D_3(y,z) = \sum_{\bar{L}\in\bar{\mathscr{L}}_{\mathcal{S}_3,1}} w(L)y^{nprl(\bar{L})} z^{\theta(L)-1}$ is a combinatorial interpretation for $CS_3(y,z)$.

Theorem 5.5. *$(\bar{\mathscr{L}}_{\mathcal{S}_3,0}, \bar{\theta}, nprl)$ is a Chung-Feller extension for $(\mathscr{N}_{\mathcal{S}_3,0}, \theta)$.*

Theorem 5.6. *Let $\bar{s}_{3;m,r}$ be the sum of weights of rooted lattice paths in the set $\bar{\mathscr{L}}_{\mathcal{S}_3,1}$ which*

(a) *have length* $m+1$,

(b) *have non-positive root length* r.

Let $s_{3;m}$ *be the sum of the weights of the lattice paths in the set* $\mathscr{N}_{\mathcal{S}_3,0}$ *with length* m. *Then* $\bar{s}_{3;m,r} = s_{3;m}$ *and* $\bar{s}_{3;m,r}$ *is independent on* r.

By Theorem 5.6, we can derive many new Chung-Feller Theorems for lattice paths. As example, we give the following corollary.

Corollary 5.7. *Let* $\mathcal{S} = \{(1,1),(1,-3),(2,0),(4,0)\}$, $w(x,y) = 1$ *for any* $(x,y) \in \mathcal{S}$, $l(1,1) = 1$, $l(1,-3) = 0$, $l(4,0) = 2$ *and* $l(2,0) = 1$. *Then the number of lattice paths in* $\bar{\mathscr{L}}_{\mathcal{S}_1,1}$ *with length* $m+1$ *and non-positive root length* r *equal to the number of lattice paths in* $\mathscr{N}_{\mathcal{S}_1,0}$ *with length* m *and independent on* r.

Example 5.8. Let $m = 3$. Let the step set $\mathcal{S}$, the weight function w and the length function l be given as those in Corollary 5.7. We draw all the lattice paths L in $\mathscr{N}_{\mathcal{S}_3,0}$ with $\theta(L) = 3$ as follows:

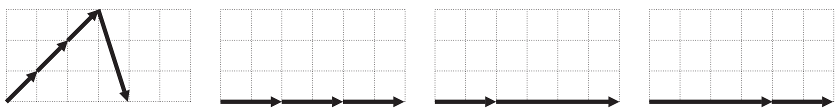

Figure 13. All the lattice paths in $\mathscr{N}_{\mathcal{S}_3,0}$ with $\theta(L) = 3$.

We draw rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_3,1}$ with $\bar{\theta}(\bar{L}) = 3$ and $nprl(\bar{L}) = r$ as follows:

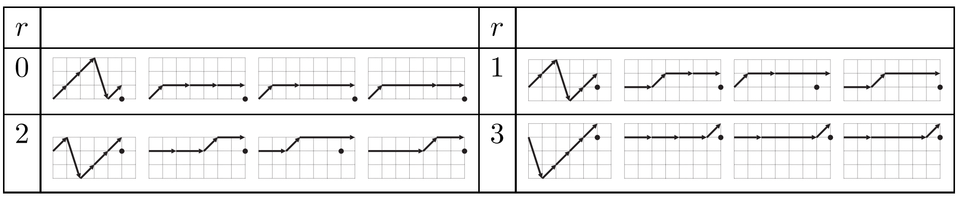

Figure 14. Rooted lattice paths $\bar{L}$ in $\bar{\mathscr{L}}_{\mathcal{S}_3,1}$ with $\bar{\theta}(\bar{L}) = 3$ and $nprl(\bar{L}) = r$.

where the notation "$\bullet$" denotes the root of the corresponding path.

6. Remarks

In this section, we give some observations and Remarks.

6.1. New Chung-Feller theorems

Narayana [10] related cycle permutations of lattice paths to the Chung-Feller theorem. Mohanty's book [9] devotes an entire section to exploring Chung-Feller theorem. We are interested in the Theorem 2 in the page 70 of the book. We state the Theorem as the following lemma.

Lemma 6.1. ([9]) *Given a positive integer n, let $Y = (y_1, \ldots, y_{n+1})$ be a sequence of integers with $1 - n \le y_i \le 1$ for all $i \in [n+1]$ such that $\sum_{i=1}^{n+1} y_i = 1$. Furthermore, let $E(Y) = |\{i \mid \sum_{j=1}^{i} y_j \le 0\}|$. Let Y_i be the i-th cyclic permutation of Y (i.e., $Y_i = (y_i, y_{i+1}, \ldots, y_{n+i+1})$ with $y_{n+r+1} = y_r$). Then there exists a permutation $i_1, \ldots, i_{n+1}$ on the set $[n+1]$ such that $E(Y_{i_1}) > E(Y_{i_2}) > \cdots > E(Y_{i_{n+1}})$.*

Many Chung-Feller theorems are consequences of Lemma 6.1. First, let ϕ be a mapping from $\mathbb{Z}$ to $\mathbb{P}$, where $\mathbb{P}$ is the set of all the positive integers. For any a sequence $Y = (y_1, \ldots, y_{n+1})$, we can obtain a sequence of vectors $(\phi(y_1), y_1)(\phi(y_2), y_2) \cdots (\phi(y_{n+1}), y_{n+1})$. The sequence can be viewed as a lattice path in the plane $\mathbb{Z} \times \mathbb{Z}$ that goes from the origin to the point $(\sum_{i=1}^{n+1} \phi(y_i), \sum_{i=1}^{n+1} y_i)$. We use $P(Y)$ to denote this path.

For example, let $\phi(y) = 1$ for all $y \in \mathbb{Z}$. Let $Y = (y_1, \ldots, y_{n+1})$ be a sequence of integers with $y_i \in \{-1, 1\}$ for all $i \in [n+1]$ such that $\sum_{i=1}^{n+1} y_i = 1$. Then $P(Y)$ is the famous Dyck path that goes from the origin to the point $(n+1, 1)$. Using Lemma 6.1, we derive a Chung-Feller theorem for Dyck paths.

If let $\phi(y) = 1$ for all $y \in \mathbb{Z}$ and let $Y = (y_1, \ldots, y_{n+1})$ be a sequence of integers with $y_i \in \{-1, 0, 1\}$ for all $i \in [n+1]$ such that $\sum_{i=1}^{n+1} y_i = 1$, then $P(Y)$ is the famous Motzkin path that goes from the origin to the point $(n+1, 1)$. Using Lemma 6.1, we derive a Chung-Feller theorem for Motzkin paths.

If let $\phi(0) = 2$, $\phi(y) = 1$ for all $y \neq 0$ and let $Y = (y_1, \ldots, y_{n+1})$ be a sequence of integers with $y_i \in \{-1, 0, 1\}$ for all $i \in [n+1]$ such that $\sum_{i=1}^{n+1} y_i = 1$, then $P(Y)$ is the Schröder path. Using Lemma 6.1, we derive a Chung-Feller theorem for Schröder paths.

On the other hand, as an example, Corollary 3.13 cannot be derived directly as a special case of Lemma 6.1 since there is no mapping ϕ such that $\phi(-1) = 1$ and $\phi(-1) = 5$. See Corollaries 4.13 and 5.7 for more examples. Hence, by the main theorems in this paper, many new Chung-Feller theorems for rooted lattice paths are derived.

6.2. Incomplete Chung-Feller phenomenents

We are given a combinatorial model $\bar{\mathscr{S}}$, two mappings $\bar{\theta}$ and $\bar{\lambda}$, where $0 \leq \bar{\lambda}(S) \leq \bar{\theta}(S)$ for every $S \in \bar{\mathscr{S}}$. For any $m \geq 0$, let A_m be a subset of $\{0, 1, \ldots, m\}$. We say that $(\bar{\mathscr{S}}, \bar{\theta}, \bar{\lambda})$ has *incomplete Chung-Feller phenomenents* if the number of combinatorial structures S in the set $\bar{\mathscr{S}}$ with $\bar{\theta}(S) = m$ and $\bar{\lambda}(S) = r$ is a constant and independent on r for all $r \in A_m$. Chung-Feller theorems which we discuss in the foregoing sections are the case $A_m = \{0, 1, \ldots, m\}$ for all $m \geq 0$.

Example 6.2. There is an example for incomplete Chung-Feller theorems. Let $D = (x_1, y_1) \cdots (x_{2m}, y_{2m})$ be a sequence of vectors in the set $\{(1,1), (1,-1)\}$ such that $\sum_{i=1}^{2m} y_i = 0$. Then D is an m-Dyck path in the plane $\mathbb{Z} \times \mathbb{Z}$. The semi-length of D is m. For every $i \in [2m-1]$, we say that a subsequence of length 2 $(x_i, y_i)(x_{i+1}, y_{i+1})$ in D is a *peak* of D if $y_i = 1$ and $y_{i+1} = -1$. We use $p(D)$ to denote the number of peaks in a m-Dyck path D. Let $s_{m,r,k}$ be the number of m-Dyck paths with r nonpositive up-steps such that $p(D) = k$ or $m-k$. By datas obtained by computer searchs, we observe that $s_{m,r,k}$ is independent on r for all $r \in [m-1]$. In [6], we proved this proposition by using the ideal of the function of Chung-Feller type. In particular, we define generating functions $S(x,y,z) = \sum_{m \geq 0} \sum_{r=1}^{m-1} \sum_{k=1}^{m-1} s_{m,r,k} x^k y^r z^m$ and $\alpha(x,z) = \sum_{m \geq 0} \sum_{k=1}^{m-1} s_{m,1,k} x^k z^m$. By simple computations, we can get formulars for $S(x, y, z)$ and $\alpha(x, z)$. Note that

$$S(x, y, z) = \frac{y\alpha(x, z) - \alpha(x, yz)}{1 - y}.$$

Hence, the proposition holds. For $r = 1, 2, 3$, and $k = 1, 2$ we draw all the 4-Dyck paths with r nonpositive up-steps which have k or $m - k$ peaks as follows.

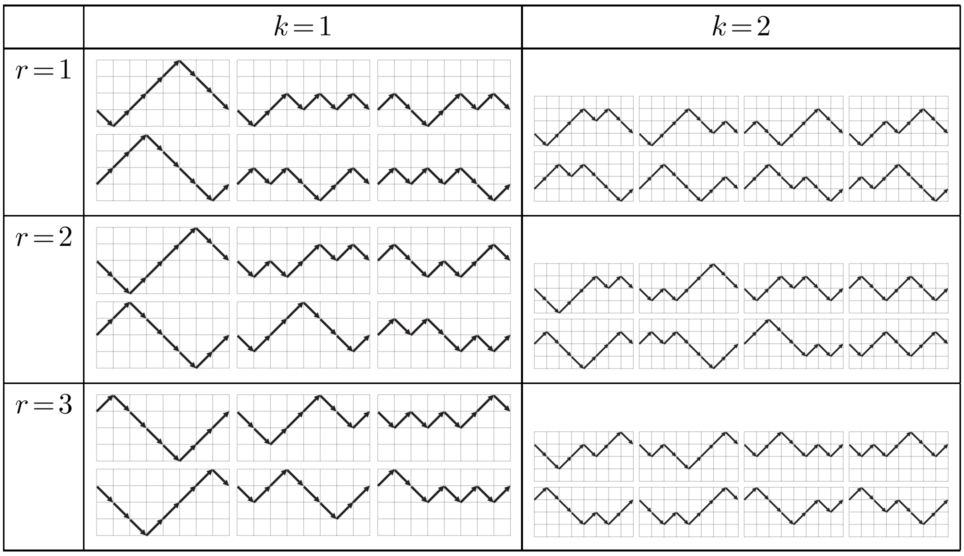

Figure 15. All the 4-Dyck paths with non-positive r which have k or $m - k$ peaks for $r = 1, 2, 3$ and $k = 1, 2$.

Example 6.3. Fix two positive integers m and n. Let $\bar{\mathscr{S}}_{m,n}$ be a set of sequences of vectors $(x_1, y_1) \cdots (x_{n+1}, y_{n+1})$ such that $\sum_{i=1}^{n+1} y_i = 1$, $1 - n \leq y_i \leq 1$, $\sum_{i=1}^{n+1} x_i = m$ and $x_i \geq 1$. Each element in $\bar{\mathscr{S}}_{m,n}$ can be viewed as a lattice path in the plane $\mathbb{Z} \times \mathbb{Z}$. For every $S \in \bar{\mathscr{S}}_{m,n}$, let $NP(S) = \{i \mid \sum_{j=1}^{i} y_j \leq 0\}$. Define the non-positive length of S, denoted by $npl(S)$, as $npl(S) = \sum_{i \in NP(S)} x_i$. Let $s_{m,n,r}$ be the number of lattice paths S in $\bar{\mathscr{S}}_{m,n}$ with $npl(S) = r$. By bijection method, we proved that $s_{m,n,r}$ is a constant and independent on r for all $r \in [m - 1]$ in [7]. For $r = 0, 1, 2, 3$, we draw 18 lattice paths in $\bar{\mathscr{S}}_{4,2}$ with non-positive length r as follows.

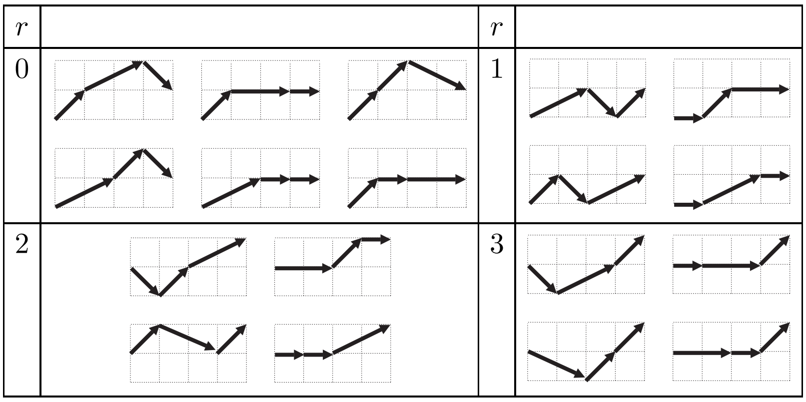

Figure 16. All the lattice paths in $\bar{\mathscr{S}}_{4,2}$ with non-positive length r for $r = 0, 1, 2, 3$.

Let $S = (x_1, y_1)\cdots(x_{n+1}, y_{n+1}) \in \bar{\mathscr{S}}_{m,n}$. Let $a_0 = 0$, $b_0 = 0$, $a_i = \sum_{j=1}^{i} x_j$ and $b_i = \sum_{j=1}^{i} y_j$ for every $i \in [n+1]$. We suppose that (a_i, b_i) is the rightmost minimum point of S. Define the rightmost minimum length of S, denoted by $rml(S)$, as $rml(S) = a_i$. Let $\bar{s}_{m,n,r}$ be the number of lattice paths S in $\bar{\mathscr{S}}_{m,n}$ with $rml(S) = r$. By bijection method, we proved that $\bar{s}_{m,n,r}$ is a constant and independent on r for all $r \in [m-1]$ in [7]. For $r = 0, 1, 2, 3$, we draw 18 lattice paths in $\bar{\mathscr{S}}_{4,2}$ with rightmost minimum length r as follows.

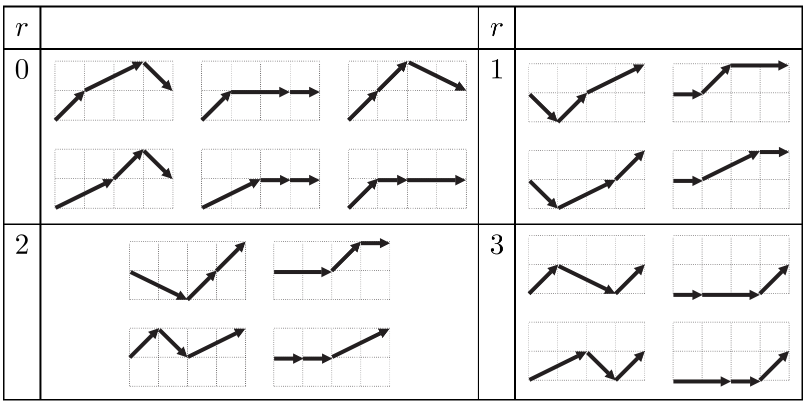

Figure 17. All the lattice paths in $\bar{\mathscr{S}}_{4,2}$ with non-positive length r for $r = 0, 1, 2, 3$.

Acknowledgments

The authors would like to thank Professor Christian Krattenthaler for his valuable suggestions.

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