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Uniform partition extensions, a generating functions perspective

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Abstract In this paper, a bivariate generating function $CF(x,y) = \frac{f(x)-yf(xy)}{1-y}$ is investigated, where $f(x) = \sum_{n\geqslant 0} f_n x^n$ is a generating function satisfying the functional equation $f(x) = 1 + \sum_{j=1}^{r} \sum_{i=j-1}^{m} a_{ij} x^i f(x)^j$. In particular, we study lattice paths in which their end points are on the line $y = 1$. Rooted lattice paths are defined. It is proved that the function $CF(x,y)$ is a generating function defined on some rooted lattice paths with end point on $y = 1$. So, by a simple and unified method, from the view of lattice paths, we obtain two combinatorial interpretations of this bivariate function and derive two uniform partitions on these rooted lattice paths.

Keywords Chung-Feller theorem, Dyck path, Motzkin path, Schröder path

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1 Introduction

Let Ω be a set of some combinatorial structures and θ a mapping from Ω to $\mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. We call Ω a combinatorial model and θ a parameter defined on Ω. Let $\overline{\Omega}$ be a combinatorial model as well. Let $\bar{\theta}$ and δ be two parameters defined on $\overline{\Omega}$ such that $0 \leqslant \delta(X) \leqslant \bar{\theta}(X)$ for every $X \in \overline{\Omega}$. A partition of a given set is said to be uniform if all the partition classes have the same cardinality.

Definition 1.1. $(\overline{\Omega}, \bar{\theta}, \delta)$ is a *uniform partition extension* for (Ω, θ) if the number of combinatorial structures $X \in \overline{\Omega}$ such that $\bar{\theta}(X) = n$ and $\delta(X) = k$ is equal to the number of combinatorial structures $X \in \Omega$ such that $\theta(X) = n$ for any $k \in \{0, 1, \ldots, n\}$. We say that δ is a uniform partition parameter for $(\overline{\Omega}, \bar{\theta})$.

The uniform partition extension is also called *the Chung-Feller extension* in [10]. In communications, some experts suggest the authors of the paper to use the name "uniform partition extension" since this name can describe the nature of problems in the paper. Hence, we rename "Chung-Feller extension" as "uniform partition extension".

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The combinatorial structures discussed in this paper are lattice paths. Let $\mathbb{Z}$ be the set of integers. An n-lattice path is a sequence $L=(x_i,y_i)_{i=1}^n=(x_1,y_1)(x_2,y_2)\cdots(x_n,y_n)$ of two-dimensional vectors, where $(x_i,y_i)\in\mathbb{N}\times\mathbb{Z}$. Every vector (x_i,y_i) is called a step of L. Let $a_0=0$, $h_0=0$, and

$$a_i=\sum_{j=1}^{i}x_j,\quad h_i=\sum_{j=1}^{i}y_j,\quad 1\leqslant i\leqslant n,$$

then L corresponds to the following sequence of points

$$(a_0,h_0)(a_1,h_1)\cdots(a_n,h_n).$$

We call (a_n,h_n) the end point of L. Let $\mathscr{N}(L)$ be the set of positions where the path L is on or below the x-axis, i.e.,

$$\mathscr{N}(L)=\{i\mid h_i\leqslant 0 \text{ and } i\in\{1,2,\ldots,n\}\}.$$

Moreover, we use $m(L)$ to denote the rightmost position of minimum vertical position in the path L, i.e.,

$$m(L)=\max\{i\mid h_i\leqslant h_j \text{ for any } 0\leqslant j\leqslant n \text{ and } i\in\{0,1,\ldots,n\}\}.$$

Let $S\subseteq\mathbb{N}\times\mathbb{Z}$ be a finite set. An S-path is a lattice path $L=(x_i,y_i)_{i=1}^n$ such that $(x_i,y_i)\in S$ for any $i\in\{1,2,\ldots,n\}$. An (S,h)-path is an S-path if $\sum_{i=1}^n y_i=h$. An (S^+,h)-path is an (S,h)-path if such a path is in the first quadrant.

Example 1.2. Let $S=\{(1,1),(1,-1)\}$. Then $(S^+,0)$-paths are the famous Dyck paths. Let Ω be the set of Dyck paths. For any Dyck path $L=(x_i,y_i)_{i=1}^{2n}$, the end point of L is $(2n,0)$. So the integer n is called the semi-length of L and denoted by $\theta(L)$. It is well known that the number c_n of Dyck paths with semi-length n is

$$\frac{1}{n+1}\binom{2n}{n},$$

which is called the n-th Catalan number. Its generating function $C(x)=\sum_{n\geqslant 0}c_nx^n$ satisfies the functional equation

$$C(x)=1+xC(x)^2.$$

$(S,0)$-paths are also called free Dyck paths. Let $\bar{\Omega}$ be the set of free Dyck paths. For any free Dyck path $L=(x_i,y_i)_{i=1}^{2n}$, let $\bar{\theta}(L)=n$ and $\delta(L)$ the number of steps (x_i,y_i) in L such that $i\in\mathscr{N}(L)$ and $(x_i,y_i)=(1,1)$. Chung-Feller theorem [4] shows a uniform partition property for free Dyck paths: the number of free Dyck paths L with $\bar{\theta}(L)=n$ and $\delta(L)=k$ is the same as the n-th Catalan number c_n for any $k\in\{0,1,\ldots,n\}$. Hence $(\bar{\Omega},\bar{\theta},\delta)$ is a uniform partition extension for (Ω,θ), and δ is a uniform partition parameter for $(\bar{\Omega},\bar{\theta})$.

Chung-Feller theorem was proved by many different methods. Please refer to [1, 2, 4, 9, 12, 14–16, 20]. Researchers also attempted to generalize this theorem, see also [3, 5–8, 11].

Example 1.3. Let $S=\{(1,1),(1,-1),(5,-1)\}$. There are exactly 6 $(S^+,0)$-paths with the end point $(6,0)$ (see Figure 1).

Let Ω be the set of $(S^+,0)$-paths. For every $L=(x_i,y_i)_{i=1}^n\in\Omega$, let

$$\theta(L)=\frac{1}{2}\sum_{i=1}^{n}x_i.$$

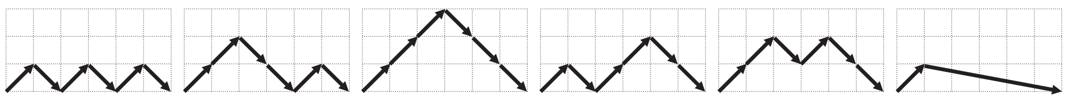

Figure 1 The 6 $(S^+,0)$-paths with the end point $(6,0)$. The starting point of each path is $(0,0)$. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0,0)$

What is a uniform partition extension for (Ω, θ)? Similar to the situation for Dyck paths and free Dyck paths, are $(S, 0)$-paths a uniform partition extension for $(S^+, 0)$-paths? We find that there are exactly 22 $(S, 0)$-paths with the end point $(6, 0)$ (see Figure 2), but 22 cannot be divided by 6. So $(S, 0)$-paths are not a uniform partition extension for $(S^+, 0)$-paths.

Are $(S, 1)$-paths a uniform partition extension for $(S^+, 0)$-paths? There are exactly 23 $(S, 1)$-paths with the end point $(6, 0)$, but 23 cannot be divided by 6 as well. So $(S, 1)$-paths are not a uniform partition extension for $(S^+, 0)$-paths.

In this paper, we always let $f(x)$ satisfy the following functional equation

$$f(x) = 1 + \sum_{j=1}^{r} \sum_{i=j-1}^{m} a_{ij} x^i f(x)^j. \tag{1.1}$$

Example 1.4. Let us consider $S = \{(1, 1), (1, -1), (5, -1)\}$. Let $f_0 = 1$ and f_n be the number of $(S^+, 0)$-paths $L = (x_i, y_i)_{i=1}^n$ such that $\frac{1}{2}\sum_{i=1}^n x_i = n$. Define a generating function $f(x)$ to be

$$f(x) = \sum_{n \geqslant 0} f_n x^n.$$

Let L be a nonempty $(S^+, 0)$-path. Clearly $(x_1, y_1) = (1, 1)$. Let (x_j, y_j) be the first step which returns to the x-axis. So the path L is decomposed into $L = (x_1, y_1)L_1(x_j, y_j)L_2$, where L_1 and L_2 are $(S^+, 0)$-paths. Hence $f(x)$ satisfies the functional equation

$$f(x) = 1 + (x + x^3)f(x)^2,$$

which is a special case of (1.1).

We investigate the following bivariate generating function

$$CF(x, y) = \frac{f(x) - yf(xy)}{1 - y},$$

which is called a function of Chung-Feller type for $f(x)$ in [10]. From the view of lattice paths, by a simple and unified method, we give two combinatorial interpretations of the function $CF(x, y)$ of Chung-Feller type for $f(x)$. So many uniform partition extensions for lattice paths can be derived by the main results of this paper.

Lattice path interpretations for the generating function $f(x)$ which satisfies the functional equation (1.1) go back at least to the references [17, 19]. Note that we have $1 \leqslant j \leqslant r$ and $j - 1 \leqslant i \leqslant m$ in the functional equation (1.1). In this paper, to give combinatorial interpretations for $f(x)$, we let

$$S = \{(\beta, 1)\} \cup \{(\alpha_i, -j + 1) \mid 1 \leqslant j \leqslant r \text{ and } j - 1 \leqslant i \leqslant m\},$$

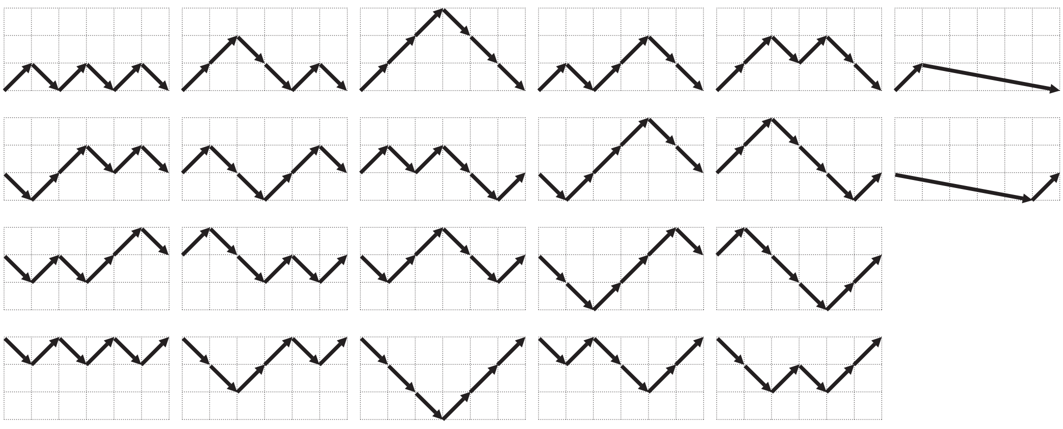

Figure 2 The 22 $(S, 0)$-paths with the end point $(6, 0)$. The starting point of each path is $(0, 0)$. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0, 0)$

where β and α_i are nonnegative integers, and $\alpha_i \neq 0$ if $-j+1=0$.

Let $w : S \mapsto \mathbb{R}$ and $\theta : S \mapsto \mathbb{N}$ be two functions, where $\mathbb{R}$ is the set of real numbers. We call w a weight function of S, and θ a length function of S. For any S-path $L = (x_i, y_i)_{i=1}^n$, let

$$w(L) = \prod_{i=1}^n w(x_i, y_i) \quad \text{and} \quad \theta(L) = \sum_{i=1}^n \theta(x_i, y_i).$$

Use $\Omega_n = \Omega_n(S)$ to denote the set of $(S^+, 0)$-paths L such that $\theta(L) = n$ and let $\Omega = \Omega(S) = \bigcup_{n\geqslant 0} \Omega_n(S)$. Let $f_0 = 1$ and f_n be the sum of weights of paths in $\Omega_n(S)$ for $n \geqslant 1$, i.e.,

$$f_n = \sum_{L\in\Omega_n} w(L).$$

Note that f_n is the number of $(S^+, 0)$-paths in Ω_n if $w(L) = 1$ for any $L \in \Omega_n$. Define a generating function $f(x)$ to be

$$f(x) = \sum_{L\in\Omega} w(L)x^{\theta(L)} = \sum_{n\geqslant 0} f_n x^n.$$

Let us consider a nonempty $(S^+, 0)$-path $L = (x_i, y_i)_{i=1}^n$. Clearly $(x_n, y_n) = (\alpha_i, -j-1)$ for some $1 \leqslant j \leqslant r$ and $j-1 \leqslant i \leqslant m$. Let (x_{i_k}, y_{i_k}) be the last step $(\beta, 1)$ which goes from the line $y = k-1$ to the line $y = k$ for any $1 \leqslant k \leqslant j-1$. L can be decomposed into

$$L = L_0(x_{i_1}, y_{i_1})L_1(x_{i_2}, y_{i_2})L_2 \cdots (x_{i_{j-1}}, y_{i_{j-1}})L_{j-1}(\alpha_i, -j+1),$$

where $L_0, L_1, \dots, L_{j-1}$ are $(S^+, 0)$-paths.

By the above decomposition, in this paper, we suppose that w and θ satisfy

$$\begin{aligned} &w(\beta, 1) \neq 0, \quad w(\alpha_i, -j+1)(w(\beta, 1))^{j-1} = a_{ij}, \\ &\theta(\beta, 1) = 1, \quad \theta(\alpha_i, -j+1) = i-j+1, \end{aligned}$$

for any $j \in \{1, 2, \dots, r\}$ and $i \in \{j-1, j, \dots, m\}$. So we obtain the term

$$(w(\beta, 1))^{j-1} w(\alpha_i, -j+1) x^{j-1} x^{i-j+1} f(x)^j = a_{ij} x^i f(x)^j.$$

This proves the following proposition.

Proposition 1.5. *The generating function $f(x)$ satisfies the functional equation*

$$f(x) = 1 + \sum_{j=1}^r \sum_{i=j-1}^m a_{ij} x^i f(x)^j.$$

Example 1.6. Let $f(x) = \sum_{n\geqslant 0} f_n x^n$ satisfy the functional equation

$$f(x) = 1 + (x + x^3) f(x)^2.$$

Note that $a_{ij} = 0$ when $j = 1$, or $j = 2$ and $i \in \{0, 2\}$. Take $\beta = 1$, $\alpha_1 = 1$ and $\alpha_3 = 5$. So, we have

$$\begin{aligned} &S = \{(1, 1), (1, -1), (5, -1)\}, \\ &w(1, 1) = w(1, -1) = w(5, -1) = 1, \\ &\theta(1, 1) = 1, \quad \theta(1, -1) = 0, \quad \theta(5, -1) = 2. \end{aligned}$$

So, f_n is the number of $(S^+, 0)$-paths L such that $\theta(L) = n$ for any $n \geqslant 1$. For any $(S^+, 0)$-path L, the end point of L is $(2\theta(L), 0)$. We have $f_3 = 6$ by Example 1.3.

Definition 1.7. Let $L = (x_i, y_i)_{i=1}^n$ be an S-path. For any $s \in \{0, 1, \dots, \theta(x_n, y_n) - 1\}$, we call the pair (L, s) a θ-rooted $(S^+, 0)$-path if L is an $(S^+, 0)$-path, and a θ-rooted $(S, 1)$-path if L is an $(S, 1)$-path.

Example 1.8. Let $S = \{(1,1),(1,-1),(5,-1)\}$. The weight function w and the length function θ of S are given as those in Example 1.6. Let

$$L = (1,-1)(1,1)(1,1)(1,-1)(1,1)(1,1)(5,-1).$$

The end point of L is $(11,1)$. By Definition 1.7, there are two θ-rooted $(S,1)$-paths $(L,0)$ and $(L,1)$.

In order to denote a θ-rooted S-path (L,s) in the plane, we draw the sequence of vectors as a path which starts at the origin $(0,0)$, and add a black node "$\bullet$" at the point $(a-s,0)$ if the end point of L is (a,h) (see Figure 3).

Let us consider the path

$$L = (1,1)(1,1)(5,-1)(1,-1)(1,1)(1,1)(1,-1).$$

Since $\theta(1,-1) = 0$, no θ-rooted $(S,1)$-path (L,s) exists for any number s by Definition 1.7.

Definition 1.9. For any S-path $L = (x_i,y_i)_{i=1}^n$, let

$$\mathscr{N}_\theta(L) = \sum_{i\in\mathscr{N}(L)} \theta(x_i,y_i).$$

For any $s \in \{0,1,\ldots,\theta(x_n,y_n)-1\}$, let

$$\delta(L,s) = \mathscr{N}_\theta(L) + s.$$

Example 1.10. Let $S = \{(1,1),(1,-1),(5,-1)\}$. The weight function w and the length function θ of S are given as those in Example 1.6. For the S-path L in Example 1.8, we have

$$\mathscr{N}(L) = \{1,2,4\}, \quad \mathscr{N}_\theta(L) = 1, \quad \delta(L,0) = 1, \quad \delta(L,1) = 2.$$

Use $\overline{\Omega}_n = \overline{\Omega}_n(S)$ to denote the set of θ-rooted $(S,1)$-paths (L,s) such that $\theta(L) = n+1$ and let $\overline{\Omega} = \overline{\Omega}(S) = \bigcup_{n\geqslant 0}\overline{\Omega}_n(S)$.

One of the main results of this paper is stated as the following theorem.

Theorem 1.11. *Let $f(x) = \sum_{n\geqslant 0} f_n x^n$ satisfy the functional equation*

$$f(x) = 1 + \sum_{j=1}^{r}\sum_{i=j-1}^{m} a_{ij}x^i f(x)^j.$$

(1) *Let*

$$CF(x,y) = \frac{yf(xy) - f(x)}{y-1}.$$

Then

$$CF(x,y) = \sum_{(L,k)\in\overline{\Omega}} w(L) y^{\delta(L,k)} x^{\theta(L)-1}.$$

(2) *Let $f_{n,k}$ be the sum of weights of θ-rooted $(S,1)$-paths (L,s) such that $\theta(L) = n+1$ and $\delta(L,s) = k$. Then $f_{n,k} = f_n$ for any $k \in \{0,1,\ldots,n\}$.*

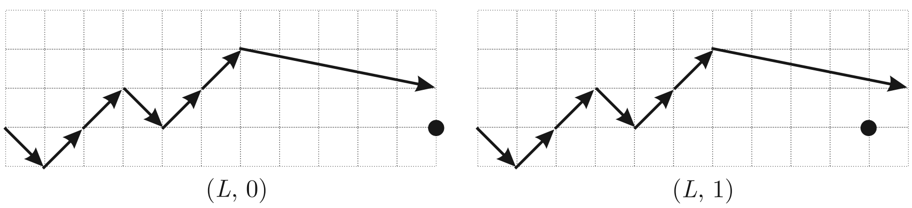

Figure 3 The two θ-rooted S-paths. The starting point of each path is $(0,0)$. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0,0)$

For any $L \in \overline{\Omega}_n$, notice that the line $y = 0$ partitions L into two parts of up and down. We say that the uniform partition extension obtained by Theorem 1.11 is of *up-down type* and δ is a uniform partition parameter of up-down type.

Example 1.12. Let $S = \{(1,1), (1,-1), (5,-1)\}$. The weight function w and the length function θ of S are given as those in Example 1.6. There are exactly 6 $(S^+, 0)$-paths L such that $\theta(L) = 3$ (see Figure 1). Using the above results, we have the number of θ-rooted $(S,1)$-paths (L, s) such that $\theta(L) = 4$ and $\delta(L, s) = k$ is 6 for any $k \in \{0, 1, 2, 3\}$ (see Figure 4).

So, θ-rooted $(S,1)$-paths are a uniform partition extension for $(S^+, 0)$-paths and δ is a uniform partition parameter.

Definition 1.13. For any S-path $L = (x_i, y_i)_{i=1}^n$, let

$$h_0 = 0 \quad \text{and} \quad h_i = \sum_{j=1}^{i} y_j$$

for any $1 \leqslant i \leqslant n$. Recall that $m(L)$ denotes the largest index i with $h_i = \min_{0 \leqslant j \leqslant n} h_j$. Let

$$\mathscr{M}_\theta(L) = \sum_{i=1}^{m(L)} \theta(x_i, y_i).$$

For any $s \in \{0, 1, \ldots, \theta(x_n, y_n) - 1\}$, let

$$\delta'(L, s) = \mathscr{M}_\theta(L) + s.$$

Example 1.14. Let $S = \{(1,1), (1,-1), (5,-1)\}$. The weight function w and the length function θ of S are given as those in Example 1.6. For the S-path L in Example 1.8, we have

$$m(L) = 1, \quad \mathscr{M}_\theta(L) = 0, \quad \delta'(L, 0) = 0, \quad \delta'(L, 1) = 1.$$

We state another main result of this paper as the following theorem.

Theorem 1.15. *Let $f(x) = \sum_{n \geqslant 0} f_n x^n$ satisfy the functional equation*

$$f(x) = 1 + \sum_{j=1}^{r} \sum_{i=j-1}^{m} a_{ij} x^i f(x)^j.$$

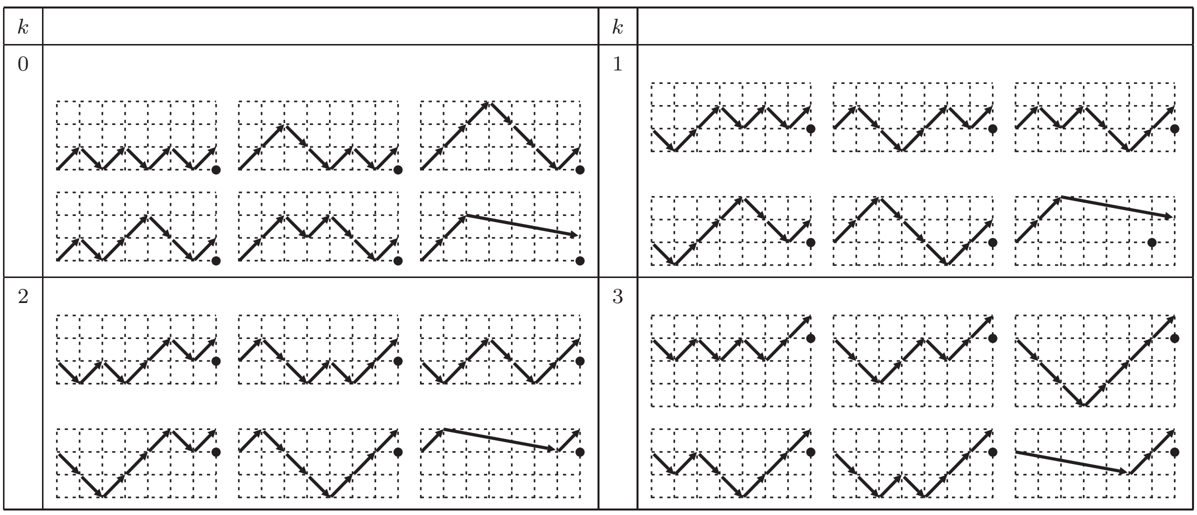

Figure 4 The 24 θ-rooted $(S,1)$-paths (L, s) such that $\theta(L) = 4$. The starting point of each path is $(0,0)$. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0,0)$

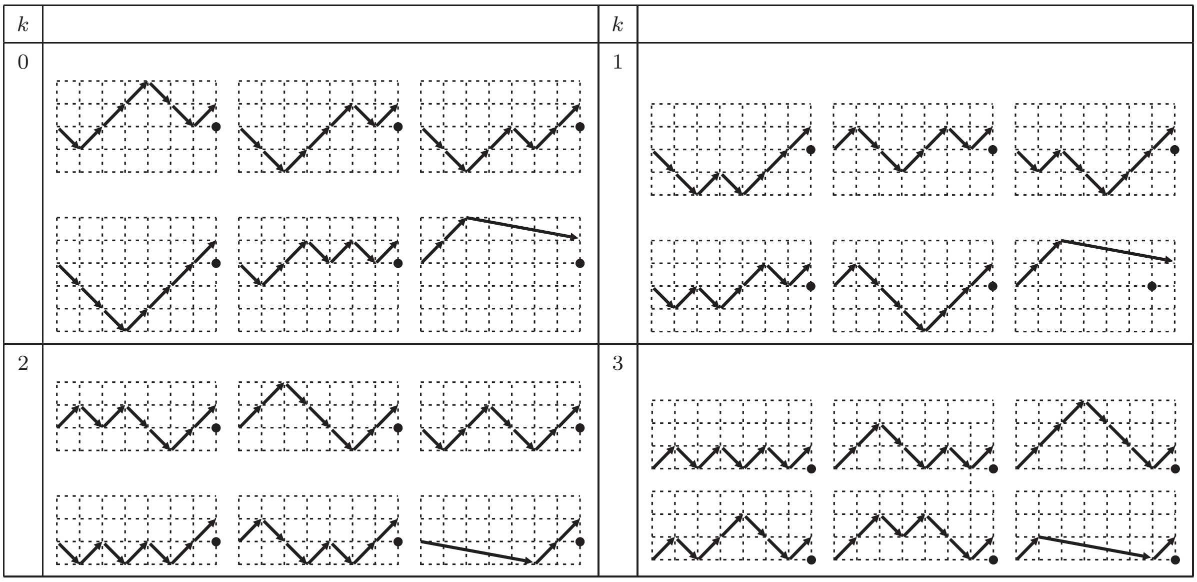

Figure 5 The 24 θ-rooted $(S,1)$-paths (L,s) such that $\theta(L)=4$. The starting point of each path is $(0,0)$. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0,0)$

(1) *Let*

$$CF(x,y)=\frac{yf(xy)-f(x)}{y-1}.$$

Then

$$CF(x,y)=\sum_{(L,s)\in\overline{\Omega}} w(L)y^{\delta'(L,s)}x^{\theta(L)-1}.$$

(2) *Let* $f'_{n,k}$ *be the sum of weights of* θ*-rooted* $(S,1)$*-paths* (L,s) *such that* $\theta(L)=n+1$ *and* $\delta'(L,s)=k$. *Then* $f'_{n,k}=f_n$ *for any* $k\in\{0,1,\dots,n\}$.

For any $L\in\overline{\Omega}_n$, notice that the line $x=\sum_{i=1}^{m(L)}x_i$ partitions L into two parts of left and right. We say that the uniform partition extension obtained by Theorem 1.15 is of *left-right type* and δ' is a uniform partition parameter of left-right type.

Example 1.16. Let $S=\{(1,1),(1,-1),(5,-1)\}$. The weight function w and the length function θ of S are given as those in Example 1.6. Using the above results, we have the number of θ-rooted $(S,1)$-paths (L,s) such that $\theta(L)=4$ and $\delta'(L,s)=k$ is 6 for any $k\in\{0,1,2,3\}$ (see Figure 5).

So, θ-rooted $(S,1)$-paths are a uniform partition extension for $(S^+,0)$-paths and δ' is a uniform partition parameter.

The rest of this paper is organized as follows. In Section 2, we study the uniform partition parameter δ and prove Theorem 1.11. In Section 3, we study the uniform partition parameter δ' and prove Theorem 1.15. In Section 4, we give some conclusions and remarks.

2 Uniform partition parameter δ of up-down type

In this section, we will prove Theorem 1.11. The simple expression of $CF(x,y)$ hides many pieces of information. To obtain these information, we need the following identical transformations for $CF(x,y)$.

Lemma 2.1. *Let* $f(x)$ *satisfy the functional equation*

$$f(x)=1+\sum_{j=1}^{r}\sum_{i=j-1}^{m}a_{ij}x^if(x)^j.$$

Then

$$CF(x,y)=\frac{1+\sum_{j=1}^{r}\sum_{i=j-1}^{m}\sum_{s=0}^{i-j}a_{ij}x^iy^sf(x)^j}{1-\sum_{j=1}^{r}\sum_{i=j-1}^{m}\sum_{k=0}^{j-1}a_{ij}x^iy^{i-k}f(x)^kf(xy)^{j-1-k}}.$$

Proof. Since

$$f(x)=1+\sum_{j=1}^{r}\sum_{i=j-1}^{m}a_{ij}x^if(x)^j,$$

we have

$$yf(xy)=y+\sum_{j=1}^{r}\sum_{i=j-1}^{m}a_{ij}x^iy^{i+1}f(xy)^j.$$

Thus,

$$f(x)-yf(xy)=1-y+\sum_{j=1}^{r}\sum_{i=j-1}^{m}a_{ij}x^i(f(x)^j-y^{i+1}f(xy)^j). \tag{2.1}$$

Note that

$$\begin{aligned}f(x)^j-y^{i+1}f(xy)^j&=f(x)^j-y^{i-j+1}f(x)^j+y^{i-j+1}f(x)^j-y^{i+1}f(xy)^j\\&=f(x)^j(1-y)\sum_{k=0}^{i-j}y^k+y^{i-j+1}(f(x)^j-y^jf(xy)^j)\\&=(1-y)\left(f(x)^j\sum_{k=0}^{i-j}y^k+y^{i-j+1}\frac{f(x)-yf(xy)}{1-y}\sum_{k=0}^{j-1}f(x)^k[yf(xy)]^{j-1-k}\right)\\&=(1-y)\left(f(x)^j\sum_{k=0}^{i-j}y^k+y^{i-j+1}CF(x,y)\sum_{k=0}^{j-1}f(x)^k[yf(xy)]^{j-1-k}\right).\end{aligned}$$

Substituting it into (2.1), we obtain

$$CF(x,y)=1+\sum_{j=1}^{r}\sum_{i=j-1}^{m}\sum_{k=0}^{i-j}a_{ij}x^if(x)^jy^k+CF(x,y)\sum_{j=1}^{r}\sum_{i=j-1}^{m}\sum_{k=0}^{j-1}a_{ij}x^iy^{i-k}f(x)^kf(xy)^{j-1-k}.$$

Rearranging it gives rise to the desired formula. This completes the proof. □

Let $\mathcal{G}_n=\mathcal{G}_n(S)$ be the set of θ-rooted $(S^+,0)$-paths (L,s) such that $\theta(L)=n$ and $\mathcal{G}=\mathcal{G}(S)=\bigcup_{n\geqslant 0}\mathcal{G}_n$.

Lemma 2.2. *Let*

$$G(x,y)=\sum_{(L,s)\in\mathcal{G}}w(L)y^sx^{\theta(L)}=\sum_{n\geqslant 0}\sum_{(L,s)\in\mathcal{G}_n}w(L)y^sx^n.$$

Then

$$G(x,y)=1+\sum_{j=1}^{r}\sum_{i=j-1}^{m}\sum_{s=0}^{i-j}a_{ij}x^iy^sf(x)^j.$$

Proof. Let $(L,s)\in\mathcal{G}$ be a nonempty θ-rooted $(S^+,0)$-path. Suppose $L=(x_i,y_i)_{i=1}^n$. Then

$$(x_n,y_n)=(\alpha_i,-j+1)$$

for some $j\in\{1,2,\ldots,r\}$ and $i\in\{j-1,j,\ldots,m\}$.

For any $k\in\{1,2,\ldots,j-1\}$, let

$$i_k=\max\left\{u\;\middle|\;\sum_{t=0}^{u}y_t=k\text{ and }(x_u,y_u)=(\beta,1)\right\}.$$

So L is decomposed into

$$L=L_0(x_{i_1},y_{i_1})L_1(x_{i_2},y_{i_2})L_2\cdots(x_{i_{j-1}},y_{i_{j-1}})L_{j-1}(\alpha_i,-j+1),$$

where $L_0, L_1, \ldots, L_{j-1}$ are $(S^+, 0)$-paths.

In Figure 6, we draw the decomposition for θ-rooted $(S^+, 0)$-paths.

Note that $(w(\beta,1))^{j-1}w(\alpha_i, -j+1) = a_{ij}$. This provides the term

$$(w(\beta,1))^{j-1}w(\alpha_i,-j+1)x^{j-1}x^{i-j+1}y^s f(x)^j = a_{ij}x^i y^s f(x)^j.$$

Since $0 \leqslant s \leqslant i-j$, we have

$$G(x,y) = 1 + \sum_{j=1}^{r}\sum_{i=j-1}^{m}\sum_{s=0}^{i-j} a_{ij}x^i y^s f^j(x). \qquad \square$$

Definition 2.3. A lattice path $L = (x_i, y_i)_{i=1}^n$ is *prime* if it satisfies $\sum_{i=1}^{j} y_i \neq 0$ for every $j \in \{1, 2, \ldots, n-1\}$. Let $\mathcal{P}_n = \mathcal{P}_n(S)$ be the set of prime $(S, 0)$-paths L such that $\theta(L) = n$ and $\mathcal{P} = \mathcal{P}(S) = \bigcup_{n\geqslant 1}\mathcal{P}_n$.

Lemma 2.4. *Let*

$$P(x,y) = \sum_{L\in\mathcal{P}} w(L)y^{\mathcal{N}_\theta(L)}x^{\theta(L)} = \sum_{n\geqslant 1}\sum_{L\in\mathcal{P}_n} w(L)y^{\mathcal{N}_\theta(L)}x^n.$$

Then

$$P(x,y) = \sum_{j=1}^{r}\sum_{i=j-1}^{m}\sum_{k=0}^{j-1} a_{ij}x^i y^{i-s} f(x)^k f(xy)^{j-1-k}.$$

Proof. First, for any lattice path $L = (x_i, y_i)_{i=1}^n$, use $\overleftarrow{L}$ to denote the following lattice path

$$(x_n, y_n)(x_{n-1}, y_{n-1})\cdots(x_1, y_1).$$

Let $L \in \mathcal{P}$ be a nonempty prime $(S, 0)$-path. Let

$$l = \min\left\{u \,\middle|\, \sum_{t=1}^{u} y_t \leqslant 0\right\}.$$

Then $(x_l, y_l) = (\alpha_i, -j+1)$ for some $j \in \{1, 2, \ldots, r\}$ and $i \in \{j-1, j, \ldots, m\}$. Suppose $\sum_{t=1}^{l-1} y_t = k$. Then

$$k \geqslant 0 \quad \text{and} \quad \sum_{t=1}^{l} y_t = -j+1+k \leqslant 0.$$

For any $t \in \{1, 2, \ldots, k\}$, let

$$u_t = \max\left\{u \,\middle|\, \sum_{s=1}^{u} y_s = t \text{ and } (x_u, y_u) = (\beta, 1)\right\}.$$

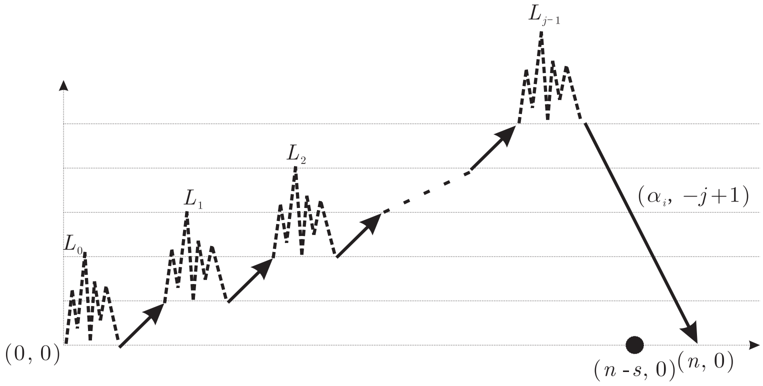

Figure 6 The decomposition for θ-rooted $(S^+, 0)$-paths. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0, 0)$

For any $t \in \{1, 2, \dots, j-1-k\}$, let

$$u'_t = \min\left\{u \,\middle|\, \sum_{s=1}^{u} y_s = -t+1 \text{ and } (x_u, y_u) = (\beta, 1)\right\}.$$

So L is decomposed into

$$L = (x_{u_1}, y_{u_1})L_1 \cdots (x_{u_k}, y_{u_k})L_k(\alpha_i, -j+1)R_{j-1-k}(x_{u'_{j-1-k}}, y_{u'_{j-1-k}}) \cdots R_1(x_{u'_1}, y_{u'_1}),$$

where $L_1, \dots, L_k, \overleftarrow{R}_1, \dots, \overleftarrow{R}_{j-1-k}$ are $(S^+, 0)$-paths (see Figure 7).

This provides the term

$$\begin{aligned}&(w(\beta,1))^{j-1} w(\alpha_i, -j+1) x^k (xy)^{(j-1-k)+i-j+1} f(x)^k f(xy)^{j-1-k}\\ &\quad = a_{ij} x^i y^{i-k} f(x)^k f(xy)^{j-1-k}.\end{aligned}$$

Hence, we have

$$P(x,y) = \sum_{j=1}^{r} \sum_{i=j-1}^{m} \sum_{k=0}^{j-1} a_{ij} x^i y^{i-k} f(x)^k f(xy)^{j-1-k}. \qquad \square$$

Let $\mathcal{H}_n = \mathcal{H}_n(S)$ be the set of $(S,0)$-paths L such that $\theta(L) = n$ and $\mathcal{H} = \mathcal{H}(S) = \bigcup_{n\geqslant 0} \mathcal{H}_n$.

Lemma 2.5. *Let*

$$H(x,y) = \sum_{L\in\mathcal{H}} w(L) y^{\mathcal{N}_\theta(L)} x^{\theta(L)} = \sum_{n\geqslant 0} \sum_{L\in\mathcal{H}_n} w(L) y^{\mathcal{N}_\theta(L)} x^n.$$

Then

$$H(x,y) = \frac{1}{1-P(x,y)}.$$

Proof. Let $L = (x_i, y_i)_{i=1}^n$ be a nonempty $(S,0)$-path. Let $i_0 = 0$ and

$$D = \left\{u \,\middle|\, \sum_{t=1}^{u} y_t = 0\right\}.$$

Suppose $D = \{i_1, i_2, \dots, i_k\}$ where $1 \leqslant i_1 < i_2 \cdots < i_k = n$. Then for any $t \in \{0, 1, \dots, k-1\}$, the lattice path

$$(x_{i_t+1}, y_{i_t+1})(x_{i_t+2}, y_{i_t+2}) \cdots (x_{i_{t+1}}, y_{i_{t+1}})$$

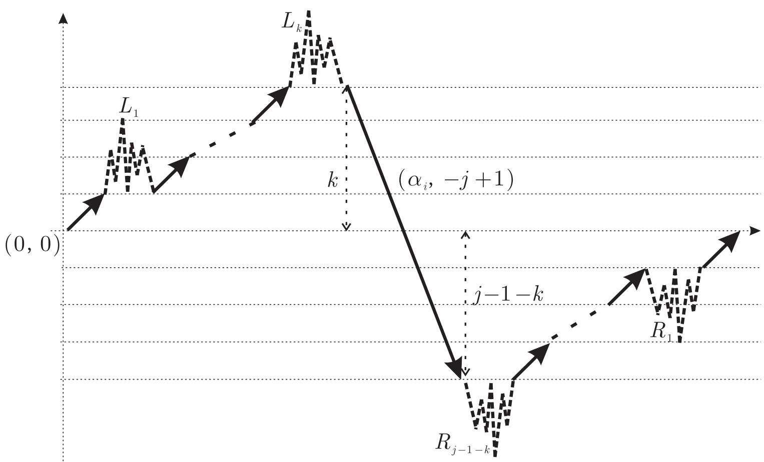

Figure 7 The decomposition for prime $(S,0)$-paths. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0,0)$

is a prime $(S,0)$-path. This provides the term $P(x,y)^k$. Hence, we have

$$H(x,y)=\sum_{k=0}^{\infty}P(x,y)^k=\frac{1}{1-P(x,y)}. \qquad \square$$

Proof of Theorem 1.11. (1) Let

$$\widetilde{CF}(x,y)=\sum_{(L,s)\in\overline{\Omega}}w(L)y^{\delta(L,s)}x^{\theta(L)-1}.$$

Let $(L,s)\in\bar{\Omega}$ be a nonempty θ-rooted $(S,1)$-path and

$$k=\max\left\{u\,\middle|\,\sum_{t=1}^{u}y_t=1 \text{ and } (x_u,y_u)=(\beta,1)\right\}.$$

So we can decompose L into $L=L_1(x_k,y_k)L_2$ such that $L_1\in\mathcal{H}$ is an $(S,0)$-path and $(L_2,s)\in\mathcal{G}$ is a θ-rooted $(S^+,0)$-path (see Figure 8).

By Lemmas 2.2 and 2.5, we have

$$\widetilde{CF}(x,y)=G(x,y)H(x,y).$$

It is easy to see that

$$CF(x,y)=G(x,y)H(x,y)$$

by Lemma 2.1. Hence, we have

$$CF(x,y)=\sum_{(L,s)\in\overline{\Omega}}w(L)y^{\delta(L,s)}x^{\theta(L)-1}.$$

(2) For any θ-rooted $(S,1)$-path (L,s) with $\theta(L)=n+1$, we have $0\leqslant\delta(L,s)\leqslant n$. By Theorem 1.11(1), we have

$$CF(x,y)=\sum_{(L,s)\in\overline{\Omega}}w(L)y^{\delta(L,s)}x^{\theta(L)-1}=\sum_{n\geqslant0}\sum_{k=0}^{n}f_{n,k}y^kx^n.$$

By the definition of the function of Chung-Feller type, we have

$$CF(x,y)=\frac{f(x)-yf(xy)}{1-y}=\sum_{n\geqslant0}\sum_{k=0}^{n}f_ny^kx^n.$$

Hence, $f_{n,k}=f_n$ for any $k\in\{0,1,\ldots,n\}$. $\square$

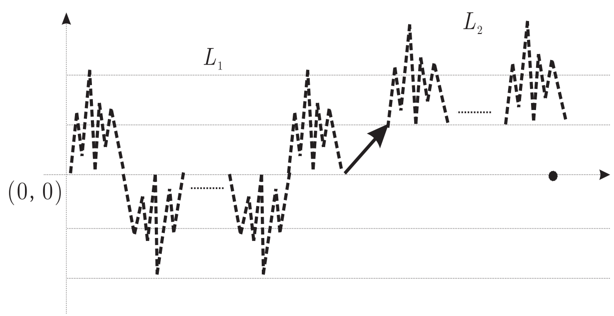

Figure 8 The decomposition for θ-rooted $(S,1)$-paths. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0,0)$

3 Uniform partition parameter δ' of left-right type

In this section, we will prove Theorem 1.15.

Lemma 3.1. *Let*

$$H'(x,y)=\sum_{L\in\mathcal{H}} w(L)y^{\mathscr{M}_\theta(L)}x^{\theta(L)}=\sum_{n\geqslant 0}\sum_{L\in\mathcal{H}_n} w(L)y^{\mathscr{M}_\theta(L)}x^{n}.$$

Then

$$H'(x,y)=\frac{1}{1-P(x,y)}.$$

Proof. Let $L=(x_i,y_i)_{i=1}^n$ be a nonempty $(S,0)$-path. Let $(x_0,y_0)=(0,0)$ and $(x_{n+1},y_{n+1})=(\beta,1)$. Let

$$D=\left\{i\,\middle|\,\sum_{t=0}^{i} y_t<\sum_{t=0}^{j} y_t \text{ for any } j\in\{0,1,\dots,i-1\} \text{ and } i\in\{0,1,\dots,n+1\}\right\}.$$

Suppose that $D=\{i_1,i_2,\dots,i_k\}$ where $1\leqslant i_1<i_2<\cdots<i_k$. Then $i_k\leqslant m(L)$. Let $i_0=0$ and $j_0=n+1$. For any $t\in\{1,\dots,k\}$, let

$$j_t=\max\left\{j\,\middle|\,\sum_{s=0}^{j} y_s=1+\sum_{s=0}^{i_t} y_s \text{ and } (x_j,y_j)=(\beta,1)\right\}.$$

Then $j_k=m(L)+1<j_{k-1}<\cdots<j_1<j_0=n+1$.

For any $s\in\{1,2,3,\dots,k\}$, let

$$L_s=(x_{i_{s-1}+1},y_{i_{s-1}+1})\cdots(x_{i_s},y_{i_s})(x_{j_s+1},y_{j_s+1})\cdots(x_{j_{s-1}},y_{j_{s-1}}).$$

Then L_s is an $(S,0)$-path and $(x_{i_s},y_{i_s})=(\alpha_i,-j+1)$ for some $j\in\{1,2,\dots,r\}$ and $i\in\{j-1,j,\dots,m\}$. Suppose $\sum_{t=i_{s-1}+1}^{i_s-1} y_t=h$. Then

$$h\geqslant 0 \quad\text{and}\quad \sum_{t=i_{s-1}+1}^{i_s} y_t=-j+1+h<0.$$

For any $t\in\{1,2,\dots,h\}$, let

$$u_t=\max\left\{u\,\middle|\,\sum_{v=i_{s-1}+1}^{u} y_v=t, i_{s-1}+1\leqslant u<i_s \text{ and } (x_u,y_u)=(\beta,1)\right\}.$$

For any $t\in\{1,2,\dots,j-1-h\}$, let

$$u'_t=\max\left\{u\,\middle|\,\sum_{v=j_s+1}^{u} y_v=-t+1, j_s+1\leqslant u\leqslant j_{s-1} \text{ and } (x_u,y_u)=(\beta,1)\right\}.$$

So L_s is decomposed into

$$L_s=Q_0(x_{u_1},y_{u_1})Q_1\cdots(x_{u_h},y_{u_h})Q_h(x_{i_s},y_{i_s})R_{j-1-h}(x_{u'_{j-1-h}},y_{u'_{j-1-h}})\cdots R_1(x_{u'_1},y_{u'_1}),$$

where $Q_0,\dots,Q_h,R_1,\dots,R_{j-1-h}$ are $(S^+,0)$-paths (see Figure 9). Then

$$(x_{u_1},y_{u_1})Q_1\cdots(x_{u_h},y_{u_h})Q_h(x_{i_s},y_{i_s})\overleftarrow{R}_{j-1-h}(x_{u'_{j-1-h}},y_{u'_{j-1-h}})\cdots\overleftarrow{R}_1(x_{u'_1},y_{u'_1})$$

is a prime $(S,0)$-path. Since $i_s<m(L)$, this provides the term

$$\sum_{j=1}^{r}\sum_{i=j-1}^{m}\sum_{h=0}^{j-2} a_{ij}x^iy^{i-j+1+h}f(x)^{j-1-h}f(xy)^{h+1}$$

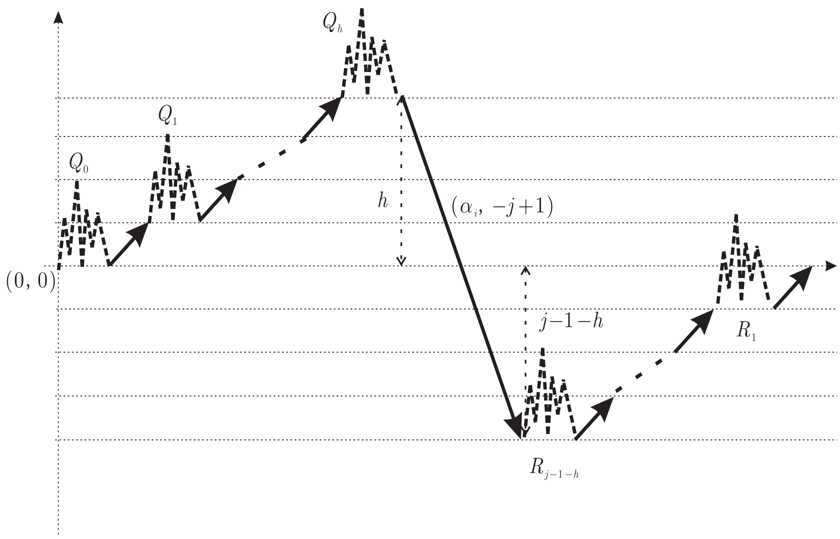

Figure 9 The decomposition for L_s. The x-axis/y-axis is the horizontal/vertical line which passes through the point $(0, 0)$

$$
\begin{aligned}
&= \sum_{j=1}^{r} \sum_{i=j-1}^{m} \sum_{h=1}^{j-1} a_{ij} x^i y^{i-h} f(x)^h f(xy)^{j-h} \\
&= f(xy)p(x,y) - f(xy) + 1.
\end{aligned}
$$

Let

$$L_{k+1} = (x_{i_k+1}, y_{i_k+1}) \cdots (x_{j_k-1}, y_{j_k-1}).$$

Then L_{k+1} is an $(S^+, 0)$-path. Since $j_k = m(L) + 1$, this provides the term $f(xy)$. Hence,

$$H'(x,y) = \sum_{k=0}^{\infty} (f(xy)p(x,y) - f(xy) + 1)^k f(xy) = \frac{1}{1 - P(x,y)}. \qquad \square$$

Proof of Theorem 1.15. (1) Let

$$\widetilde{CF}(x,y) = \sum_{(L,s)\in\overline{\Omega}} w(L) y^{\delta'(L,s)} x^{\theta(L)-1}.$$

Let $(L, s) \in \bar{\Omega}$ be a nonempty θ-rooted $(S, 1)$-path and

$$k = \max\left\{u \,\middle|\, \sum_{t=1}^{u} y_t = 1 \text{ and } (x_u, y_u) = (\beta, 1)\right\}.$$

So we can decompose L into $L = L_1(x_k, y_k)L_2$ such that $L_1 \in \mathcal{H}$ is an $(S, 0)$-path and $(L_2, s) \in \mathcal{G}$ is a θ-rooted $(S^+, 0)$-path.

By Lemmas 2.2 and 3.1, we have

$$\widetilde{CF}(x,y) = G(x,y)H'(x,y).$$

It is easy to see that

$$\frac{f(x) - yf(xy)}{1-y} = G(x,y)H'(x,y)$$

by Lemma 2.1. Hence, we have

$$CF(x,y) = \sum_{(L,s)\in\overline{\Omega}} w(L) y^{\delta'(L,s)} x^{\theta(L)-1}.$$

(2) The proof is similar to that given in (2) of Theorem 1.11. $\square$

4 Conclusions

By giving combinatorial interpretations of the functions of Chung-Feller type for a generating function $f(x)$, Liu et al. [10] obtained uniform partition extensions of some classical lattice paths, such as Dyck paths, Motzkin paths, large and little Schröder paths. Defined similarly as Dyck paths, the data of generating functions of some $(S^+, 0)$-paths are listed in Table 1.

The functional equations in Table 1 are special cases of the functional equation (1.1). In [10], three combinatorial interpretations of the functions of Chung-Feller type for $C(x)$, $M(x)$ and $S(x)$ were derived by different methods. In Theorems 1.11 and 1.15 of this paper, from the view of lattice paths, by a simple and unified method, we give two classes of combinatorial interpretations of the function $CF(x, y)$ of Chung-Feller type for $f(x)$, where $f(x)$ satisfies the functional equation (1.1).

In this paper, we focus on lattice paths in the two-dimensional plane. However, there are many other combinatorial models which have uniform partition extensions and uniform partition parameters, see [3, 18].

Example 4.1. There are n drivers which are labelled by $\{1, 2, \ldots, n\}$ and $n + 1$ parking spaces which are arranged in a cycle and labelled by $\{0, 1, \ldots, n\}$ clockwise. Each driver i has an initial parking preference a_i. We call such a sequence $X = (a_1, \ldots, a_n)$ a preference function of length n. Drivers enter the parking area in the order in which they are labelled. Each driver proceeds to his preferred parking space and parks there if it is free, or moves clockwise to the first unoccupied parking space and parks there. Every preference function X leaves one parking space unoccupied. We denote this unoccupied parking space by $\bar{\delta}(X)$.

A preference function X of length n is a parking function if $\bar{\delta}(X) = 0$. Let Ω_n be the set of parking functions of length n and $\Omega = \bigcup_{n\geqslant 0} \Omega_n$. Use $\theta(X)$ to denote the length of parking function X for every $X \in \Omega$.

Let $\bar{\Omega}_n$ be the set of preference functions of length n and $\bar{\Omega} = \bigcup_{n\geqslant 0} \bar{\Omega}_n$. For every $X \in \bar{\Omega}$, let $\bar{\theta}(X)$ be the length of S. Riordan [18] proved that the number of preference functions X in $\bar{\Omega}$ such that $\bar{\theta}(X) = n$ and $\bar{\delta}(X) = r$ is equal to the number of parking functions X of length n in Ω for all $r = 0, 1, \ldots, n$. Hence, $(\bar{\Omega}, \bar{\theta}, \bar{\delta})$ is a uniform partition extension for (Ω, θ), and $\bar{\delta}$ is a uniform partition parameter for $(\bar{\Omega}, \bar{\theta})$.

Example 4.2. Given an undirected multigraph G with loops, and a positive integer n, we say that a map

$$c : V(G) \to \{0, 1, 2, \ldots, n\}$$

is a proper vertex-colouring of G if $c(u) \neq c(v)$ when u and v are adjacent vertices in G. Let $\Omega_n(G)$ be the set of proper vertex-colourings of G. For any coloring $c \in \Omega_n(G)$, let $\theta_G(c) = n$.

Suppose that e is an edge connecting the vertex v to the vertex u in G. Denote by G/e the graph obtained by contracting the vertices u and v and by $G - e$ the graph obtained by removing the edge e from G.

Suppose that $e = \{u, v\}$ is a bridge of the graph G, i.e., an edge removing which increases the number of connected component of the graph. For any $c \in \Omega_n(G - e)$, let $\delta(c) \equiv c(u) - c(v) \pmod{n+1}$ and $\delta(c) \in \{0, 1, \ldots, n\}$. Then the number of proper vertex-colourings c of the graph $G - e$ such that $\theta_{G-e}(c) = n$ and $\delta(c) = k$ is equal to the number of proper vertex-colourings c of the graph G/e such that $\theta_{G/e}(c) = n$ for all $k = 0, 1, \ldots, n$. Hence, the triple $(\Omega_n(G - e), \theta_{G-e}, \delta_{G-e})$ is a uniform partition

Table 1 Generating functions of some $(S^+, 0)$-paths

Lattice path	Step set S	End point	Generating function and functional equation
n-Dyck path	$(1, 1), (1, -1)$	$(2n, 0)$	$C(x)\colon C(x) = 1 + xC(x)^2$
n-Motzkin path	$(1, 1), (1, -1), (1, 0)$	$(n, 0)$	$M(x)\colon M(x) = 1 + xM(x) + x^2M(x)^2$
n-Schröder path	$(1, 1), (1, -1), (2, 0)$	$(2n, 0)$	$S(x)\colon S(x) = 1 + xS(x) + xS(x)^2$
$(S^+, 0)$-path	$(1, 1), (1, -1), (5, -1)$	$(2n, 0)$	$f(x)\colon f(x) = 1 + (x + x^3)f(x)^2$

extension for the pair $(\Omega_n(G/e), \theta_{G/e})$, and the number δ_{G-e} is a uniform partition parameter for the pair $(\Omega_n(G-e), \theta_{G-e})$.

In more general, let Ω and $\bar{\Omega}$ be two combinatorial models. Let θ and $\bar{\theta}$ be two parameters defined on Ω and $\bar{\Omega}$, respectively. Suppose that the number of combinatorial structures $X \in \bar{\Omega}$ such that $\bar{\theta}(X) = n$ is $N(n)$ times the number of combinatorial structures $X \in \Omega$ such that $\theta(X) = n$, where $N(n) : \mathbb{N} \mapsto \mathbb{N}$ is a polynomial. A problem is

- can we define a natural and interesting uniform partition parameter $\bar{\delta}$ on $\bar{\Omega}$ such that the number of combinatorial structures $X \in \bar{\Omega}$ with $\bar{\theta}(X) = n$ and $\bar{\delta}(X) = k$ is equal to the number of combinatorial structures $X \in \Omega$ with $\theta(X) = n$ for any $k \in \{0, 1, \ldots, N(n) - 1\}$?

Example 4.3. In the 3-dimensional coordinate space, consider lattice paths that use the unit steps

$$X := (1,0,0), \quad Y := (0,1,0), \quad Z := (0,0,1).$$

Let $\bar{\Omega}_n$ denote the set of lattice paths running from $(0,0,0)$ to (n,n,n). It is easy to see that the number of lattice paths in $\bar{\Omega}_n$ is

$$\binom{3n}{n,n,n}.$$

Let Ω_n denote the set of lattice paths in $\bar{\Omega}_n$ lying in the region $\{(x,y,z) \mid 0 \leqslant x \leqslant y \leqslant z\}$. The paths in Ω_n are also called ballot paths for 3 candidates, or lattice permutations as in [13]. It can be derived from [13] that the number of lattice paths in Ω_n is

$$\frac{2}{(n+1)^2(n+2)}\binom{3n}{n,n,n}.$$

Consider the polynomial

$$N(n) = \frac{(n+1)^2(n+2)}{2}.$$

Then the number of lattice paths in $\bar{\Omega}_n$ is $N(n)$ times the number of lattice paths in Ω_n. The problem is

- how to define a natural and interesting uniform partition parameter $\bar{\delta}$ on $\bar{\Omega}_n$ such that the number of lattice paths L in $\bar{\Omega}_n$ with $\bar{\delta}(L) = k$ is equal to the number of lattice paths in Ω_n for any $k \in \{0, 1, \ldots, N(n) - 1\}$?

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