Stirling permutations, cycle structure of permutations and perfect matchings

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Abstract

In this paper we provide constructive proofs that the following three statistics are equidistributed: the number of ascent plateaus of Stirling permutations of order n, a weighted variant of the number of excedances in permutations of length n and the number of blocks with even maximal elements in perfect matchings of the set $\{1, 2, 3, \ldots, 2n\}$.

Keywords: Stirling permutations; Excedances; Perfect matchings; Eulerian polynomials

1 Introduction

In this paper we will study the relationship between Stirling permutations, the cycle structure of permutations and perfect matchings, and will give constructive proofs for the equidistribution of some combinatorial statistics on these combinatorial structures.

Stirling permutations were introduced by Gessel and Stanley [6]. A *Stirling permutation* of order n is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that every element between the two occurrences of i is greater than i for each $i \in [n]$, where $[n] = \{1, 2, \ldots, n\}$. We refer the reader to [1, 7, 8, 11] for some recent results on Stirling permutations.

Let $\mathcal{Q}_n$ be the set of Stirling permutations of order n. For any $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$, an occurrence of an *ascent* (resp. *a plateau*) is an index i such that $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i = \sigma_{i+1}$). We say that an index $i \in [2n-1]$ is an *ascent plateau* if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$,

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where $\sigma_0 = 0$ (see [11]). Let $\mathrm{ap}\,(\sigma)$ be the number of the ascent plateaus of σ. Then as an example, $\mathrm{ap}\,(\mathbf{2}211\mathbf{3}3) = 2$. Let

$$P_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)} = \sum_{k=1}^{n} P(n,k)x^k.$$

Then $P_n(1) = \#\mathcal{Q}_n = (2n-1)!!$, where $(2n-1)!!$ is the double factorial of $2n-1$. By [11, Theorem 3], the numbers $P(n,k)$ satisfy the recurrence relation

$$P(n+1,k) = 2kP(n,k) + (2n-2k+3)P(n,k-1), \tag{1}$$

with the initial values $P(1,1) = 1$ and $P(1,k) = 0$ for $k \leqslant 0$ or $k \geqslant 2$.

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$ and $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. An *excedance* in π is an index i such that $\pi_i > i$. Let $\mathrm{exc}\,(\pi)$ denote the number of excedances in π. The classical *Eulerian polynomials* $A_n(x)$ are defined by

$$A_0(x) = 1,\ A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} \text{ for } n \geqslant 1,$$

and have been extensively investigated (see [4, 5] for instance). In [5], Foata and Schützenberger introduced a q-analog of the Eulerian polynomials defined by

$$A_n(x;q) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)},$$

where $\mathrm{cyc}\,(\pi)$ is the number of cycles in π. Brenti [2, 3] further studied q-Eulerian polynomials and established the link with q-symmetric functions arising from plethysm. In particular, Brenti [3, Proposition 7.3] obtained the exponential generating function for $A_n(x;q)$:

$$1 + \sum_{n \geqslant 1} A_n(x;q)\frac{z^n}{n!} = \left(\frac{1-x}{e^{z(x-1)} - x}\right)^q.$$

For any $k \geqslant 1$, the $1/k$-*Eulerian polynomials* $A_n^{(k)}(x)$ are defined by

$$\sum_{n \geqslant 0} A_n^{(k)}(x)\frac{z^n}{n!} = \left(\frac{1-x}{e^{kz(x-1)} - x}\right)^{\frac{1}{k}}. \tag{2}$$

Let $\mathbf{e} = (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n$. Let $I_{n,k} = \{\mathbf{e} \mid 0 \leqslant e_i \leqslant (i-1)k\}$, which is known as the set of n-dimensional k-*inversion sequences*. The number of *ascents* of $\mathbf{e}$ is defined by

$$\mathrm{asc}\,(\mathbf{e}) = \#\left\{ i : 1 \leqslant i \leqslant n-1 \mid \frac{e_i}{(i-1)k+1} < \frac{e_{i+1}}{ik+1} \right\}.$$

Recently, Savage and Viswanathan [12] discovered that

$$A_n^{(k)}(x) = \sum_{\mathbf{e} \in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})} = k^n A_n(x; 1/k). \tag{3}$$

A *perfect matching* of $[2n]$ is a set partition of $[2n]$ with blocks (disjoint nonempty subsets) of size exactly 2. Let $\mathcal{M}_{2n}$ be the set of matchings of $[2n]$, and let $\mathrm{M} \in \mathcal{M}_{2n}$. The *standard form* of M is a list of blocks $\{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$ such that $i_r < j_r$ for all $1 \leqslant r \leqslant n$ and $1 = i_1 < i_2 < \cdots < i_n$. Throughout this paper we always write M in standard form. It is well known that M can be regarded as a fixed-point-free involution on $[2n]$. We call (a, b) a *marked block* (resp. an *unmarked block*) if b is even (resp. odd) and large than a. Let mark(M) be the number of marked blocks of M.

Let

$$N_n(x) = \sum_{\mathrm{M} \in \mathcal{M}_{2n}} x^{\mathrm{mark(M)}} = \sum_{k=1}^{n} N(n,k)x^k.$$

Then $N_n(1) = \#\mathcal{M}_{2n} = (2n-1)!!$. The first few terms of $N_n(x)$ are

$$N_1(x) = x,\ N_2(x) = 2x + x^2,\ N_3(x) = 4x + 10x^2 + x^3,\ N_4(x) = 8x + 60x^2 + 36x^3 + x^4.$$

The main purpose of the paper is to prove constructively that the following three statistics are equidistributed:

(m_1) $\mathrm{ap}\,(\sigma)$, the number of ascent plateaus of $\sigma \in \mathcal{Q}_n$;

(m_2) a weighted variant of $\mathrm{exc}\,(\pi)$, where the variant is $n - \mathrm{exc}\,(\pi)$ and the weight is $2^{n-\mathrm{cyc}\,(\pi)}$;

(m_3) mark(M), the number of marked blocks of $\mathrm{M} \in \mathcal{M}_{2n}$.

The paper is organized as follows. In Section 2, we first derive the recurrence relation of the numbers $N(n, k)$ and then we give the connection between the three combinatorial objects studied in this paper on the level of generating functions. In Section 3, we give constructive proofs of the main results by using the $\mathcal{SPM}$-sequences.

2 Recurrence formula for $N(n, k)$

Proposition 1. *The numbers $N(n, k)$ satisfy the recurrence relation*

$$N(n+1,k) = 2kN(n,k) + (2n-2k+3)N(n,k-1), \tag{4}$$

for $n, k \geqslant 1$, where $N(1, 1) = 1$ and $N(1, k) = 0$ for $k \leqslant 0$ or $k \geqslant 2$.

Proof. For each $\mathrm{M} \in \mathcal{M}_{2n}$ and for each block (a, b) of M, let $\varphi_1(\mathrm{M}, (a, b)) \in \mathcal{M}_{2n+2}$ be obtained from M by replacing the block (a, b) with two blocks $(a, 2n + 1), (b, 2n + 2)$, and let $\varphi_2(\mathrm{M}, (a, b)) \in \mathcal{M}_{2n+2}$ be obtained from M by replacing the block (a, b) with two blocks $(a, 2n + 2), (b, 2n + 1)$. Moreover, let $\varphi(\mathrm{M}) \in \mathcal{M}_{2n+2}$ be obtained from M by adding the block $(2n + 1, 2n + 2)$. It is obvious that for any $\mathrm{M}' \in \mathcal{M}_{2n+2}$, there is an $\mathrm{M} \in \mathcal{M}_{2n}$ such that either $\mathrm{M}' = \varphi(\mathrm{M})$ or there is a block (a, b) of M such that $\mathrm{M}' = \varphi_1(\mathrm{M}, (a, b))$ or $\mathrm{M}' = \varphi_2(\mathrm{M}, (a, b))$. Moreover, it follows from the definition that $\mathrm{mark}(\varphi(\mathrm{M})) = \mathrm{mark}(\mathrm{M}) + 1$ and

(i) if (a,b) is a marked block, then $\mathrm{mark}(\varphi_i(\mathrm{M},(\mathrm{a},\mathrm{b}))) = \mathrm{mark}(\mathrm{M})$ for $i \in \{1,2\}$;

(ii) if (a,b) is an unmarked block, then $\mathrm{mark}(\varphi_i(\mathrm{M},(\mathrm{a},\mathrm{b}))) = \mathrm{mark}(\mathrm{M})+1$ for $i \in \{1,2\}$.

Assume $\mathrm{M}' = \varphi(\mathrm{M})$ or $\varphi_i(\mathrm{M},(a,b))$ $(i = 1,2)$. If $\mathrm{mark}(\mathrm{M}') = k$, then $\mathrm{mark}(\mathrm{M}) = k$ or $\mathrm{mark}(\mathrm{M}) = k-1$. Each matching $\mathrm{M} \in \mathcal{M}_{2n}$ with $\mathrm{mark}(\mathrm{M}) = k$ corresponds to $2k$ matchings $\mathrm{M}' \in \mathcal{M}_{2n+2}$ with $\mathrm{mark}(\mathrm{M}') = k$, and each matching $\mathrm{M} \in \mathcal{M}_{2n}$ with $\mathrm{mark}(\mathrm{M}) = k-1$ corresponds to $2(n-(k-1))+1$ matchings $\mathrm{M}' \in \mathcal{M}_{2n+2}$ with $\mathrm{mark}(\mathrm{M}') = k$. So

$$N(n+1,k) = 2kN(n,k) + (2(n-(k-1))+1)N(n,k-1).$$

This completes the proof of (4). □

Comparing (1) with (4), we see that the numbers $P(n,k)$ satisfy the same recurrence relation and initial conditions as $N(n,k)$. Hence they agree. The bijection $(a,b) \to (b',a')$ on $\mathcal{M}_{2n}$ defined by $a' = 2n+1-a, b' = 2n+1-b$ shows that $N(n,k)$ is also the number of perfect matchings of $[2n]$ in which exactly k blocks with odd minimal elements. Now it is well known that the exponential generating function for $N_n(x)$ is given by (see [9, 11]):

$$N(x,z) = \sum_{n\geqslant 0} N_n(x)\frac{z^n}{n!} = \sqrt{\frac{1-x}{1-xe^{2z(1-x)}}}. \tag{5}$$

By (2) and (5),

$$A_n^{(2)}(x) = x^n N_n\left(\frac{1}{x}\right). \tag{6}$$

Therefore, it follows from (3) and (6) that

$$N_n(x) = \sum_{\pi\in\mathfrak{S}_n} 2^{n-\mathrm{cyc}\,(\pi)} x^{n-\mathrm{exc}\,(\pi)}.$$

3 The $\mathcal{SPM}$-sequences

Let $Y_n = (y_1, y_2, \ldots, y_n)$ be a sequence of integers of length n. For $1 \leqslant k \leqslant n$, let $\mathrm{POS}\,(Y_k)$ (resp. $\mathrm{NPOS}\,(Y_k)$) denote the set of positive (resp. non-positive) entries of $Y_k = (y_1, y_2, \ldots, y_k)$. We define

$$\mathrm{pos}\,(Y_k) = \#\mathrm{POS}\,(Y_k),\ \mathrm{npos}\,(Y_k) = \#\mathrm{NPOS}\,(Y_k).$$

Let P_k be the set $\{1,2,3,\ldots,2k\}$ and let N_k be the set $\{0,-1,-2,\ldots,-2k\}$. In particular, $P_0 = \emptyset$, $P_1 = \{1,2\}$, $N_0 = \{0\}$ and $N_1 = \{0,-1,-2\}$.

Definition 2. *Let $Y_n = (y_1, y_2, \ldots, y_n)$ be a sequence of integers of length n. We call the sequence Y_n an $\mathcal{SPM}$-sequence of length n if $y_1 = 0$, and $y_{k+1} \in P_{\mathrm{npos}\,(Y_k)} \cup N_{\mathrm{pos}\,(Y_k)}$ for $k = 1,2,3,\ldots,n-1$.*

For example, $(0,1,-1,2,0)$ is a $\mathcal{SPM}$-sequence, while $(0,1,-1,-4,2)$ is not since $y_4=-4\notin P_2\cup N_1$. Let $\mathcal{SPM}_n$ denote the set of $\mathcal{SPM}$-sequences of length n. Note that $\mathrm{npos}\,(Y_{n-1})+\mathrm{pos}\,(Y_{n-1})=n-1$. This implies that the set $P_{\mathrm{npos}\,(Y_{n-1})}\cup N_{\mathrm{pos}\,(Y_{n-1})}$ contains $2n-1$ elements. So there are $2n-1$ choices for y_n and hence

$$\#\mathcal{SPM}_n=(2n-1)\#\mathcal{SPM}_{n-1}=(2n-1)!!.$$

We now present the first main result of this paper.

Theorem 3. *For any $n\geqslant 1$,*

$$\sum_{\mathrm{M}\in\mathcal{M}_{2n}}x^{\mathrm{mark(M)}}=\sum_{Y_n\in\mathcal{SPM}_n}x^{\mathrm{npos}\,(Y_n)}=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{ap}\,(\sigma)}. \tag{7}$$

Let $\mathrm{neg}\,(Y_n)$ and $\mathrm{zero}\,(Y_n)$ be the number of negative entries and 0's in Y_n, respectively. Then $\mathrm{npos}\,(Y_n)=\mathrm{neg}\,(Y_n)+\mathrm{zero}\,(Y_n)$. The second main result is the following.

Theorem 4. *For any $n\geqslant 1$,*

$$\sum_{Y_n\in\mathcal{SPM}_n}x^{\mathrm{neg}\,(Y_n)}y^{\mathrm{zero}\,(Y_n)}=\sum_{\pi\in\mathfrak{S}_n}2^{n-\mathrm{cyc}\,(\pi)}x^{n-\mathrm{cyc}\,(\pi)-\mathrm{exc}\,(\pi)}y^{\mathrm{cyc}\,(\pi)}. \tag{8}$$

In particular,

$$\sum_{Y_n\in\mathcal{SPM}_n}x^{\mathrm{npos}\,(Y_n)}=\sum_{\pi\in\mathfrak{S}_n}2^{n-\mathrm{cyc}\,(\pi)}x^{n-\mathrm{exc}\,(\pi)}.$$

In the following subsections, we will give constructive proofs for the main results.

3.1 A bijection between $\mathcal{SPM}$-sequences and Perfect matchings

We define

$$\mathcal{SPM}_{n,k}=\{Y_n\in\mathcal{SPM}_n\mid \mathrm{npos}\,(Y_n)=k\},$$
$$\mathcal{M}_{2n,k}=\{\mathrm{M}\in\mathcal{M}_{2n}\mid \mathrm{mark(M)}=k\}.$$

Now we start to construct a bijection, denoted by Φ_1, between $\mathcal{SPM}_{n,k}$ and $\mathcal{M}_{2n,k}$. When $n=1$, we have $y_1=0$. Set $\Phi_1(Y_1)=(1,2)$. This gives a bijection between $\mathcal{SPM}_{1,1}$ and $\mathcal{M}_{2,1}$. Let $n=m$. Suppose Φ_1 is a bijection between $\mathcal{SPM}_{m,k}$ and $\mathcal{M}_{2m,k}$ for all k. Consider the case $n=m+1$. Let $Y_{m+1}=(y_1,y_2,\ldots,y_m,y_{m+1})\in\mathcal{SPM}_{m+1}$. Then $Y_m=(y_1,y_2,\ldots,y_m)\in\mathcal{SPM}_{m,k}$ for some k. Assume $\Phi_1(Y_m)\in\mathcal{M}_{2m,k}$. Consider the following five cases:

(i) If $y_{m+1}=2i-1>0$, then let $\Phi_1(Y_{m+1})$ be obtained from $\Phi_1(Y_m)$ by replacing the ith marked block (c,d) by two blocks $(c,2m+1),(d,2m+2)$. In this case, $(c,2m+1)$ is an unmarked block and $(d,2m+2)$ is a marked block. Hence, $\mathrm{mark}(\Phi_1(\mathrm{Y_{m+1}}))=\mathrm{mark}(\Phi_1(\mathrm{Y_m}))=\mathrm{k}$;

(*ii*) If $y_{m+1} = 2i > 0$, then let $\Phi_1(Y_{m+1})$ be obtained from $\Phi_1(Y_m)$ by replacing the ith marked block (c, d) by two blocks $(c, 2m+2), (d, 2m+1)$. Hence, $\mathrm{mark}(\Phi_1(\mathrm{Y_{m+1}})) = \mathrm{mark}(\Phi_1(\mathrm{Y_m})) = \mathrm{k}$;

(*iii*) If $y_{m+1} = -(2i-1) < 0$, then let $\Phi_1(Y_{m+1})$ be obtained from $\Phi_1(Y_m)$ by replacing the ith unmarked block (e, f) by two blocks $(e, 2m+1), (f, 2m+2)$. In this case, $\mathrm{mark}(\Phi_1(\mathrm{Y_{m+1}})) = \mathrm{mark}(\Phi_1(\mathrm{Y_m})) + 1 = \mathrm{k} + 1$;

(*iv*) If $y_{m+1} = -2i < 0$, then let $\Phi_1(Y_{m+1})$ be obtained from $\Phi_1(Y_m)$ by replacing the ith unmarked block (e, f) by two blocks $(e, 2m+2), (f, 2m+1)$. In this case, $\mathrm{mark}(\Phi_1(\mathrm{Y_{m+1}})) = \mathrm{mark}(\Phi_1(\mathrm{Y_m})) + 1 = \mathrm{k} + 1$;

(*v*) If $y_{m+1} = 0$, then let $\Phi_1(Y_{m+1})$ be obtained from $\Phi_1(Y_m)$ by appending the marked block $(2m+1, 2m+2)$ right after $\Phi_1(Y_m)$. In this case,

$$\mathrm{mark}(\Phi_1(\mathrm{Y_{m+1}})) = \mathrm{mark}(\Phi_1(\mathrm{Y_m})) + 1 = \mathrm{k} + 1.$$

After the above step, we write the obtained perfect matching $\Phi_1(Y_{m+1})$ in standard form. It is easy to verify that if $y_{m+1} \in P_{\mathrm{npos}\,(Y_m)}$ (resp. $y_{m+1} \in N_{\mathrm{pos}\,(Y_m)}$), then $\Phi_1(Y_{m+1}) \in \mathcal{M}_{2m+2,k}$ (resp. $\Phi_1(Y_{m+1}) \in \mathcal{M}_{2m+2,k+1}$). By induction, we see that Φ_1 is the desired bijection between $\mathcal{SPM}_{n,k}$ and $\mathcal{M}_{2n,k}$, which also gives a constructive proof of the left equality of (7).

In the following example, if (a, b) is a marked block, then we put the entries a and b into a square bracket, i.e., replace (a, b) by $[a, b]$. Otherwise, the parenthesis that contains a and b unchanged.

Example 5. *Let $Y_4 = (0, 1, -2, 4)$. The correspondence between Y_4 and $\Phi_1(Y_4)$ is built up as follows:*

$$\begin{aligned}
0 &\Leftrightarrow \{[1,2]\};\\
1 &\rightarrow \{[1,2]\} \Leftrightarrow \{(1,3),[2,4]\};\\
-2 &\rightarrow \{(1,3),[2,4]\} \Leftrightarrow \{[1,6],(3,5),[2,4]\} = \{[1,6],[2,4],(3,5)\};\\
4 &\rightarrow \{[1,6],[2,4],(3,5)\} \Leftrightarrow \{[1,6],[2,8],(4,7),(3,5)\} = \{[1,6],[2,8],(3,5),(4,7)\}.
\end{aligned}$$

3.2 A bijection between $\mathcal{SPM}$-sequences and Stirling permutations

Let$\mathcal{Q}\left(P_{\mathrm{npos}\,(Y_k)} \cup N_{\mathrm{pos}\,(Y_k)}\right)$ be the set of Stirling permutations of order k with exactly $\mathrm{npos}\,(Y_k)$ ascent plateaus. In particular, $\mathcal{Q}\,(P_1 \cup N_0) = \{11\}$, $\mathcal{Q}\,(P_1 \cup N_1) = \{2211, 1221\}$ and $\mathcal{Q}\,(P_2 \cup N_0) = \{1122\}$. We now introduce a definition of *labeled Stirling permutations.*

Definition 6. *Let $\sigma \in \mathcal{Q}\left(P_{\mathrm{npos}\,(Y_k)} \cup N_{\mathrm{pos}\,(Y_k)}\right)$. If $i_1 < i_2 < \ldots < i_{\mathrm{npos}\,(Y_k)}$ are the ascent plateaus of σ, then we put the superscript labels $2\ell - 1$ before i_ℓ and 2ℓ after it, where $1 \leqslant \ell \leqslant \mathrm{npos}\,(Y_k)$. In the remaining positions, we put the superscript labels $-1, -2, \ldots, -2\mathrm{pos}\,(Y_k)$ and 0 from left to right.*

As an example, the labeled version of 13324421 is given by ${}^{-1}1^{1}3^{2}3^{-2}2^{3}4^{4}4^{-3}2^{-4}1^{0}$. We define

$$\mathcal{Q}_{n,k} = \{\sigma \in \mathcal{Q}_n \mid \mathrm{ap}\,(\sigma) = k\}.$$

Now we start to construct a bijection, denoted by Φ_2, between $\mathcal{SPM}_{n,k}$ and $\mathcal{Q}_{n,k}$. When $n = 1$, we have $y_1 = 0$. Set $\Phi_2(Y_1) = 11$. This gives a bijection between $\mathcal{SPM}_{1,1}$ and $\mathcal{Q}_{1,1}$. Let $n = m$. Suppose Φ_2 is a bijection between $\mathcal{SPM}_{m,k}$ and $\mathcal{Q}_{m,k}$ for all k. Consider the case $n = m + 1$. Let $Y_{m+1} = (y_1, y_2, \ldots, y_m, y_{m+1}) \in \mathcal{SPM}_{m+1}$. Then $Y_m = (y_1, y_2, \ldots, y_m) \in \mathcal{SPM}_{m,k}$ for some k. Assume $\Phi_2(Y_m) \in \mathcal{Q}_{m,k}$. Consider the following three cases:

(*i*) If $y_{m+1} = i > 0$, then let $\Phi_2(Y_{m+1})$ be obtained from $\Phi_2(Y_m)$ by inserting the pair $(m + 1)(m + 1)$ to the position of $\Phi_2(Y_m)$ with label i. Hence $Y_{m+1} \in \mathcal{SPM}_{m+1,k}$ and $\Phi_2(Y_{m+1}) \in \mathcal{Q}_{m+1,k}$.

(*ii*) If $y_{m+1} = -i < 0$, then let $\Phi_2(Y_{m+1})$ be obtained from $\Phi_2(Y_m)$ by inserting the pair $(m+1)(m+1)$ to the position of $\Phi_2(Y_m)$ with label $-i$. Hence $Y_{m+1} \in \mathcal{SPM}_{m+1,k+1}$ and $\Phi_2(Y_{m+1}) \in \mathcal{Q}_{m+1,k+1}$.

(*iii*) If $y_{m+1} = 0$, then let $\Phi_2(Y_{m+1})$ be obtained from $\Phi_2(Y_m)$ by appending the pair $(m + 1)(m + 1)$ right after $\Phi_2(Y_m)$. Hence $Y_{m+1} \in \mathcal{SPM}_{m+1,k+1}$ and $\Phi_2(Y_{m+1}) \in \mathcal{Q}_{m+1,k+1}$.

By induction, we see that Φ_2 is the desired bijection between $\mathcal{SPM}_{n,k}$ and $\mathcal{Q}_{n,k}$ for all k, which also gives a constructive proof of the right equality of (7).

Example 7. *Let $Y_4 = (0, 2, -1, 3)$. The correspondence between Y_4 and $\Phi_2(Y_4)$ is built up as follows:*

$$\begin{aligned} 0 &\Leftrightarrow 11; \\ 2 &\to^{1} 1^{2}1^{0} \Leftrightarrow 1221; \\ -1 &\to^{-1} 1^{1}2^{2}2^{-2}1^{0} \Leftrightarrow 331221; \\ 3 &\to^{1} 3^{2}3^{-1}1^{3}2^{4}2^{-2}1^{0} \Leftrightarrow 33144221. \end{aligned}$$

3.3 A map from $\mathcal{SPM}$-sequences to permutations

In the following discussion, we always write $\pi \in \mathfrak{S}_n$ by its *standard cycle decomposition*, in which each cycle is written with its smallest entry first and the cycles are written in ascending order of their smallest entry. We now introduce a definition of *labeled permutations.*

Definition 8. *Let $\pi \in \mathfrak{S}_n$ with p excedances. If $i_1 < i_2 < \cdots < i_p$ are the excedances, then we put superscript labels $-k$ between i_k and $\pi(i_k)$, where $1 \leqslant k \leqslant p$. In the remaining positions except the first position of each cycle, we put the superscript labels $1, 2, \ldots, n-p$ from left to right.*

The number of *anti-excedances* of $\pi \in \mathfrak{S}_n$ is defined by $\operatorname{aexc}(\pi) = n - \operatorname{exc}(\pi)$. Let

$$\mathfrak{S}_{n,k} = \{\pi \in \mathfrak{S}_n \mid \operatorname{aexc}(\pi) = k\}.$$

Now we start to construct a map, denoted by Φ_3, from $\mathcal{SPM}_{n,k}$ to $\mathfrak{S}_{n,k}$. When $n = 1$, we have $y_1 = 0$. Set $\Phi_3(Y_1) = (1)$. This gives a map from $\mathcal{SPM}_{1,1}$ to $\mathfrak{S}_{1,1}$. Let $n = m$. Suppose Φ_3 is a map from $\mathcal{SPM}_{m,k}$ to $\mathfrak{S}_{m,k}$ for all k and there are 2^{m-t} sequences in $\mathcal{SPM}_{m,k}$ are mapped to one permutation $\pi \in \mathfrak{S}_{m,k}$ with $\operatorname{cyc}(\pi) = t$. Consider the case $n = m + 1$. Let $Y_{m+1} = (y_1, y_2, \ldots, y_m, y_{m+1}) \in \mathcal{SPM}_{m+1}$. Then $Y_m = (y_1, y_2, \ldots, y_m) \in \mathcal{SPM}_{m,k}$ for some k. Assume $\Phi_3(Y_m) \in \mathfrak{S}_{m,k}$. Consider the following three cases:

(*i*) If $y_{m+1} = 2i - 1 > 0$ or $y_{m+1} = 2i > 0$, then let $\Phi_3(Y_{m+1})$ be obtained from $\Phi_3(Y_m)$ by inserting the entry $m + 1$ to the position of $\Phi_3(Y_m)$ with label i. Since $\operatorname{exc}(\Phi_3(Y_{m+1}) = \operatorname{exc}(\Phi_3(Y_m) + 1 = m - k + 1$, we have $\operatorname{aexc}(\Phi_3(Y_{m+1})) = k$. Moreover, $\operatorname{cyc}(\Phi_3(Y_{m+1})) = \operatorname{cyc}(\Phi_3(Y_m))$. Therefore, we have $2 \cdot 2^{m-t} = 2^{m+1-t}$ sequences in $\mathcal{SPM}_{m+1,k}$ that are mapped to one permutation $\pi \in \mathfrak{S}_{m+1,k}$ with $\operatorname{cyc}(\pi) = t$.

(*ii*) If $y_{m+1} = -(2i - 1) < 0$ or $y_{m+1} = -2i < 0$, then let $\Phi_3(Y_{m+1})$ be obtained from $\Phi_3(Y_m)$ by inserting the entry $m + 1$ to the position of $\Phi_3(Y_m)$ with label $-i$. Since $\operatorname{exc}(\Phi_3(Y_{m+1}) = \operatorname{exc}(\Phi_3(Y_m) = m - k$, we have $\operatorname{aexc}(\Phi_3(Y_{m+1})) = k + 1$. Moreover, $\operatorname{cyc}(\Phi_3(Y_{m+1})) = \operatorname{cyc}(\Phi_3(Y_m))$. Therefore, we have $2 \cdot 2^{m-t} = 2^{m+1-t}$ sequences in $\mathcal{SPM}_{m+1,k+1}$ that are mapped to one permutation $\pi \in \mathfrak{S}_{m+1,k+1}$ with $\operatorname{cyc}(\pi) = t$.

(*iii*) If $y_{m+1} = 0$, then let $\Phi_3(Y_{m+1})$ be obtained from $\Phi_3(Y_m)$ by appending a new cycle $(m + 1)$ right after $\Phi_3(Y_m)$. Since $\operatorname{exc}(\Phi_3(Y_{m+1}) = \operatorname{exc}(\Phi_3(Y_m) = m - k$, we have $\operatorname{aexc}(\Phi_3(Y_{m+1})) = k + 1$. Moreover, $\operatorname{cyc}(\Phi_3(Y_{m+1})) = \operatorname{cyc}(\Phi_3(Y_m)) + 1$. Therefore, we have $1 \cdot 2^{m-t} = 2^{m-t}$ sequences in $\mathcal{SPM}_{m+1,k+1}$ that are mapped to one permutation $\pi \in \mathfrak{S}_{m+1,k+1}$ with $\operatorname{cyc}(\pi) = t + 1$.

After the above step, we first write the obtained $\Phi_3(Y_{m+1})$ in standard cycle decomposition, and then label it. By induction, we see that Φ_3 is the desired map from $\mathcal{SPM}_{n,k}$ to $\mathfrak{S}_{n,k}$, which also gives a constructive proof of (8).

Example 9. *Given $\pi = (1\ 3\ 5\ 2)(4)$. The map Φ_3 can be done if you proceed as follows:*

$$\begin{aligned}
y_1 = 0 \ &\rightarrow (1^1);\\
y_2 = 1 \ \mathit{or}\ y_2 = 2 \ &\rightarrow (1\ {}^{-1}\ 2\ {}^{1});\\
y_3 = -1 \ \mathit{or}\ y_3 = -2 \ &\rightarrow (1\ {}^{-1}\ 3\ {}^{1}\ 2\ {}^{2});\\
y_4 = 0 \ &\rightarrow (1\ {}^{-1}\ 3\ {}^{1}\ 2\ {}^{2})(4\ {}^{3});\\
y_5 = 1 \ \mathit{or}\ y_5 = 2 \ &\rightarrow (1^{-1}3^{-2}5^{1}2^{2})(4^{3}).
\end{aligned}$$

Hence, by Φ_3, there are eight $\mathcal{SPM}$-sequences that are mapped to π:

$$(0, 1, -1, 0, 1); (0, 1, -1, 0, 2); (0, 1, -2, 0, 1); (0, 1, -2, 0, 2);$$

$$(0, 2, -1, 0, 1); (0, 2, -1, 0, 2); (0, 2, -2, 0, 1); (0, 2, -2, 0, 2).$$

4 Concluding remarks

The key observation of the paper is the recurrence relation (4), from which we discover the relationship among Stirling permutations, the cycle structure of permutations and perfect matchings. We believe that the techniques developed in the paper can be used to deal with various combinatorial structures. Moreover, it would be interesting to study pattern avoidance properties of the $\mathcal{SPM}$-sequences.

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