Eulerian polynomials, Stirling permutations of the second kind and perfect matchings

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Abstract

In this paper, we introduce Stirling permutations of the second kind. In particular, we count Stirling permutations of the second kind by their cycle ascent plateaus, fixed points and cycles. Moreover, we get an expansion of the ordinary derangement polynomials in terms of the Stirling derangement polynomials. Finally, we present constructive proofs of a kind of combinatorial expansions of the Eulerian polynomials of types A and B.

Keywords: Eulerian polynomials; Stirling permutations of the second kind; Stirling derangements

1 Introduction

Let $[n]$ denote the set $\{1,2,\ldots,n\}$. Let $\mathfrak{S}_n$ be the set of all permutations of $[n]$ and let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathfrak{S}_n$. Let $\mathcal{B}_n$ be the hyperoctahedral group of rank n. Elements π of $\mathcal{B}_n$ are signed permutations of the set $\pm[n]$ such that $\pi(-i)=-\pi(i)$ for all i, where $\pm[n]=\{\pm1,\pm2,\ldots,\pm n\}$. Let $\#S$ denote the cardinality of a set S. We define

$$\operatorname{des}_A(\pi):=\#\{i\in\{1,2,\ldots,n-1\}|\pi(i)>\pi(i+1)\},$$
$$\operatorname{des}_B(\pi):=\#\{i\in\{0,1,2,\ldots,n-1\}|\pi(i)>\pi(i+1)\},$$

where $\pi(0)=0$. The Eulerian polynomials of types A and B are respectively defined by

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\operatorname{des}_A(\pi)},\ B_n(x)=\sum_{\pi\in\mathcal{B}_n}x^{\operatorname{des}_B(\pi)}.$$

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There is a larger literature devoted to $A_n(x)$ and $B_n(x)$ (see, e.g., [5, 10, 13, 19, 26] and references therein). Let $s = (s_1, s_2, \ldots)$ be a sequence of positive integers. Let

$$I_n^{(s)} = \{(e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n |\ 0 \leqslant e_i < s_i\},$$

which is known as the set of s-inversion sequences. The reader is referred to Savage [28] for recent progress on this subject. The number of *ascents* of an s-inversion sequence $e = (e_1, e_2, \ldots, e_n) \in I_n^{(s)}$ is defined by

$$\operatorname{asc}(e) = \# \left\{ i \in [n-1] : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\} \cup \{0 : \text{if } e_1 > 0\}.$$

Let $E_n^s(x) = \sum_{e \in I_n^{(s)}} x^{\operatorname{asc}(e)}$. Following [25] and [26], we have

$$A_n(x) = E_n^{(1,2,\ldots,n)}(x), B_n(x) = E_n^{(2,4,\ldots,2n)}(x).$$

Let $M_n(x)$ be a sequence of polynomials defined by

$$M(x,z) = \sum_{n \geqslant 0} M_n(x) \frac{z^n}{n!} = \sqrt{\frac{x-1}{x - e^{2z(x-1)}}}. \tag{1}$$

Combining (1) and an explicit formula of the Ehrhart polynomial of the s-lecture hall polytope, Savage and Viswanathan [27] proved that $M_n(x) = E_n^{(1,3,\ldots,2n-1)}(x)$.

A *perfect matching* of $[2n]$ is a partition of $[2n]$ into n blocks of size 2. Let $\mathcal{M}_{2n}$ be the set of perfect matchings of $[2n]$. Let $\operatorname{el}(\mathrm{M})$ be the number of blocks of $\mathrm{M} \in \mathcal{M}_{2n}$ with even larger entries. We define

$$N(n,k) = \#\{\mathrm{M} \in \mathcal{M}_{2n} : \operatorname{el}(\mathrm{M}) = k\}.$$

The numbers $N(n,k)$ satisfy the recurrence relation

$$N(n+1,k) = 2kN(n,k) + (2n-2k+3)N(n,k-1)$$

for $n, k \geqslant 1$, where $N(1,1) = 1$ and $N(1,k) = 0$ for $k \geqslant 2$ or $k \leqslant 0$ (see [24, Proposition 1]). Let $N_n(x) = \sum_{k=1}^{n} N(n,k)x^k$. The first few of the polynomials $N_n(x)$ are

$$N_0(x) = 1, N_1(x) = x, N_2(x) = 2x + x^2, N_3(x) = 4x + 10x^2 + x^3.$$

The exponential generating function for $N_n(x)$ is given as follows (see [21, Eq. (25)]):

$$N(x,z) = \sum_{n \geqslant 0} N_n(x) \frac{z^n}{n!} = \sqrt{\frac{1-x}{1 - xe^{2z(1-x)}}}. \tag{2}$$

Combining (1) and (2), we get $M_n(x) = x^n N_n(\frac{1}{x})$ for $n \geqslant 0$. Let $\operatorname{ol}(\mathrm{M})$ be the number of blocks of $\mathrm{M} \in \mathcal{M}_{2n}$ with odd larger entries. Then

$$M_n(x) = \sum_{\mathrm{M} \in \mathcal{M}_{2n}} x^{\operatorname{ol}(\mathrm{M})}.$$

Context-free grammar was introduced by Chen [6] and it is a powerful tool for studying exponential structures in combinatorics. We refer the reader to [8, 11, 22] for further information. Using [22, Theorem 10], one can present a grammatical proof of the following result.

Proposition 1. *For $n \geqslant 0$, we have*

$$2^n xA_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x)N_{n-k}(x), \tag{3}$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x)M_{n-k}(x). \tag{4}$$

Recall that the exponential generating function for $xA_n(x)$ is

$$A(x,z) = 1 + \sum_{n\geqslant 1} xA_n(x)\frac{t^n}{n!} = \frac{1-x}{1-xe^{z(1-x)}}.$$

An equivalent formula of (3) is given as follows:

$$N^2(x,z) = A(x,2z).$$

Motivated by Proposition 1, the main purpose of this paper is to introduce some cycle structure related to $N(x,z)$ or $M(x,z)$. This paper is organized as follows. In Section 2, we introduce the Stirling permutations of the second kind, which is a disjoint union of Stirling permutations. In Section 3, we count Stirling permutations of the second kind by their cycle ascent plateaus, fixed points and cycles. In Section 4, we present a constructive proof of Proposition 1.

2 Stirling permutations of the second kind

Stirling permutations were introduced by Gessel and Stanley [14]. A *Stirling permutation* of order n is a permutation of the multiset $[n]_2$ such that every element between the two occurrences of i is greater than i for each $i \in [n]$, where $[n]_2 = \{1,1,2,2\ldots,n,n\}$. Let $\mathcal{Q}_n$ be the set of Stirling permutations of $[n]_2$. For example, $\mathcal{Q}_2 = \{1122, 1221, 2211\}$. Let $\sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. An index i is a *descent* of σ if $\sigma_i > \sigma_{i+1}$ or $i = 2n$. Let $C(n,k)$ be the number of Stirling permutations of $[n]_2$ with k descents. Following [14, Eq. (6)], the numbers $C(n,k)$ satisfy the recurrence relation

$$C(n,k) = kC(n-1,k) + (2n-k)C(n-1,k-1) \tag{5}$$

for $n \geqslant 2$, with the initial conditions $C(1,1) = 1$ and $C(1,0) = 0$. The *second-order Eulerian polynomial* is defined by

$$C_n(x) = \sum_{i=1}^{n} C(n,k)x^k.$$

In recent years, there has been much work on Stirling permutations (see [2, 12, 15, 16, 23]). In particular, Bóna [2] introduced the plateau statistic on Stirling permutations, and proved that descents and plateaus have the same distribution over $\mathcal{Q}_n$. Given $\sigma \in \mathcal{Q}_n$, the index i is called a *plateau* if $\sigma_i = \sigma_{i+1}$. We say that an index $i \in [2n-1]$ is an *ascent plateau* if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $\sigma_0 = 0$. Let $\mathrm{ap}\,(\sigma)$ be the number of the ascent plateaus of σ. For example, $\mathrm{ap}\,(\mathbf{2}211\mathbf{3}3) = 2$. Very recently, we present a combinatorial proof of the following identity (see [24, Theorem 3]):

$$\sum_{\mathrm{M} \in \mathcal{M}_{2n}} x^{\mathrm{el}\,(\mathrm{M})} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)}. \tag{6}$$

Motivated by Proposition 1 and (6), we shall introduce Stirling permutations of the second kind.

Let $[k]^n$ denote the set of words of length n in the alphabet $[k]$. For $\omega = \omega_1\omega_2\cdots\omega_n \in [k]^n$, the reduction of ω, denoted by $\mathrm{red}\,(\omega)$, is the unique word of length n obtained by replacing the ith smallest entry by i. For example, $\mathrm{red}\,(33224547) = 22113435$.

Definition 2. A permutation σ of the multiset $[n]_2$ is a *Stirling permutation of the second kind* of order n whenever σ can be written as a nonempty disjoint union of its distinct cycles and σ has a standard cycle form satisfying the following conditions:

(i) For each $i \in [n]$, the two copies of i appear in exactly one cycle;

(ii) Each cycle is written with one of its smallest entry first and the cycles are written in increasing order of their smallest entry;

(iii) The reduction of the word formed by all entries of each cycle is a Stirling permutation. In other words, if $(c_1, c_2, \ldots, c_{2k})$ is a cycle of σ, then $\mathrm{red}\,(c_1c_2\cdots c_{2k}) \in \mathcal{Q}_k$.

Let $\mathcal{Q}_n^2$ denote the set of Stirling permutations of the second kind of order n. In the following discussion, we always write $\sigma \in \mathcal{Q}_n^2$ in standard cycle form.

Example 3.

$$\begin{aligned}
&\mathcal{Q}_1^2 = \{(11)\},\ \mathcal{Q}_2^2 = \{(11)(22), (1122), (1221)\},\\
&\mathcal{Q}_3^2 = \{(11)(22)(33), (11)(2233), (11)(2332), (1133)(22), (1331)(22), (1122)(33), (112233),\\
&\quad (112332), (113322), (133122), (1221)(33), (122133), (122331), (123321), (133221)\}.
\end{aligned}$$

Let $(c_1, c_2, \ldots, c_{2k})$ be a cycle of σ. An entry c_i is called a *cycle plateau* (resp. *cycle ascent*) if $c_i = c_{i+1}$ (resp. $c_i < c_{i+1}$), where $1 \leqslant i < 2k$. Let $\mathrm{cplat}\,(\pi)$ and $\mathrm{casc}\,(\pi)$ be the number of cycle plateaus and cycle ascents of π, respectively. For example, $\mathrm{cplat}\,((1\mathbf{2}21)(\mathbf{3}3)) = 2$ and $\mathrm{casc}\,((\mathbf{1}221)(33)) = 1$. Now we present a dual result of [2, Proposition 1].

Proposition 4. *For $n \geqslant 1$, we have*

$$C_n(x) = \sum_{\pi \in \mathcal{Q}_n^2} x^{\mathrm{cplat}\,(\pi)} = \sum_{\pi \in \mathcal{Q}_n^2} x^{\mathrm{casc}\,(\pi)+1}.$$

Proof. There are two ways in which a permutation $\sigma' \in \mathcal{Q}_n^2$ with k cycle plateaus can be obtained from a permutation $\sigma \in \mathcal{Q}_{n-1}^2$. If $\mathrm{cplat}\,(\sigma) = k$, then we can put the two copies of n right after a cycle plateau of σ. This gives k possibilities. If $\mathrm{cplat}\,(\sigma) = k-1$, then we can append a new cycle (nn) right after σ or insert the two copies of n into any of the remaining $2n-2-(k-1) = 2n-k-1$ positions. This gives $2n-k$ possibilities. Comparing with (5), this completes the proof of $C_n(x) = \sum_{\pi \in \mathcal{Q}_n^2} x^{\mathrm{cplat}\,(\pi)}$. Along the same lines, one can easily prove the assertion for cycle ascents. This completes the proof. □

Let $(c_1, c_2, \ldots, c_{2k})$ be a cycle of σ, where $k \geqslant 2$. An entry c_i is called a *cycle ascent plateau* if $c_{i-1} < c_i = c_{i+1}$, where $2 \leqslant i \leqslant 2k-1$. Denote by $\mathrm{cap}\,(\sigma)$ (resp. $\mathrm{cyc}\,(\sigma)$) the number of cycle ascent plateaus (resp. cycles) of σ. For example, $\mathrm{cap}\,((1\mathbf{2}21)(33)) = 1$. We define

$$Q_n(x,q) = \sum_{\sigma \in \mathcal{Q}_n^2} x^{\mathrm{cap}\,(\sigma)} q^{\mathrm{cyc}\,(\sigma)},$$

$$Q(x,q;z) = 1 + \sum_{n \geqslant 1} Q_n(x,q) \frac{z^n}{n!}.$$

Our main result of this section is the following.

Theorem 5. *The polynomials $Q_n(x,q)$ satisfy the recurrence relation*

$$Q_{n+1}(x,q) = (q + 2nx) Q_n(x,q) + 2x(1-x) \frac{\partial}{\partial x} Q_n(x,q) \tag{7}$$

for $n \geqslant 0$, with the initial condition $Q_0(x) = 1$. Moreover,

$$Q(x,q;z) = \left(\sqrt{\frac{x-1}{x - e^{2z(x-1)}}} \right)^q. \tag{8}$$

Proof. Given $\sigma \in \mathcal{Q}_n^2$. Let σ_i be an element of $\mathcal{Q}_{n+1}^2$ obtained from σ by inserting the two copies of $n+1$, in the standard cycle decomposition of σ, right after $i \in [n]$ or as a new cycle $(n+1, n+1)$ if $i = n+1$. It is clear that

$$\mathrm{cyc}\,(\sigma_i) = \begin{cases} \mathrm{cyc}\,(\sigma), & \text{if } i \in [n]; \\ \mathrm{cyc}\,(\sigma) + 1, & \text{if } i = n+1. \end{cases}$$

Therefore, we have

$$\begin{aligned}
Q_{n+1}(x,q) &= \sum_{\pi \in \mathcal{Q}_{n+1}^2} x^{\mathrm{cap}\,(\pi)} q^{\mathrm{cyc}\,(\pi)} \\
&= \sum_{i=1}^{n+1} \sum_{\sigma_i \in \mathcal{Q}_n^2} x^{\mathrm{cap}\,(\sigma_i)} q^{\mathrm{cyc}\,(\sigma_i)} \\
&= \sum_{\sigma \in \mathcal{Q}_n^2} x^{\mathrm{cap}\,(\sigma)} q^{\mathrm{cyc}\,(\sigma)+1} + \sum_{i=1}^{n} \sum_{\sigma_i \in \mathcal{Q}_n^2} x^{\mathrm{cap}\,(\sigma_i)} q^{\mathrm{cyc}\,(\sigma_i)} \\
&= qQ_n(x,q) + \sum_{\sigma \in \mathcal{Q}_n^2} (2\mathrm{cap}\,(\sigma) x^{\mathrm{cap}\,(\sigma)} + (2n - 2\mathrm{cap}\,(\sigma)) x^{\mathrm{cap}\,(\sigma)+1}) q^{\mathrm{cyc}\,(\sigma)}
\end{aligned}$$

and (7) follows. By rewriting (7) in terms of the exponential generating function $Q(x,q;z)$, we have

$$(1-2xz)\frac{\partial}{\partial z}Q(x,q;z) = qQ(x,q;z) + 2x(1-x)\frac{\partial}{\partial x}Q(x,q;z). \tag{9}$$

It is routine to check that the generating function

$$\widetilde{Q}(x,q;z) = \left(\sqrt{\frac{x-1}{x-e^{2z(x-1)}}}\right)^{q}$$

satisfies (9). Also, this generating function gives $\widetilde{Q}(x,q;0) = 1, \widetilde{Q}(x,0;z) = 1$ and $\widetilde{Q}(0,q;z) = e^{qz}$. Hence $Q(x,q;z) = \widetilde{Q}(x,q;z)$. □

Combining (1) and (8), we get $Q(x,q;z) = M^q(x,z)$. Thus $Q_n(x,1) = M_n(x)$. Moreover, it follows from (7) that

$$Q_{n+1}(1,q) = (q+2n)Q_n(1,q).$$

So the following corollary is immediate.

Corollary 6. *For $n \geqslant 1$, we have*

$$\sum_{\sigma\in\mathcal{Q}_n^2} q^{\operatorname{cyc}(\sigma)} = q(q+2)\cdots(q+2n-2).$$

Let $A_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$. The numbers $\left\langle {n \atop k} \right\rangle$ are called Eulerian numbers of type A and satisfy the recurrence relation

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle,$$

with the initial conditions $\left\langle {0 \atop 0} \right\rangle = 1$ and $\left\langle {0 \atop k} \right\rangle = 0$ for $k \geqslant 1$ (see [3, 5] for instance). Hence

$$A_{n+1}(x) = (1+nx)A_n(x) + x(1-x)A_n'(x), \tag{10}$$

Let $\mathcal{CQ}_n$ denote the set of Stirling permutations of $\mathcal{Q}_n^2$ with only one cycle, which can be named as the set of *cyclic Stirling permutations*. Define

$$Y_n(x) = \sum_{\sigma\in\mathcal{CQ}_n} x^{\operatorname{cap}(\sigma)}.$$

Comparing (7) with (10), we get the following corollary.

Corollary 7. *For $n \geqslant 1$, we have*

$$Y_{n+1}(x) = 2^n x A_n(x).$$

3 The joint distribution of cycle ascent plateaus and fixed points on $\mathcal{Q}_n^2$

Given $\sigma \in \mathcal{Q}_n^2$. Let the entry $k \in [n]$ be called a *fixed point* of σ if (kk) is a cycle of σ. The number of fixed points of σ is defined by $\mathrm{fix}\,(\sigma) = \#\{k \in [n] : (kk) \text{ is a cycle of } \sigma\}$. For example, $\mathrm{fix}\,((1133)(22)) = 1$. Define

$$P_n(x,y,q) = \sum_{\sigma \in \mathcal{Q}_n^2} x^{\mathrm{cap}\,(\sigma)} y^{\mathrm{fix}\,(\sigma)} q^{\mathrm{cyc}\,(\pi)},$$

$$P(x,y,q;z) = \sum_{n \geqslant 0} P_n(x,y,q) \frac{z^n}{n!}.$$

Now we present the main result of this section.

Theorem 8. *For $n \geqslant 1$, the polynomials $P_n(x,y,q)$ satisfy the recurrence relation*

$$P_{n+1}(x,y,q) = qyP_n(x,y,q) + qx\sum_{k=0}^{n-1}\binom{n}{k}P_k(x,y,q)2^{n-k}A_{n-k}(x), \tag{11}$$

with the initial conditions $P_0(x,y,q) = 1, P_1(x,y,q) = yq$. Moreover,

$$P_{n+1}(x,y,q) = (2nx+qy)P_n(x,y,q)+2x(1-x)\frac{\partial}{\partial x}P_n(x,y,q)+2x(1-y)\frac{\partial}{\partial y}P_n(x,y,q). \tag{12}$$

Furthermore,

$$P(x,y,q;z) = e^{qz(y-1)}Q(x,q;z). \tag{13}$$

In the following, we shall prove Theorem 8 by using context-free grammars. For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. Following [6], a context-free grammar over A is a function $G : A \to \mathbb{Q}[[A]]$ that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. More precisely, the derivative $D = D_G$: $\mathbb{Q}[[A]] \to \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation, and for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity.

Lemma 9. *If $A = \{a,b,c,d\}$ and $G = \{a \to qab^2, b \to b^{-1}c^2d^2, c \to cd^2, d \to c^2d\}$, then*

$$D^n(a) = a\sum_{\sigma \in \mathcal{Q}_n^2} q^{\mathrm{cyc}\,(\sigma)} b^{2\mathrm{fix}\,(\sigma)} c^{2\mathrm{cap}\,(\sigma)} d^{2n-2\mathrm{fix}\,(\sigma)-2\mathrm{cap}\,(\sigma)}. \tag{14}$$

Proof. Let $\mathcal{Q}_n^2(i,j,k) = \{\sigma \in \mathcal{Q}_n^2 : \mathrm{cyc}\,(\sigma) = i, \mathrm{fix}\,(\sigma) = j, \mathrm{cap}\,(\sigma) = k\}$. Given $\sigma \in \mathcal{Q}_n^2(i,j,k)$. We now introduce a labeling scheme for σ:

(i) Put a superscript label a at the end of σ and a superscript q before each cycle of σ;

(*ii*) If k is a fixed point of σ, then we put a superscript label b right after each k;

(*iii*) Put superscript labels c immediately before and right after each cycle ascent plateau;

(*iv*) In each of the remaining positions except the first position of each cycle, we put a superscript label d.

When $n = 1$, we have $\mathcal{Q}_1^2(1,1,0) = \{^q(1^b1^b)^a\}$. When $n = 2$, we have $\mathcal{Q}_2^2(2,2,0) = \{^q(1^b1^b)^q(2^b2^b)^a\}$ and $\mathcal{Q}_2^2(1,0,1) = \{^q(1^d1^c2^c2^d)^a,\ ^q(1^c2^c2^d1^d)^a\}$. Let $n = m$. Suppose we get all labeled permutations in $\mathcal{Q}_m^2(i,j,k)$ for all i,j,k, where $m \geqslant 2$. We consider the case $n = m+1$. Let $\sigma' \in \mathcal{Q}_{m+1}^2$ be obtained from $\sigma \in \mathcal{Q}_m^2(i,j,k)$ by inserting two copies of the entry $m+1$ into σ. Now we construct a correspondence, denoted by ϑ, between σ and σ'. Consider the following cases:

(c_1) If the two copies of $m+1$ are put at the end of σ as a new cycle $((m+1)(m+1))$, then we leave all labels of σ unchanged except the last cycle. In this case, the correspondence ϑ is defined by

$$\sigma = \cdots(\cdots)^a \overset{\vartheta}{\longleftrightarrow} \sigma' = \cdots(\cdots)^q((m+1)^b(m+1)^b)^a,$$

which corresponds to the operation $a \to qab^2$. Moreover, $\sigma' \in \mathcal{Q}_{m+1}^2(i+1,j+1,k)$.

(c_2) If the two copies of $m+1$ are inserted to a position of σ with label b, then ϑ corresponds to the operation $b \to b^{-1}c^2d^2$. In this case, $\sigma' \in \mathcal{Q}_{m+1}^2(i,j-1,k+1)$.

(c_3) If the two copies of $m+1$ are inserted to a position of σ with label c, then ϑ corresponds to the operation $c \to cd^2$. In this case, $\sigma' \in \mathcal{Q}_{m+1}^2(i,j,k)$.

(c_4) If the two copies of $m+1$ are inserted to a position of σ with label d, then ϑ corresponds to the operation $d \to c^2d$. In this case, $\sigma' \in \mathcal{Q}_{m+1}^2(i,j,k+1)$.

By induction, we see that ϑ is the desired correspondence between permutations in $\mathcal{Q}_m^2$ and $\mathcal{Q}_{m+1}^2$, which also gives a constructive proof of (14). □

Lemma 10. *If $A = \{b,c,d\}$ and $G = \{b \to b^{-1}c^2d^2, c \to cd^2, d \to c^2d\}$, then*

$$D^n(b^2) = 2^n \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle c^{2k+2}d^{2n-2k} = 2^n d^{2n} c^2 A_n\left(\frac{c^2}{d^2}\right) \text{ for } n \geqslant 1.$$

Proof. Note that $D(b^2) = 2c^2d^2$. Hence $D^n(b^2) = 2D^{n-1}(c^2d^2)$ for $n \geqslant 1$. Note that $D(c^2d^2) = 2(c^2d^4 + c^4d^2)$. Assume that

$$D^n(b^2) = 2^n \sum_{k=0}^{n-1} F(n,k)c^{2n-2k}d^{2k+2}.$$

Since

$$D(D^n(b^2)) = 2^{n+1}\sum_{k=0}^{n-1}(n-k)F(n,k)c^{2n-2k}d^{2k+4} + 2^{n+1}\sum_{k=0}^{n-1}(k+1)F(n,k)c^{2n-k+2}d^{2k+2},$$

there follows

$$F(n+1,k) = (k+1)F(n,k) + (n-k+1)F(n,k-1).$$

We see that the coefficients $F(n,k)$ satisfy the same recurrence relation and initial conditions as $\left\langle {n \atop k} \right\rangle$, so they agree. □

Proof of Theorem 8:
By Lemma 9 and Lemma 10, we get

$$\begin{aligned} D^{n+1}(a) =& qD^n(ab^2) \\ =& q\sum_{k=0}^{n}\binom{n}{k}D^k(a)D^{n-k}(b^2) \\ =& qb^2D^n(a) + q\sum_{k=0}^{n-1}\binom{n}{k}D^k(a)2^{n-k}d^{2n-2k}c^2A_{n-k}\left(\frac{c^2}{d^2}\right). \end{aligned}$$

Taking $c^2 = x, b^2 = y$ and $d^2 = 1$ in both sides of the above identity, we immediately get (11). Set $S_n(i,j,k) = \#\mathcal{Q}_n^2(i,j,k)$. The following recurrence relation follows easily from the proof of Lemma 9:

$$\begin{aligned} S_{n+1}(i,j,k) = S_n(i-1,j-1,k) + 2(j+1)S_n(i,j+1,k-1) + \\ 2kS_n(i,j,k) + 2(n-j-k+1)S_n(i,j,k-1). \end{aligned}$$

Multiplying both sides of the last recurrence relation by $q^iy^jx^k$ and summing for all i,j,k, we immediately get (12).

Note that

$$\begin{aligned} P_n(x,y,q) &= \sum_{i=0}^{n}\binom{n}{i}(yq-q)^i\sum_{\sigma\in\mathcal{Q}_n^2}x^{\mathrm{cap}\,(\sigma)}q^{\mathrm{cyc}\,(\pi)} \\ &= \sum_{i=0}^{n}\binom{n}{i}(yq-q)^iQ_{n-i}(x,q). \end{aligned}$$

Thus $P(x,y,q;z) = e^{qz(y-1)}Q(x,q;z)$. This completes the proof of Theorem 8.

It should be noted that there exists a straightforward proof of (13). Note that each object of $\mathcal{Q}_n^2$ is a disjoint union of one object counted by $P(x,0,q;z)$ and some fixed points. Since each fixed point contributes no cycle ascent plateau but one cycle, by rules of exponential generating function one has $Q(x,q;z) = e^{qz}P(x,0,q;z)$ and $P(x,y,q;z) = e^{yqz}P(x,0,q;z)$.

Given $\sigma \in \mathcal{Q}_n^2$. We say that σ is a *Stirling derangement* if σ has no fixed points. Let $\mathcal{DQ}_n$ be the set of Stirling derangements of $[n]_2$. Let $R_{n,k}(x,q)$ be the coefficient of y^k in $P_n(x,y,q)$. Note that $R_{n,0}(x,q)$ is the corresponding enumerative polynomials on $\mathcal{DQ}_n$.

Set $R_n(x,q)=R_{n,0}(x,q)$. Note that

$$\begin{aligned} R_{n,k}(x,q) &= \sum_{\substack{\sigma\in\mathcal{Q}_n^2\\ \mathrm{fix}\,(\sigma)=k}} x^{\mathrm{cap}\,(\sigma)}q^{\mathrm{cyc}\,(\pi)} \\ &= \binom{n}{k}q^k \sum_{\sigma\in\mathcal{DQ}_{n-k}} x^{\mathrm{cap}\,(\sigma)}q^{\mathrm{cyc}\,(\pi)} \\ &= \binom{n}{k}q^k R_{n-k}(x,q). \end{aligned}$$

Comparing the coefficients of both sides of (12), we get the following result.

Theorem 11. *For $n\geqslant 1$, the polynomials $R_n(x,q)$ satisfy the recurrence relation*

$$R_{n+1}(x,q)=2nxR_n(x,q)+2x(1-x)\frac{\partial}{\partial x}R_n(x,q)+2nxqR_{n-1}(x,q), \tag{15}$$

with the initial conditions $R_1(x,q)=0, R_2(x,q)=2qx, R_3(x,q)=4qx(1+x)$.

Let $q_n=\#\mathcal{DQ}_n$. Then the following corollary is immediate.

Corollary 12. *For $n\geqslant 1$, the numbers q_n satisfy the recurrence relation*

$$q_{n+1}=2n(q_n+q_{n-1}),$$

with the initial conditions $q_0=1, q_1=0$ and $q_2=2$.

Note that $\#\mathcal{Q}_n^2=\mathcal{Q}_n=(2n-1)!!$. Then $\sum_{n\geqslant 0}\#\mathcal{Q}_n^2\frac{z^n}{n!}=\frac{1}{\sqrt{1-2z}}$. Thus

$$\sum_{n\geqslant 0} q_n\frac{z^n}{n!}=\frac{e^{-z}}{\sqrt{1-2z}},$$

which can be easily proved by using the *exponential formula* (see [3, Theorem 3.50]). It should be noted that q_{n+1} is also the number of minimal number of 1-factors in a $2n$-connected graph having at least one 1-factor (see [1]). It would be interesting to study the relationship between Stirling permutations of the second kind and $2n$-connected graphs.

Recently, there has been much work on derangements polynomials (see [7, 9, 18, 19, 31] for instance). For each $\pi\in\mathfrak{S}_n$, let the index i be called an *excedance* (resp. *anti-excedance*) if $\pi(i)>i$ (resp. $\pi(i)<i$). Let $\mathrm{exc}\,(\pi)$ be the number of excedances of π. A permutation $\pi\in\mathfrak{S}_n$ is a *derangement* if $\pi(i)\neq i$ for any $i\in[n]$. Let $\mathcal{D}_n$ denote the set of derangements of $\mathfrak{S}_n$. The ordinary *derangements polynomial* is defined by $d_n(x)=\sum_{\pi\in\mathcal{D}_n}x^{\mathrm{exc}\,(\pi)}$. Brenti [4, Proposition 5] derived that

$$\sum_{n\geqslant 0} d_n(x)\frac{z^n}{n!}=\frac{1-x}{e^{xz}-xe^z}. \tag{16}$$

Let $d_n(x,q)=\sum_{\pi\in\mathcal{D}_n} x^{\mathrm{exc}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}$ and let $d(x,q;z)=\sum_{n\geqslant 0} d_n(x,q)\frac{z^n}{n!}$. Following [17], we have

$$d(x,q;z)=\left(\frac{1-x}{e^{xz}-xe^{z}}\right)^q.$$

Taking $y=0$ in (13), we have

$$P(x,0,q;z)=e^{-qz}Q(x,q;z)=\left(\sqrt{\frac{x-1}{xe^{2z}-e^{2xz}}}\right)^q. \tag{17}$$

Thus $P^2(x,0,q;z)=d(x,q,2z)$, which is a dual result of (1). So the following result is immediate.

Theorem 13. *For $n\geqslant 0$, we have*

$$2^n d_n(x,q)=\sum_{k=0}^{n}\binom{n}{k}R_k(x,q)R_{n-k}(x,q). \tag{18}$$

Let $R_n(x)=\sum_{\mathcal{DQ}_n} x^{\mathrm{cap}\,(\sigma)}$ be the *Stirling derangement polynomials*. Combining (15), (17) and [20, Corollary 2.4], we immediately get the following result.

Proposition 14. *For $n\geqslant 2$, the polynomial $R_n(x)$ is symmetric and has only simple real zeros.*

Let $\imath^2=\sqrt{-1}$. From (17), putting $x=-1$, we deduce the expression

$$S(-1,z)=\sqrt{\frac{2}{e^{2z}+e^{-2z}}}=\sqrt{\sec(2\imath z)}. \tag{19}$$

Note that $\sec(z)$ is an even function. Therefore, for $n\geqslant 1$, we have

$$\sum_{\sigma\in\mathcal{DQ}_n}(-1)^{\mathrm{cap}\,(\sigma)}=\begin{cases}0, & \text{if } n=2k-1;\\ (-1)^k h_k, & \text{if } n=2k,\end{cases}$$

where the number h_n is defined by the following series expansion:

$$\sqrt{\sec(2\imath z)}=\sum_{n\geqslant 0}(-1)^n h_n\frac{z^{2n}}{(2n)!}.$$

The first few of the numbers h_n are $h_0=1,h_1=2,h_2=28,h_3=1112,h_4=87568$. It should be noted that the numbers h_n also count permutations of $\mathfrak{S}_{4n}$ having the following properties:

(a) The permutation can be written as a product of disjoint cycles with only two elements;

(b) For $i\in[2n]$, indices $2i-1$ and $2i$ are either both excedances or both anti-excedances.

For example, when $n=1$, there are only two permutations having the desired properties: $(1,3)(2,4)$ and $(1,4)(2,3)$. This kind of permutations was introduced by Sukumar and Hodges [30].

4 Perfect matchings and a constructive proof of Proposition 1

Given $\mathrm{M} \in \mathcal{M}_{2n}$. The *standard form* of the perfect matching M is a list of blocks $\{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$ such that $i_r < j_r$ for all $1 \leqslant r \leqslant n$ and $1 = i_1 < i_2 < \cdots < i_n$. In the following discussion we always write M in standard form. For convenience, we call (i, j) a *marked block* (resp. an *unmarked block*) if j is even (resp. odd) and large than i.

4.1 Permutations and pairs of perfect matchings

Let the entry $\pi(i)$ be called a *descent* (resp. an *ascent*) of π if $\pi(i) > \pi(i+1)$ (resp. $\pi(i) < \pi(i+1)$). By using the reverse map, it is evident that ascent and descent are equidistributed. Let $\mathrm{asc}\,(\pi)$ be the number of ascents of π. Hence

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)}. \tag{20}$$

Throughout this subsection, we shall always use (20) as the definition of the Eulerian polynomial $A_n(x)$.

We now constructively define a set of decorated permutations on $[n]$ with some entries of permutations decorated with hats and circles, denoted by $\mathcal{P}_n$. Let $w = w_1 w_2 \cdots w_n \in \mathcal{P}_n$. We say that w_i with a hat (resp. circle) if $w_i = \widehat{k}$ or $w_i = \textcircled{$\widehat{k}$}$ (resp. $w_i = \textcircled{$k$}$ or $w_i = \textcircled{$\widehat{k}$}$) for some $k \in [n]$. Start with $\mathcal{P}_1 = \{1, \widehat{1}\}$. Suppose we have get $\mathcal{P}_{n-1}$, where $n \geqslant 2$. Given $v = v_1 v_2 \cdots v_{n-1} \in \mathcal{P}_{n-1}$. We now construct entries of $\mathcal{P}_n$ by inserting $n, \textcircled{$n$}, \widehat{n}$ or $\textcircled{$\widehat{n}$}$ into v according the following rules:

- (r_1) We can only put n or $\widehat{n}$ at the end of v;
- (r_2) For $1 \leqslant i \leqslant n-1$, if v_i with no hat, then we can only put n or $\textcircled{$n$}$ immediately before v_i; if v_i with a hat, then we can only put $\widehat{n}$ or $\textcircled{$\widehat{n}$}$ immediately before v_i.

It is clear that there are $2n$ elements in $\mathcal{P}_n$ that can be generated from any $v \in \mathcal{P}_{n-1}$. By induction, we obtain $|\mathcal{P}_n| = 2n|\mathcal{P}_{n-1}| = 2^n n!$. Let $\varphi(w) = \varphi(w_1)\varphi(w_2)\cdots\varphi(w_n)$ be a permutation of $\mathfrak{S}_n$ obtained from $w \in \mathcal{P}_n$ by deleting the hats and circles of all w_i. For example, $\varphi(\widehat{3}\ \widehat{1}\textcircled{4}\ 2) = 3142$. Let $\mathcal{P}_n(\pi) = \{w \in \mathcal{P}_n : \varphi(w) = \pi\}$. Let $k\ell$ be a consecutive subword of $\pi \in \mathfrak{S}_n$. We see that if $k < \ell$, then $k\ell$ can be decorated as $k\ell, k\widehat{\ell}, \widehat{k}\ell, \widehat{k}\ \widehat{\ell}$. If $k > \ell$, then $k\ell$ can be decorated as $k\ell, \textcircled{$k$}\ell, \widehat{k}\ \widehat{\ell}$ or $\textcircled{$\widehat{k}$}\widehat{\ell}$. Therefore, $|\mathcal{P}_n(\pi)| = 2^n$ for any $\pi \in \mathfrak{S}_n$. It should be noted that $k\widehat{\ell}$ or $\widehat{k}\ell$ is a consecutive subword of $w \in \mathcal{P}_n$ if and only if $k < \ell$. Let the entry w_i be called an *ascent* (resp. a *descent*) of w if $\varphi(w_i) < \varphi(w_{i+1})$ $(\varphi(w_i) > \varphi(w_{i+1}))$. Also a conventional ascent is counted at the beginning of w. That is, we identify a decorated permutation $w = w_1 \cdots w_n$ with the word $w_0 w_1 \cdots w_n$, where $w_0 = 0$. Let $\mathrm{asc}\,(w)$ be the number of ascents of w. Therefore, we obtain

$$2^n x A_n(x) = \sum_{w \in \mathcal{P}_n} x^{\mathrm{asc}\,(w)}.$$

Example 15. The following decorated permutations are generated from $\widehat{3}\,\widehat{1}4\,2$:

$$\widehat{3}\,\widehat{1}425,\ \widehat{3}\,\widehat{1}4\,2\,\widehat{5},\ \widehat{3}\,\widehat{1}4\,5\,2,\ \widehat{3}\,\widehat{1}4\,⑤\,2, \widehat{3}\,\widehat{1}542,$$
$$\widehat{3}\,\widehat{1}\,⑤42,\ \widehat{3}\,\widehat{5}\,\widehat{1}42,\ \widehat{3}\,\widehat{⑤}\,\widehat{1}\,4\,2, \widehat{5}\,\widehat{3}\,\widehat{1}42,\ \widehat{⑤}\,\widehat{3}\,\widehat{1}42.$$

Example 16. We have $\mathcal{P}_2 = \{12, 1\,\widehat{2}, \widehat{1}\,2, \widehat{1}\,\widehat{2}, 21, ②\,1, \widehat{2}\,\widehat{1}, \widehat{②}\,\widehat{1}\}$.

Let $\mathrm{I}_{n,k}$ be a set of subsets of $[n]$ with cardinality k. Let $\mathrm{Hat}\,(w)$ be a set of entries of w with hats and let $\mathrm{hat}\,(w) = \#\mathrm{Hat}\,(w)$. Let $\varphi(\mathrm{Hat}\,(w))$ be a subset of $[n]$ obtained from $\mathrm{Hat}\,(w)$ by deleting all hats and circles of all entries of $\mathrm{Hat}\,(w)$. For example, if $w = \widehat{⑤}\,\widehat{3}\,\widehat{1}42$, then $\mathrm{Hat}\,(w) = \{\widehat{1}, \widehat{3}, \widehat{⑤}\}$ and $\varphi(\mathrm{Hat}\,(w)) = \{1, 3, 5\}$. We define

$$\mathcal{P}_{n,k} = \{w \in \mathcal{P}_n : \mathrm{hat}\,(w) = k\},$$
$$\mathcal{PM}_{n,k} = \{(S_1, S_2, I_{n,k}) : S_1 \in \mathcal{M}_{2k},\ S_2 \in \mathcal{M}_{2n-2k},\ I_{n,k} \in \mathrm{I}_{n,k}\}.$$

In this subsection, we always assume that the weight of $w \in \mathcal{P}_{n,k}$ is $x^{\mathrm{asc}\,(w)}$ and that of the pair of matchings (S_1, S_2) is $x^{\mathrm{el}\,(S_1)+\mathrm{el}\,(S_2)}$.

Now we start to construct a bijection, denoted by Φ, between $\mathcal{P}_{n,k}$ and $\mathcal{PM}_{n,k}$. When $n = 1$, set $\Phi(1) = (\varnothing, (1,2), \varnothing)$ and $\Phi(\widehat{1}) = ((1,2), \varnothing, \{1\})$. This gives a bijection between $\mathcal{P}_{1,k}$ and $\mathcal{PM}_{1,k}$. When $n = 2$, the bijection Φ between $\mathcal{P}_{2,k}$ and $\mathcal{PM}_{2,k}$ is given as follows:

$$\Phi(12) = (\varnothing, (1,2)(3,4), \varnothing),\ \Phi(21) = (\varnothing, (1,3)(2,4), \varnothing)$$
$$\Phi(②1) = (\varnothing, (1,4)(2,3), \varnothing),\ \Phi(\widehat{1}2) = ((1,2), (1,2), \{1\}),$$
$$\Phi(1\widehat{2}) = ((1,2), (1,2), \{2\}),\ \Phi(\widehat{1}\,\widehat{2}) = ((1,2)(3,4), \varnothing, \{1,2\}),$$
$$\Phi(\widehat{2}\,\widehat{1}) = ((1,3)(2,4), \varnothing, \{1,2\}),\ \Phi(\widehat{②}\,\widehat{1}) = ((1,4)(2,3), \varnothing, \{1,2\}).$$

Suppose Φ is a bijection between $\mathcal{P}_{m-1,k}$ and $\mathcal{PM}_{m-1,k}$ for all k, where $m \geqslant 3$. Assume that $w = w_1 w_2 \cdots w_{m-1} \in \mathcal{P}_{m-1,k}$, $\mathrm{asc}\,(w) = i + j$ and $\mathrm{Hat}\,(w) = \{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$. Let $\Phi(w) = (S_1, S_2, I_{m-1,k})$, where $S_1 \in \mathcal{M}_{2k},\ S_2 \in \mathcal{M}_{2m-2k-2},\ I_{m-1,k} = \varphi(\mathrm{Hat}\,(w)),\ \mathrm{el}\,(S_1) = i$ and $\mathrm{el}\,(S_2) = j$.

Consider the case $n = m$. Let w' be a decorated permutation generated from w. We first distinguish two cases: If $w' = wm$, then let $\Phi(w') = (S_1, S_2(2m-2k-1, 2m-2k), I_{m,k})$, where $I_{m,k} = \varphi(\mathrm{Hat}\,(w))$; If $w' = w\widehat{m}$, then let $\Phi(w') = (S_1(2k+1, 2k+2), S_2, I_{m,k+1})$, where $I_{m,k+1} = \varphi(\mathrm{Hat}\,(w)) \cup \{m\}$.

Now let $\ell_1\ell_2$ be a consecutive subword of w. Firstly, suppose that ℓ_2 with no hat. We say that ℓ_2 is a *unhat-ascent-top* (resp. *unhat-descent-bottom*) if $\varphi(\ell_1) < \varphi(\ell_2)$ (resp. $\varphi(\ell_1) > \varphi(\ell_2)$). Consider the following two cases:

(c_1) If $w' = \cdots \ell_1 m \ell_2$ (resp. $w' = \cdots \ell_1 ⓜ \ell_2 \cdots$) and ℓ_2 is the pth unhat-ascent-top of w, then let $\Phi(w') = (S_1, S_2', I_{m,k})$, where $I_{m,k} = \varphi(\mathrm{Hat}\,(w))$ and S_2' is obtained from S_2 by replacing pth marked block (a,b) by two blocks $(a, 2m-2k-1), (b, 2m-2k)$ (resp. $(a, 2m-2k), (b, 2m-2k-1)$).

(c_2) If $w' = \cdots \ell_1 m \ell_2 \cdots$ (resp. $w' = \cdots \ell_1 \textcircled{m} \ell_2 \cdots$) and ℓ_2 is the pth unhat-descent-bottom of w, then let $\Phi(w') = (S_1, S_2', I_{m,k})$, where $I_{m,k} = \varphi(\mathrm{Hat}\,(w))$ and S_2' is obtained from S_2 by replacing pth unmarked block (a, b) by two blocks $(a, 2m - 2k - 1), (b, 2m - 2k)$ (resp. $(a, 2m - 2k), (b, 2m - 2k - 1)$).

Secondly, suppose that ℓ_2 with a hat. We say that ℓ_2 is a *hat-ascent-top* (resp. *hat-descent-bottom*) if $\varphi(\ell_1) < \varphi(\ell_2)$ (resp. $\varphi(\ell_1) > \varphi(\ell_2)$). Consider the following two cases:

(c_1) If $w' = \cdots \ell_1 \widehat{m}\ \ell_2 \cdots$ (resp. $w' = \cdots \ell_1 \textcircled{\widehat{m}}\ \ell_2 \cdots$) and ℓ_2 is the pth hat-ascent-top of w, then let $\Phi_1(w') = (S_1', S_2, I_{m,k+1})$, where $I_{m,k+1} = \varphi(\mathrm{Hat}\,(w)) \cup \{m\}$ and S_1' is obtained from S_1 by replacing the pth marked block (a, b) by two blocks $(a, 2k + 1), (b, 2k + 2)$ (resp. $(a, 2k + 2), (b, 2k + 1)$).

(c_2) If $w' = \cdots \ell_1 \widehat{m}\ \ell_2 \cdots$ (resp. $w' = \cdots \ell_1 \textcircled{\widehat{m}}\ \ell_2 \cdots$) and ℓ_2 is the pth hat-descent-bottom of w, then let $\Phi(w') = (S_1', S_2, I_{m,k+1})$, where $I_{m,k+1} = \varphi(\mathrm{Hat}\,(w)) \cup \{m\}$ and S_1' is obtained from S_1 by replacing the pth unmarked block (a, b) by two blocks $(a, 2k + 1), (b, 2k + 2)$ (resp. $(a, 2k + 2), (b, 2k + 1)$).

After the above step, we write the obtained perfect matchings in standard form. Suppose that $w \in \mathcal{P}_{n,k}$ and $\Phi(w) = (S_1, S_2, I_{n,k})$. Then $\mathrm{asc}\,(w) = i + j$ if and only if $\mathrm{el}\,(S_1) + \mathrm{el}\,(S_2) = i + j$. By induction, we see that Φ is the desired bijection between $\mathcal{P}_{n,k}$ and $\mathcal{PM}_{n,k}$ for all k, which also gives a constructive proof of (3).

Example 17. Let $w = \widehat{3}\ \widehat{1}42\textcircled{\widehat{6}}\widehat{5} \in \mathcal{P}_{6,4}$. The correspondence between w and $\Phi(w)$ is built up as follows:

$$\begin{aligned}
\widehat{1} &\Leftrightarrow ((1,2), \varnothing, \{1\});\\
\widehat{1}2 &\Leftrightarrow ((1,2), (1,2), \{1\});\\
\widehat{3}\ \widehat{1}2 &\Leftrightarrow ((1,3)(2,4), (1,2), \{1,3\});\\
\widehat{3}\ \widehat{1}42 &\Leftrightarrow ((1,3)(2,4), (1,3)(2,4), \{1,3\});\\
\widehat{3}\ \widehat{1}42\widehat{5} &\Leftrightarrow ((1,3)(2,4)(5,6), (1,3)(2,4), \{1,3,5\});\\
\widehat{3}\ \widehat{1}42\textcircled{\widehat{6}}\widehat{5} &\Leftrightarrow ((1,3)(2,4)(5,8)(6,7), (1,3)(2,4), \{1,3,5,6\}).
\end{aligned}$$

4.2 Signed permutations and pairs of perfect matchings

In this subsection, we write signed permutations of $\mathcal{B}_n$ as $\pi = \pi(0)\pi(1)\pi(2)\cdots\pi(n)$, where some elements are associated with the minus sign and $\pi(0) = 0$. As usual, we denote by $\bar{i}$ the negative element $-i$. For $\pi \in \mathcal{B}_n$, let

$$\mathrm{RLMIN}\,(\pi) = \{\pi(i) : |\pi(i)| < |\pi(j)| \text{ for all } j > i\}.$$

For example, $\mathrm{RLMIN}\,(\bar{3}\ \bar{1}42\bar{6}7\bar{5}) = \{\bar{1}, 2, \bar{5}\}$. For $\pi \in \mathcal{B}_n$, we let $\mathrm{rlmin}\,(\pi) = \#\mathrm{RLMIN}\,(\pi)$. For $\pi \in \mathfrak{S}_n$, we let $\mathrm{rlmin}\,(\pi) = \#\{\pi(i) : \pi(i) < \pi(j) \text{ for all } j > i\}$. For $n \geqslant 1$, we have

$$\sum_{\pi \in \mathcal{B}_n} x^{\mathrm{rlmin}\,(\pi)} = 2^n \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{rlmin}\,(\pi)} = 2^n x(x+1)(x+2)\cdots(x+n-1).$$

Definition 18. A block of π is a maximal subsequence of consecutive elements of π ending with $\pi(i) \in \mathrm{RLMIN}(\pi)$ and not contain any other element of $\mathrm{RLMIN}(\pi)$.

It is clear that any π has a unique decomposition as a sequence of its blocks. If $\mathrm{rlmin}(\pi) = k$, then we write $\pi \mapsto B_1 B_2 \cdots B_k$, where B_i is ith block of π. A *bar-block* (resp. *unbar-block*) is a block ending with a negative (resp. positive) element. Let $\mathrm{Bar}(\pi)$ be the union of elements of bar-blocks of π and let $\mathrm{bar}(\pi) = \#\mathrm{Bar}(\pi)$. We define a map θ by

$$\theta(\mathrm{Bar}(\pi)) = \{|\pi(i)| : \pi(i) \in \mathrm{Bar}(\pi)\}.$$

Set $\mathrm{NBar}(\pi) = [n]/\mathrm{Bar}(\pi)$. For example, if $\pi = \overline{3}\ \overline{1}42\overline{6}7\overline{5}$, then $\pi \mapsto [\overline{3}\ \overline{1}][42][\overline{6}7\overline{5}]$, $[\overline{3}\ \overline{1}]$ and $[\overline{6}7\overline{5}]$ are bar-blocks of π, $\mathrm{Bar}(\pi) = \{\overline{6}, \overline{5}, \overline{3}, \overline{1}, 7\}, \mathrm{bar}(\pi) = 5$, $\theta(\mathrm{Bar}(\pi)) = \{1, 3, 5, 6, 7\}$. and $\mathrm{NBar}(\pi) = \{2, 4\}$.

We define $\mathcal{B}_{n,k} = \{\pi \in \mathcal{B}_n : \mathrm{bar}(\pi) = k\}$ and

$$\mathcal{BM}_{n,k} = \{(T_1, T_2, I_{n,k}) : T_1 \in \mathcal{M}_{2k},\ T_2 \in \mathcal{M}_{2n-2k},\ I_{n,k} \in \mathrm{I}_{n,k}\},$$

where $\mathrm{I}_{n,k}$ is the set of subsets of $[n]$ with the cardinality k. In this subsection, we always assume that the weight of $\pi \in \mathcal{B}_{n,k}$ is $x^{\mathrm{des}_B(\pi)}$ and that of the pair of matchings (T_1, T_2) is $x^{\mathrm{el}(T_1) + \mathrm{ol}(T_2)}$.

Along the same lines as the proof of (3), we start to construct a bijection, denoted by Ψ, between $\mathcal{B}_{n,k}$ and $\mathcal{BM}_{n,k}$. When $n = 1$, set $\Psi(1) = (\varnothing, (1, 2), \varnothing)$ and $\Psi(\overline{1}) = ((1, 2), \varnothing, \{1\})$. This gives a bijection between $\mathcal{B}_{1,k}$ and $\mathcal{BM}_{1,k}$. When $n = 2$, the bijection Ψ between $\mathcal{B}_{2,k}$ and $\mathcal{BM}_{2,k}$ is given as follows:

$$\begin{aligned}
&\Psi(12) = (\varnothing, (1, 2)(3, 4), \varnothing),\ \Psi(21) = (\varnothing, (1, 3)(2, 4), \varnothing)\\
&\Psi(\overline{2}1) = (\varnothing, (1, 4)(2, 3), \varnothing),\ \Psi(\overline{1}2) = ((1, 2), (1, 2), \{1\}),\\
&\Psi(1\overline{2}) = ((1, 2), (1, 2), \{2\}),\ \Psi(\overline{1}\ \overline{2}) = ((1, 2)(3, 4), \varnothing, \{1, 2\}),\\
&\Psi(2\overline{1}) = ((1, 3)(2, 4), \varnothing, \{1, 2\}),\ \Psi(\overline{2}\ \overline{1}) = ((1, 4)(2, 3), \varnothing, \{1, 2\}).
\end{aligned}$$

Suppose Ψ is a bijection between $\mathcal{B}_{m-1,k}$ and $\mathcal{BM}_{m-1,k}$ for all k, where $m \geqslant 3$. Assume that $\pi = \pi(1)\pi(2)\cdots\pi(m-1) \in \mathcal{B}_{m-1,k}$, $\mathrm{des}_B(\pi) = i + j$ and $\mathrm{Bar}(\pi) = \{\pi(i_1), \pi(i_2) \ldots, \pi(i_k)\}$. Let $\Psi(\pi) = (T_1, T_2, I_{m-1,k})$, where

$$T_1 \in \mathcal{M}_{2k},\ T_2 \in \mathcal{M}_{2m-2k-2},\ I_{m-1,k} = \theta(\mathrm{Bar}(\pi)),\ \mathrm{el}(T_1) = i, \mathrm{ol}(T_2) = j.$$

Consider the case $n = m$. Let π' be obtained from π by inserting the entry m (resp. $\overline{m}$) into π. We first distinguish two cases: If $\pi' = \pi m$, then let

$$\Psi(\pi') = (T_1, T_2(2m - 2k - 1, 2m - 2k), I_{m,k}),$$

where $I_{m,k} = \theta(\mathrm{Bar}(\pi))$; If $\pi' = \pi\overline{m}$, then let $\Psi(\pi') = (T_1(2k + 1, 2k + 2), T_2, I_{m,k+1})$, where $I_{m,k+1} = \theta(\mathrm{Bar}(\pi)) \cup \{m\}$.

For $0 \leqslant i \leqslant m - 2$, consider the consecutive subword $\pi(i)\pi(i + 1)$ of π. Firstly, suppose that $\pi(i + 1) \in \mathrm{NBar}(\pi)$. We say $\pi(i + 1)$ is a *unbar-ascent-top* (resp. *unbar-descent-bottom*) if $\pi(i) < \pi(i + 1)$ (resp. $\pi(i) > \pi(i + 1)$). Consider the following two cases:

(c_1) If $\pi' = \cdots \pi(i)m\pi(i+1)\cdots$ (resp. $\pi' = \cdots \pi(i)\overline{m}\pi(i+1)\cdots$) and $\pi(i+1)$ is the pth unbar-ascent-top of π, then let $\Psi(\pi') = (T_1, T_2', I_{m,k})$, where T_2' is obtained from T_2 by replacing the pth marked block (a,b) by two blocks $(a, 2m-2k-1), (b, 2m-2k)$ (resp. $(a, 2m-2k), (b, 2m-2k-1)$) and $I_{m,k} = \theta(\mathrm{Bar}\,(\pi))$.

(c_2) If $\pi' = \cdots \pi(i)m\pi(i+1)\cdots$ (resp. $\pi' = \cdots \pi(i)\overline{m}\pi(i+1)\cdots$) and $\pi(i+1)$ is the pth unbar-descent-bottom of π, then let $\Psi(\pi') = (T_1, T_2', I_{m,k})$, where T_2' is obtained from T_2 by replacing the pth unmarked block (a,b) by two blocks $(a, 2m-2k-1), (b, 2m-2k)$ (resp. $(a, 2m-2k), (b, 2m-2k-1)$) and $I_{m,k} = \theta(\mathrm{Bar}\,(\pi))$.

Secondly, suppose that $\pi(i+1) \in \mathrm{Bar}\,(\pi)$. We say $\pi(i+1)$ is a *bar-ascent-top* (resp. *bar-descent-bottom*) if $\pi(i) < \pi(i+1)$ (resp. $\pi(i) > \pi(i+1)$). Consider the following two cases:

(c_1) If $\pi' = \cdots \pi(i)m\pi(i+1)\cdots$ (resp. $\pi' = \cdots \pi(i)\overline{m}\pi(i+1)\cdots$) and $\pi(i+1)$ is the pth bar-ascent-top of π, then let $\Psi(\pi') = (T_1', T_2, I_{m,k+1})$, where T_1' is obtained from T_1 by replacing the pth unmarked block (a,b) by two blocks $(a, 2k+1), (b, 2k+2)$ (resp. $(a, 2k+2), (b, 2k+1)$) and $I_{m,k+1} = \theta(\mathrm{Bar}\,(\pi)) \cup \{m\}$.

(c_2) If $\pi' = \cdots \pi(i)m\pi(i+1)\cdots$ (resp. $\pi' = \cdots \pi(i)\overline{m}\pi(i+1)\cdots$) and $\pi(i+1)$ is the pth bar-descent-bottom of π, then let $\Psi(\pi') = (T_1', T_2, I_{m,k+1})$, where T_1' is obtained from T_1 by replacing the pth marked block (a,b) by two blocks $(a, 2k+1), (b, 2k+2)$ (resp. $(a, 2k+2), (b, 2k+1)$) and $I_{m,k+1} = \theta(\mathrm{Bar}\,(\pi)) \cup \{m\}$.

After the above step, we write the obtained perfect matching in standard form. Suppose that $\pi \in \mathcal{B}_{n,k}$ and $\Psi(\pi) = (T_1, T_2, I_{n,k})$. Then $\mathrm{des}_B(\pi) = i + j$ if and only if $\mathrm{el}\,(T_1) + \mathrm{ol}\,(T_2) = i + j$. By induction, we see that Ψ is the desired bijection between $\mathcal{B}_{n,k}$ and $\mathcal{BM}_{n,k}$ for all k, which also gives a constructive proof of (4).

Example 19. The correspondence between $\pi = \overline{3}\,\overline{1}42\overline{6}7\overline{5}$ and $\Psi(\pi)$ is built up as follows:

$$\begin{aligned}
\overline{1} &\Leftrightarrow ((1,2), \varnothing, \{1\});\\
\overline{1}2 &\Leftrightarrow ((1,2), (1,2), \{1\});\\
\overline{3}\,\overline{1}2 &\Leftrightarrow ((1,4)(2,3), (1,2), \{1,3\});\\
\overline{3}\,\overline{1}42 &\Leftrightarrow ((1,4)(2,3), (1,3)(2,4), \{1,3\});\\
\overline{3}\,\overline{1}42\overline{5} &\Leftrightarrow ((1,4)(2,3)(5,6), (1,3)(2,4), \{1,3,5\});\\
\overline{3}\,\overline{1}42\overline{6}\,\overline{5} &\Leftrightarrow ((1,3)(2,4)(5,8)(6,7), (1,3)(2,4), \{1,3,5,6\});\\
\overline{3}\,\overline{1}42\overline{6}7\overline{5} &\Leftrightarrow ((1,3)(2,4)(5,8)(6,9)(7,10), (1,3)(2,4), \{1,3,5,6,7\}).
\end{aligned}$$

5 Concluding remarks

Given $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. We say that σ_i is a *left-to-right minimum* of σ if $\sigma_i < \sigma_j$ for all $1 \leqslant j < i$. We now present a bijection between $\mathcal{Q}_n$ and $\mathcal{Q}_n^2$. Define $\widehat{\sigma}$ to be the Stirling permutation of the second kind obtained from σ by inserting a left parenthesis in σ preceding every left-to-right minimum. Then insert a right parenthesis before every internal left parenthesis and at the end. For example, if $\sigma = 331221$, then $\widehat{\sigma} = (33)(1221)$. It is easy to verify that we can uniquely recover σ from $\widehat{\sigma}$ by requiring that (a) each cycle is written with its smallest element first, (b) the cycles are written in decreasing order of their smallest element, and (c) we then erase all parentheses.

A natural generalization of Stirling permutations is k-Stirling permutations. Let j^i denote the i copies of j, where $i, j \geqslant 1$. We call a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ a *k-Stirling permutation* of order n if for each i, $1 \leqslant i \leqslant n$, all entries between the two occurrences of i are at least i. One can introduce k-Stirling permutations of the second kind along the same line as in Definition 2. Moreover, it would be interesting to investigate an analog of (18) on Coxeter groups of types B and D. Furthermore, one may find some multivariate extensions of Proposition 1.

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