Discrete Mathematics 340 (2017) 3097–3103

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Hankel determinants of linear combinations of consecutive Catalan-like numbers

Lili Mu [a], Yi Wang [b,*], Yeong-Nan Yeh [b,c]

[a] School of Mathematics, Liaoning Normal University, Dalian 116024, China
[b] School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
[c] Institute of Mathematics, Academia Sinica, Taipei 10617, China

ARTICLE INFO

Article history:
Received 13 March 2017
Received in revised form 22 June 2017
Accepted 5 July 2017
Available online 2 August 2017

Keywords:
Hankel determinant
Catalan-like number
Generating function

ABSTRACT

Let $(a_n)_{n\geq 0}$ be a sequence of the Catalan-like numbers. We evaluate Hankel determinants $\det[\lambda a_{i+j}+\mu a_{i+j+1}]_{0\leq i,j\leq n}$ and $\det[\lambda a_{i+j+1}+\mu a_{i+j+2}]_{0\leq i,j\leq n}$ for arbitrary coefficients λ and μ. Our results unify many known results of Hankel determinant evaluations for classic combinatorial counting coefficients, including the Catalan, Motzkin and Schröder numbers.

1. Introduction

Given a sequence $(a_n)_{n\geq 0}$, define its *Hankel matrix* $[a_{i+j}]_{i,j\geq 0}$ and the nth *Hankel determinant* $\det[a_{i+j}]_{0\leq i,j\leq n}$. Hankel determinants occur naturally in diverse areas of mathematics. In recent years, there has been a considerable amount of interest in the evaluation of Hankel determinants $\det[a_{i+j+m}]_{0\leq i,j\leq n}$ and $\det[a_{i+j+m}+a_{i+j+m+1}]_{0\leq i,j\leq n}$ involving various combinatorial sequences [1–3,5–12,14–16,18]. As we will see in Examples 2.4 and 2.5, these combinatorial sequences, including the Catalan numbers, the Motzkin numbers and the Schröder numbers, turn out to be the so-called Catalan-like numbers (or generalized Motzkin numbers [19]). The purpose of this paper is to provide a unified framework for previous results from the viewpoint of Catalan-like numbers.

Let $\mathfrak{s}=(s_k)_{k\geq 0}$ and $\mathfrak{t}=(t_k)_{k\geq 1}$ be two sequences of nonnegative numbers and define an infinite lower triangular matrix $A=[a_{n,k}]_{n,k\geq 0}$ by the recurrence

$$a_{0,0}=1,\quad a_{n+1,k}=a_{n,k-1}+s_k a_{n,k}+t_{k+1}a_{n,k+1}, \tag{1.1}$$

where $a_{n,k}=0$ unless $n\geq k\geq 0$. Clearly, all $a_{n,n}=1$. Following Aigner [3], we say that A is the *recursive matrix* and $a_n=a_{n,0}$ are the nth *Catalan-like numbers* corresponding to $(\mathfrak{s},\mathfrak{t})$.

Example 1.1. The Catalan-like numbers unify many well-known counting coefficients, such as

(i) the Catalan numbers C_n when $\mathfrak{s}=(1,2,2,\ldots)$ and $\mathfrak{t}=(1,1,1,\ldots)$;
(ii) the shifted Catalan numbers C_{n+1} when $\mathfrak{s}=(2,2,2,\ldots)$ and $\mathfrak{t}=(1,1,1,\ldots)$;
(iii) the Motzkin numbers M_n when $\mathfrak{s}=\mathfrak{t}=(1,1,1,\ldots)$;
(iv) the central binomial coefficients $\binom{2n}{n}$ when $\mathfrak{s}=(2,2,2,\ldots)$ and $\mathfrak{t}=(2,1,1,\ldots)$;

* Corresponding author.
E-mail addresses: lly-mu@hotmail.com (L. Mu), wangyi@dlut.edu.cn (Y. Wang), mayeh@gate.sinica.edu.tw (Y.-N. Yeh).

http://dx.doi.org/10.1016/j.disc.2017.07.004
0012-365X/

(v) the central trinomial coefficients T_n when $\mathfrak{s} = (1, 1, 1, \ldots)$ and $\mathfrak{t} = (2, 1, 1, \ldots)$;
(vi) the central Delannoy numbers D_n when $\mathfrak{s} = (3, 3, 3, \ldots)$ and $\mathfrak{t} = (4, 2, 2, \ldots)$;
(vii) the large Schröder numbers r_n when $\mathfrak{s} = (2, 3, 3, \ldots)$ and $\mathfrak{t} = (2, 2, \ldots)$;
(viii) the little Schröder numbers S_n when $\mathfrak{s} = (1, 3, 3, \ldots)$ and $\mathfrak{t} = (2, 2, \ldots)$;
(ix) the Fine numbers F_n when $\mathfrak{s} = (0, 2, 2, \ldots)$ and $\mathfrak{t} = (1, 1, 1, \ldots)$;
(x) the Riordan numbers R_n when $\mathfrak{s} = (0, 1, 1, \ldots)$ and $\mathfrak{t} = (1, 1, 1, \ldots)$;
(xi) the (restricted) hexagonal numbers h_n when $\mathfrak{s} = (3, 3, 3, \ldots)$ and $\mathfrak{t} = (1, 1, 1, \ldots)$;
(xii) the Bell numbers B_n when $\mathfrak{s} = \mathfrak{t} = (1, 2, 3, 4, \ldots)$;
(xiii) the factorial $n!$ when $\mathfrak{s} = (1, 3, 5, 7, \ldots)$ and $\mathfrak{t} = (1, 4, 9, 16, \ldots)$.

The Catalan-like numbers have a nice combinatorial interpretation from the viewpoint of weighted lattice paths. A Motzkin path of length n is a lattice path from $(0, 0)$ to $(n, 0)$ consisting of up steps $(1, 1)$, down steps $(1, -1)$ and horizontal steps $(1, 0)$ that never falls below the x-axis. The height of a step in a Motzkin path is the y coordinate of the starting point. Assign a weight 1 (s_k, t_k, resp.) to all up steps (all horizontal steps, all down steps, resp.) of height k. Define the weight of a Motzkin path to be the product of weights of its steps. Then the Catalan-like number a_n counts the total weight of all Motzkin paths of length n.

The Catalan-like numbers are closely related to continued fractions and orthogonal polynomials. Let a_n be the Catalan-like numbers corresponding to $(\mathfrak{s}, \mathfrak{t})$. Then

$$\sum_{n\geq 0} a_n x^n = \cfrac{1}{1 - s_0 x - \cfrac{t_1 x^2}{1 - s_1 x - \cfrac{t_2 x^2}{1 - s_2 x - \cdots}}}.$$

Let $(p_n(x))_{n\geq 0}$ be the sequence of orthogonal polynomials with respect to the linear operator $\mathcal{L}(x^n) = a_n$. Then $\mathcal{L}(p_m(x)p_n(x)) = \delta_{m,n} t_1 \cdots t_n$ and

$$p_{n+1}(x) = (x - s_n)p_n(x) - t_n p_{n-1}(x), \quad p_0(x) = 1.$$

For an infinite matrix $M = [m_{i,j}]_{i,j\geq 0}$, let $M_n = [m_{i,j}]_{0\leq i,j\leq n}$ denote its nth leading principal submatrix and $\delta_n(M) = \det M_n$. For convenience, denote $\delta_{-1}(M) = 1$. Let

$$J = \begin{bmatrix} s_0 & 1 & & & \\ t_1 & s_1 & 1 & & \\ & t_2 & s_2 & 1 & \\ & & t_3 & s_3 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

denote the coefficient matrix of the recursive relation (1.1) and $d_n = \delta_n(J)$. Denote $T_0 = 1$ and $T_n = t_1 t_2 \cdots t_n$ for $n \geq 1$. The following result is folklore (see [3,19] for instance).

Proposition 1.2. *Let a_n be the Catalan-like numbers corresponding to $(\mathfrak{s}, \mathfrak{t})$. Then*

(i) $\det[a_{i+j}]_{0\leq i,j\leq n} = T_1 \cdots T_n$.
(ii) $\det[a_{i+j+1}]_{0\leq i,j\leq n} = T_1 \cdots T_n d_n$.
(iii) $\det[a_{i+j+2}]_{0\leq i,j\leq n} = T_1 \cdots T_n T_{n+1} \sum_{i=-1}^{n} \frac{d_i^2}{T_{i+1}}$.

For $\lambda, \mu \in \mathbb{R}$, let

$$d_n^{(\lambda,\mu)} = \det \begin{bmatrix} \lambda + \mu s_0 & \mu & & & \\ \mu t_1 & \lambda + \mu s_1 & \mu & & \\ & \mu t_2 & \ddots & \ddots & \\ & & \ddots & \lambda + \mu s_{n-1} & \mu \\ & & & \mu t_n & \lambda + \mu s_n \end{bmatrix}.$$

Then $d_n^{(1,0)} = 1$ and $d_n^{(0,1)} = d_n$. Our main results are the following general formulae.

Theorem 1.3. *Let a_n be the Catalan-like numbers corresponding to $(\mathfrak{s}, \mathfrak{t})$. Then*

(i) $\det[\lambda a_{i+j} + \mu a_{i+j+1}]_{0\leq i,j\leq n} = T_1 \cdots T_n d_n^{(\lambda,\mu)}$.
(ii) $\det[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{0\leq i,j\leq n} = T_1 \cdots T_n T_{n+1} \sum_{i=-1}^{n} \frac{d_i d_i^{(\lambda,\mu)}}{T_{i+1}} \mu^{n-i}$.

In the next section, we give the proof of the theorem and then present applications on some interesting Catalan-like numbers. Our results unify many known results of Hankel determinant evaluations for classic combinatorial counting coefficients, including the Catalan, Motzkin and Schröder numbers.

2. Proof and applications of Theorem 1.3

We first present the proof of the theorem.

Proof of Theorem 1.3. Let $A = [a_{n,k}]_{n,k\geq 0}$ be the recursive matrix corresponding to $(\mathfrak{s}, \mathfrak{t})$. Then the recurrence (1.1) is equivalent to $\bar{A} = AJ$, where

$$\bar{A} = \begin{bmatrix} a_{1,0} & a_{1,1} & & & \\ a_{2,0} & a_{2,1} & a_{2,2} & & \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & \\ \vdots & & & & \ddots \end{bmatrix}.$$

On the other hand, let

$$K = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{bmatrix}.$$

Then $KA = \bar{A}$. Thus we have $KA = AJ$.

Clearly,

$$[\lambda a_{i+j} + \mu a_{i+j+1}]_{i,j\geq 0} = \begin{bmatrix} \lambda & \mu & & & \\ & \lambda & \mu & & \\ & & \lambda & \mu & \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \\ a_2 & a_3 & a_4 & \\ \vdots & & & \ddots \end{bmatrix}.$$

Let I be the identity matrix and $H = [a_{i+j}]_{i,j\geq 0}$ the Hankel matrix of the Catalan-like numbers a_n. Then

$$[\lambda a_{i+j} + \mu a_{i+j+1}]_{i,j\geq 0} = (\lambda I + \mu K)H.$$

Recall that the Aigner's Fundamental Theorem [3]:

$$a_{m+n} = \sum_k a_{m,k}a_{n,k}T_k, \quad m, n \geq 0.$$

In other words, $H = ATA^t$, where $T = \mathrm{diag}(T_0, T_1, T_2, \ldots)$. It follows that

$$[\lambda a_{i+j} + \mu a_{i+j+1}]_{i,j\geq 0} = (\lambda I + \mu K)ATA^t = A(\lambda I + \mu J)TA^t.$$

Note that A is a lower triangular matrix with all diagonal entries equal to 1 and T is a diagonal matrix. Hence

$$\det[\lambda a_{i+j} + \mu a_{i+j+1}]_{0\leq i,j\leq n} = \delta_n(\lambda I + \mu J)T_0T_1\cdots T_n.$$

This proves (i).

Similarly, we have

$$[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{i,j\geq 0} = \begin{bmatrix} 0 & \lambda & \mu & & \\ & 0 & \lambda & \mu & \\ & & 0 & \lambda & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \\ a_2 & a_3 & a_4 & \\ \vdots & & & \ddots \end{bmatrix}.$$

Thus

$$[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{i,j\geq 0} = (\lambda K + \mu K^2)ATA^t = A(\lambda J + \mu J^2)TA^t,$$

and so

$$\det[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{0\leq i,j\leq n} = \delta_n(\lambda J + \mu J^2)T_0T_1\cdots T_n.$$

Let

$$J_i = \begin{bmatrix} s_0^{(i)} & r_1^{(i)} & & & \\ t_1^{(i)} & s_1^{(i)} & r_2^{(i)} & & \\ & t_2^{(i)} & s_2^{(i)} & r_3^{(i)} & \\ & & t_3^{(i)} & s_3^{(i)} & \ddots \\ & & & \ddots & \ddots \end{bmatrix}, \quad i = 1, 2$$

be two tridiagonal matrices. Then by the Cauchy–Binet formula,

$$\begin{aligned}\delta_n(J_1J_2) &= \delta_n(J_1)\delta_n(J_2) + \delta_{n-1}(J_1)r_{n+1}^{(1)}\delta_{n-1}(J_2)t_{n+1}^{(2)} \\ &\quad + \delta_{n-2}(J_1)r_{n+1}^{(1)}r_n^{(1)}\delta_{n-2}(J_2)t_{n+1}^{(2)}t_n^{(2)} \\ &\quad + \cdots \\ &\quad + s_0^{(1)}r_{n+1}^{(1)}r_n^{(1)}\cdots r_2^{(1)}s_0^{(2)}t_{n+1}^{(2)}t_n^{(2)}\cdots t_2^{(2)} \\ &\quad + r_{n+1}^{(1)}r_n^{(1)}\cdots r_1^{(1)}t_{n+1}^{(2)}t_n^{(2)}\cdots t_1^{(2)} \\ &= \sum_{j=-1}^{n}\delta_j(J_1)\delta_j(J_2)\frac{\prod_{k=0}^{n+1}r_k^{(1)}t_k^{(2)}}{\prod_{k=0}^{j+1}r_k^{(1)}t_k^{(2)}},\end{aligned}$$

where $\delta_{-1}(J_i) = r_0^{(1)} = t_0^{(2)} = 1$. In particular, setting $J_1 = J$ and $J_2 = \lambda I + \mu J$, we obtain

$$\delta_n(\lambda J + \mu J^2) = \delta_n(J(\lambda I + \mu J)) = \sum_{j=-1}^{n}\delta_j(J)\delta_j(\lambda I + \mu J)\frac{T_{n+1}}{T_{j+1}}\mu^{n-j}.$$

Thus (ii) follows. This completes the proof. □

In the rest of this section, we apply Theorem 1.3 to a class of particularly interesting Catalan-like numbers.

Let p, q, s, t be four nonnegative numbers. Define a recursive matrix $A(p, s; q, t) = [a_{n,k}]_{n,k\geq 0}$ by

$$a_{0,0} = 1, \quad a_{n+1,0} = pa_{n,0} + qa_{n,1}, \quad a_{n+1,k} = a_{n,k-1} + sa_{n,k} + ta_{n,k+1}. \tag{2.1}$$

In this case, $\mathfrak{s} = (p, s, s, \ldots)$ and $\mathfrak{t} = (q, t, t, \ldots)$. We denote by $a_n(p, s; q, t)$ the corresponding Catalan-like numbers. Many famous counting coefficients are such numbers. For example, the Catalan numbers $C_n = a_n(1, 2; 1, 1)$, the shifted Catalan numbers $C_{n+1} = a_n(2, 2; 1, 1)$, the Motzkin numbers $M_n = a_n(1, 1; 1, 1)$, and the large Schröder numbers $r_n = a_n(2, 3; 2, 2)$. Such Catalan-like numbers can be characterized by the generating function since $A(p, s; q, t)$ is also a Riordan array. See [13] for details.

Lemma 2.1. *If a_n are the Catalan-like numbers $a_n(p, s; q, t)$, then*

$$\sum_{n\geq 0} a_nx^n = \frac{2t}{2t - q + (qs - 2pt)x + q\sqrt{1 - 2sx + (s^2 - 4t)x^2}}.$$

The converse is also true.

Lemma 2.2. *Let*

$$\bar{d}_0 = 1, \quad \bar{d}_n = \det\begin{bmatrix} s & r & & & \\ t & s & r & & \\ & t & \ddots & \ddots & \\ & & \ddots & s & r \\ & & & t & s \end{bmatrix}_n$$

for $n \geq 1$ and

$$D_n = \det\begin{bmatrix} p & h & & & & \\ q & s & r & & & \\ & t & s & r & & \\ & & t & \ddots & \ddots & \\ & & & \ddots & s & r \\ & & & & t & s \end{bmatrix}_{n+1}$$

for $n \geq 0$. Then

$$\sum_{n\geq 0}\bar{d}_nx^n = \frac{1}{1 - sx + rtx^2}, \qquad \sum_{n\geq 0}D_nx^n = \frac{p - hqx}{1 - sx + rtx^2}.$$

Proof. We have $\bar{d}_n = s\bar{d}_{n-1} - rt\bar{d}_{n-2}$, with $\bar{d}_0 = 1$ and $\bar{d}_1 = s$. Let $\bar{d}(x) = \sum_{n\geq 0}\bar{d}_n x^n$. Then

$$\bar{d}(x) = \bar{d}_0 + \bar{d}_1 x + \sum_{n\geq 2}(s\bar{d}_{n-1} - rt\bar{d}_{n-2})x^n = 1 + sx\bar{d}(x) - rtx^2\bar{d}(x).$$

It follows that $\bar{d}(x) = 1/(1 - sx + rtx^2)$.

On the other hand, $D_n = p\bar{d}_n - hq\bar{d}_{n-1}$ for $n \geq 0$, where $\bar{d}_{-1} = 0$. Thus

$$\sum_{n\geq 0} D_n x^n = p\bar{d}(x) - hqx\bar{d}(x) = \frac{p - hqx}{1 - sx + rtx^2}.$$

This completes the proof. □

Corollary 2.3. *Let $a_n = a_n(p, s; q, t)$ be the Catalan-like numbers. Then*

(i) $\det[\lambda a_{i+j} + \mu a_{i+j+1}]_{0\leq i,j\leq n} = q^n t^{\binom{n}{2}} d_n^{(\lambda,\mu)}$ *and*
(ii) $\det[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{0\leq i,j\leq n} = q^n t^{\binom{n}{2}} \sum_{i=-1}^{n} d_i d_i^{(\lambda,\mu)}(\mu t)^{n-i}$,

where

$$\sum_{n\geq 0} d_n^{(\lambda,\mu)} x^n = \frac{(\lambda + \mu p) - \mu^2 qx}{1 - (\lambda + \mu s)x + \mu^2 tx^2},$$

and

$$\sum_{n\geq 0} d_n x^n = \frac{p - qx}{1 - sx + tx^2}.$$

Example 2.4. Consider the generalized Catalan numbers $C_n(t)$ defined in [4], which have the generating function

$$\sum_{n\geq 0} C_n(t) x^n = \frac{2}{1 - (t-1)x + \sqrt{1 - 2(t+1)x + (t-1)^2 x^2}}. \tag{2.2}$$

Therefore $C_n(t)$ are precisely the Catalan-like numbers $a_n(t, t+1; t, t)$. In particular, $C_n(1)$ are the Catalan numbers and $C_n(2)$ are the large Schröder numbers. It follows from (2.2) that

$$\sum_{n\geq 0} C_{n+1}(t) x^n = \frac{2t}{1 - (t+1)x + \sqrt{1 - 2(t+1)x + (t-1)^2 x^2}}.$$

In other words, $C_{n+1}(t)/t$ are the Catalan-like numbers $a_n(t+1, t+1; t, t)$. By Corollary 2.3(i), we have

(i) $\det[\lambda C_{i+j}(t) + \mu C_{i+j+1}(t)]_{0\leq i,j\leq n} = t^{\binom{n+1}{2}} d_n^{(\lambda,\mu)}$ and
(ii) $\det[\lambda C_{i+j+1}(t) + \mu C_{i+j+2}(t)]_{0\leq i,j\leq n} = t^{\binom{n+2}{2}} D_n^{(\lambda,\mu)}$,

where

$$\sum_{n\geq 0} d_n^{(\lambda,\mu)} x^n = \frac{(\lambda + \mu t) - \mu^2 tx}{1 - (\lambda + \mu(t+1))x + \mu^2 tx^2}$$

and

$$\sum_{n\geq 0} D_n^{(\lambda,\mu)} x^n = \frac{(\lambda + \mu(t+1)) - \mu^2 tx}{1 - (\lambda + \mu(t+1))x + \mu^2 tx^2},$$

or equivalently,

$$\sum_{n\geq 0} d_{n-1}^{(\lambda,\mu)} x^n = \frac{1 - \mu x}{1 - (\lambda + \mu(t+1))x + \mu^2 tx^2}, \quad d_{-1}^{(\lambda,\mu)} = 1$$

and

$$\sum_{n\geq 0} D_{n-1}^{(\lambda,\mu)} x^n = \frac{1}{1 - (\lambda + \mu(t+1))x + \mu^2 tx^2}, \quad D_{-1}^{(\lambda,\mu)} = 1.$$

The result in the special case $\lambda = \mu = 1$ has been obtained in [15] by an analytic approach. In particular, taking $t = 1$, we have the following result about the Catalan numbers C_n:

$$\det[C_{i+j} + C_{i+j+1}]_{0\le i,j\le n} = F_{2n+2} \text{ and } \det[C_{i+j+1} + C_{i+j+2}]_{0\le i,j\le n} = F_{2n+3},$$

where F_n are the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \ge 0$. This elegant result first occurred in [8].

A Schröder path of length n is a lattice path from $(0, 0)$ to $(2n, 0)$ using up steps $(1, 1)$, down steps $(1, -1)$, and level steps $(2, 0)$, and never passing below the x-axis. Let each up step and each down step have weight 1 and each level step have weight t. The t-Schröder number $r_n^{(t)}$ counts the total weight of all Schröder paths of length n. Sulanke and Xin [16, Proposition 2.1] gave the generating function

$$\sum_{n\ge0} r_n^{(t)} x^n = \frac{2}{1 - tx + \sqrt{1 - 2(t+2)x + t^2x^2}}.$$

Hence the t-Schröder number $r_n^{(t)}$ are precisely the generalized Catalan numbers $C_n(t + 1)$. Sulanke and Xin obtained $\det[r_{i+j}^{(t)}]_{0\le i,j\le n-1} = (1+t)^{\binom{n+1}{2}}$ and $\det[r_{i+j+1}^{(t)}]_{0\le i,j\le n-1} = (1+t)^{\binom{n+2}{2}}$. Rajković et al. [15] gave an explicit formula for $\det[r_{i+j}^{(t)} + r_{i+j+1}^{(t)}]_{0\le i,j\le n-1}$. Eu et al. [10] evaluated $\det[\lambda r_{i+j}^{(t)} + \mu r_{i+j+1}^{(t)}]_{0\le i,j\le n-1}$ and $\det[\lambda r_{i+j+1}^{(t)} + \mu r_{i+j+2}^{(t)}]_{0\le i,j\le n-1}$ by lattice path techniques. These results are immediate from our results.

Example 2.5. Sun [17] defined the generalized Motzkin numbers $M_n(s, t)$ by

$$M_n(s,t) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} C_k s^{n-2k} t^k,$$

which are common generalizations of the Motzkin numbers and the Catalan numbers:

$$M_n(1, 1) = M_n, \quad M_n(2, 1) = C_{n+1}.$$

Sun gave the generating function

$$\sum_{n\ge0} M_n(s,t) x^n = \frac{2}{1 - sx + \sqrt{1 - 2sx - (s^2 - 4t)x^2}}.$$

Therefore $M_n(s, t)$ are precisely the Catalan-like numbers $a_n(s, s; t, t)$. Thus we have

$$\det[\lambda M_{i+j}(s,t) + \mu M_{i+j+1}(s,t)]_{0\le i,j\le n} = t^{\binom{n+1}{2}} d_n^{(\lambda,\mu)}$$

and

$$\det[\lambda M_{i+j+1}(s,t) + \mu M_{i+j+2}(s,t)]_{0\le i,j\le n} = t^{\binom{n+1}{2}} \sum_{i=-1}^{n} d_i d_i^{(\lambda,\mu)} (\mu t)^{n-i},$$

where

$$\sum_{n\ge0} d_{n-1}^{(\lambda,\mu)} x^n = \frac{1}{1 - (\lambda + \mu s)x + \mu^2 t x^2}, \quad d_{-1}^{(\lambda,\mu)} = 1$$

and

$$\sum_{n\ge0} d_{n-1} x^n = \frac{1}{1 - sx + tx^2}, \quad d_{-1} = 1.$$

An s-Motzkin path is a Motzkin path in which the up step, down step, and unit level step have weights 1, 1 and s respectively. Let $M_n^{(s)}$ be the total weight of all s-Motzkin paths of length n. Then $M_n^{(s)}$ are precisely the generalized Motzkin numbers $M_n(s, 1)$. Thus the following determinant evaluations, obtained by Cameron and Yip [6] using lattice path techniques, are now immediate:

(i) $\det[M_{i+j+1}^{(s)}]_{0\le i,j\le n} = d_n(s)$;

(ii) $\det[M_{i+j+2}^{(s)}]_{0\le i,j\le n} = \sum_{i=-1}^{n} d_i^2(s)$;

(iii) $\det[M_{i+j}^{(s)} + M_{i+j+1}^{(s)}]_{0\le i,j\le n} = d_n(s + 1)$;

(iv) $\det[M_{i+j+1}^{(s)} + M_{i+j+2}^{(s)}]_{0\le i,j\le n} = \sum_{i=-1}^{n} d_i(s) d_i(s + 1)$,

where

$$\sum_{n\ge0} d_{n-1}(s) x^n = \frac{1}{1 - sx + x^2}.$$

In particular, $d_n(1) = \sin(n+2)\omega/\sin\omega$, where $\omega = \pi/3$, and $d_n(2) = n+2$. Thus the results reduce to Hankel determinants for the Motzkin numbers:

(i) $$\det[M_{i+j+1}]_{0\le i,j\le n} = \begin{cases} 1, & \text{if } n \equiv 0, 5 \pmod 6; \\ 0, & \text{if } n \equiv 1, 4 \pmod 6; \\ -1, & \text{if } n \equiv 2, 3 \pmod 6). \end{cases}$$

(ii) $$\det[M_{i+j+2}]_{0\le i,j\le n} = \begin{cases} 2m+2, & \text{if } n = 3m \text{ or } n = 3m+1; \\ 2m+3, & \text{if } n = 3m+2. \end{cases}$$

(iii) $$\det[M_{i+j+1} + M_{i+j+2}]_{0\le i,j\le n} = \begin{cases} (-1)^{m+1}, & \text{if } n = 3m+2; \\ 3(m+1)(-1)^m, & \text{if } n = 3m \text{ or } n = 3m+1. \end{cases}$$

These results for the Motzkin numbers have occurred in the literature [1,3,6,9].

3. Remarks

Given a sequence $(x_n)_{n\ge 0}$, let $h_n^{(k)} = \det[x_{i+j+k}]_{0\le i,j\le n}$. Then

$$h_{n+1}^{(k)} h_{n-1}^{(k+2)} = h_n^{(k)} h_n^{(k+2)} - [h_n^{(k+1)}]^2$$

by the famous Desnanot–Jacobi identity (see, e.g., [20]). It follows that

$$\frac{h_n^{(k+2)}}{h_{n+1}^{(k)}} = \sum_{i=-1}^{n} \frac{[h_i^{(k+1)}]^2}{h_i^{(k)} h_{i+1}^{(k)}}.$$

In other words, $h_n^{(k+2)}$ can be determined by all $h_n^{(k)}$ and $h_n^{(k+1)}$. For example, in Proposition 1.2, (iii) follows from (i) and (ii). Thus it is possible to determine $h_n^{(k+2)}$ by $h_n^{(0)}$ and $h_n^{(1)}$ recursively. In particular, we can evaluate $\det[\lambda a_{i+j+k} + \mu a_{i+j+k+1}]_{0\le i,j\le n}$ for $k \ge 2$ by Theorem 1.3.

Acknowledgments

This work was finished while the second author was visiting the Institute of Mathematics, Academia Sinica, Taipei. He is grateful for the Institute's financial support and hospitality. The second author was supported partially by NSFC under the Grant No. 11371078 and the third author was supported partially by NSC under the Grant No. 104-2115-M-001-010-MY3.

The authors thank the anonymous referee for his/her careful reading and helpful comments.

References

[1] M. Aigner, Motzkin numbers, European J. Combin. 19 (1998) 663–675.
[2] M. Aigner, Catalan-like numbers and determinants, J. Combin. Theory Ser. A 87 (1999) 33–51.
[3] M. Aigner, Catalan and other numbers: a recurrent theme, in: Algebraic Combinatorics and Computer Science, Springer, Milan, 2001, pp. 347–390.
[4] P. Barry, On integer-sequence-based constructions of generalized Pascal triangles, J. Integer Seq. 13 (2006) Article 06.2.4.
[5] R.A. Brualdi, S. Kirkland, Aztec diamonds and digraphs and Hankel determinants of Schröder numbers, J. Combin. Theory Ser. B 94 (2005) 334–351.
[6] N.T. Cameron, A.C.M. Yip, Hankel determinants of sums of consecutive Motzkin numbers, Linear Algebra Appl. 434 (2011) 712–722.
[7] M. Chamberland, C. French, Generalized Catalan numbers and generalized Hankel transformations, J. Integer Seq. 10 (2007) Article 07.1.1.
[8] A. Cvetković, P. Rajković, M. Ivković, Catalan numbers, the Hankel transform, and Fibonacci numbers, J. Integer Seq. 5 (2002) Article 02.1.3.
[9] M. Elouafi, A unified approach for the Hankel determinats of classical combinatorial numbers, J. Math. Anal. Appl. 431 (2015) 1253–1274.
[10] S.-P. Eu, T.-L. Wong, P.-L. Yen, Hankel determinants of sums of consecutive weighted Schröder numbers, Linear Algebra Appl. 437 (2012) 2285–2299.
[11] Q.-H. Hou, A. Lascoux, Y.-P. Mu, Evaluation of some Hankel determinants, Adv. Appl. Math. 34 (2005) 845–852.
[12] C. Krattenthaler, Determinants of (generalised) Catalan numbers, J. Statist. Plann. Inference 140 (2010) 2260–2270.
[13] H. Liang, L. Mu, Y. Wang, Catalan-like numbers and Stieltjes moment sequences, Discrete Math. 339 (2016) 484–488.
[14] L. Mu, Y. Wang, Hankel determinants of shifted Catalan-like numbers, Discrete Math. 340 (2017) 1389–1396.
[15] P.M. Rajković, M.D. Petković, P. Barry, The Hankel transform of the sum of consecutive generalized Catalan numbers, Integral Transforms Spec. Funct. 18 (2007) 285–296.
[16] R.A. Sulanke, G. Xin, Hankel determinants for some common lattice paths, Adv. Appl. Math. 40 (2008) 149–167.
[17] Z.-W. Sun, Congruences involving generalized central trinomial coefficients, Sci. China Math. 57 (2014) 1375–1400.
[18] U. Tamm, Some aspects of Hankel matrices in coding theory and combinatorics, Electron. J. Combin. 8 (2001) Article 1.
[19] X. Viennot, Une Théorie Combinatoire Des Polynômes Orthogonaux Généraux, UQAM, Montreal, Quebec, 1983.
[20] D. Zeilberger, Dodgson's determinant evaluation rule proved by two-timing men and women, Electron. J. Combin. 4 (1997). Research Paper 22.