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List colouring of graphs and generalized Dyck paths

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ABSTRACT

The Catalan numbers occur in various counting problems in combinatorics. This paper reveals a connection between the Catalan numbers and list colouring of graphs. Assume G is a graph and $f : V(G) \to N$ is a mapping. For a nonnegative integer m, let $f^{(m)}$ be the extension of f to the graph $G \oplus \overline{K_m}$ for which $f^{(m)}(v) = |V(G)|$ for each vertex v of $\overline{K_m}$. Let $m_c(G, f)$ be the minimum m such that $G \oplus \overline{K_m}$ is not $f^{(m)}$-choosable and $m_p(G, f)$ be the minimum m such that $G \oplus \overline{K_m}$ is not $f^{(m)}$-paintable. We study the parameter $m_c(K_n, f)$ and $m_p(K_n, f)$ for arbitrary mappings f. For $\vec{x} = (x_1, x_2, \dots, x_n)$, an $\vec{x}$-dominated path ending at (a, b) is a monotonic path P of the $a \times b$ grid from $(0, 0)$ to (a, b) such that each vertex (i, j) on P satisfies $i \le x_{j+1}$. Let $\psi(\vec{x})$ be the number of $\vec{x}$-dominated paths ending at (x_n, n). By this definition, the Catalan number C_n equals $\psi((0, 1, \dots, n-1))$. This paper proves that if $G = K_n$ has vertices $v_1, v_2, \dots, v_n$ and $f(v_1) \le f(v_2) \le \dots \le f(v_n)$, then $m_c(G, f) = m_p(G, f) = \psi(\vec{x}(f))$, where $\vec{x}(f) = (x_1, x_2, \dots, x_n)$ and $x_i = f(v_i) - i$ for $i = 1, 2, \dots, n$. Therefore, if $f(v_i) = n$, then $m_c(K_n, f) = m_p(K_n, f)$ equals the Catalan number C_n. We also show that if $G = G_1 \cup G_2 \cup \dots \cup G_p$ is the disjoint union of graphs $G_1, G_2, \dots, G_p$ and $f = f_1 \cup f_2 \cup \dots \cup f_p$, then $m_c(G, f) = \prod_{i=1}^{p} m_c(G_i, f_i)$ and $m_p(G, f) = \prod_{i=1}^{p} m_p(G_i, f_i)$. This generalizes a result in Carraher et al. (2014), where the case each G_i is a copy of K_1 is considered.

1. Introduction

The Catalan number C_n is the solution of many counting problems in combinatorics. In [6], a set of exercises describe 66 different interpretations of the Catalan numbers. This paper studies list colouring and online list colouring of graphs, and reveals a connection between the Catalan numbers and a solution of list colouring and online list colouring problem.

We denote by N the set of positive integers. Assume G is a graph and $f : V(G) \to N$ is a mapping. An f*-list assignment* of G is a list assignment L of G which assigns to each vertex v a set $L(v)$ of $f(v)$ colours. For a list assignment L of G, we say G is L*-colourable* if there is a proper colouring ϕ of G such that $\phi(v) \in L(v)$ for each vertex v. We say G is f*-choosable* if G is L-colourable for any f-list assignment L of G. We say G is k*-choosable* if G is f-choosable for the constant function $f \equiv k$. The *choice number* of G, denoted by $ch(G)$, is the minimum integer k for which G is k-choosable.

For a mapping $f : V(G) \to N$, the f*-painting game* is played by two players: Lister and Painter. Initially, each vertex v is assigned $f(v)$ tokens and no vertex is coloured. In each round, Lister chooses a set M of uncoloured vertices and removes one token from each chosen vertex. Painter chooses an independent set I of G contained in M and colours every vertex in I. If at the end of some round, there is an uncoloured vertex with no tokens left, then Lister wins the game. Otherwise at the end of some round, all vertices are coloured and Painter wins the game. We say G is f*-paintable* if Painter has a winning strategy

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in this game. We say G is *k-paintable* if G is f-paintable for the constant function $f \equiv k$ (i.e., $f(v) = k$ for all $v \in V(G)$). The *paint number* of G, denoted by $\chi_P(G)$, is the minimum integer k for which G is k-paintable. The list colouring and the painting game (also known as online list colouring) of graphs have been studied extensively in the literature [2,3,5,7–9].

Assume M is a subset of $V(G)$. We denote by δ_M the characteristic function of M, i.e, $\delta(v) = 1$ if $v \in M$ and $\delta(v) = 0$ if $v \notin M$. As our proofs use induction, we shall frequently use the following recursive definition of f-paintability.

Definition 1. Assume $f : V(G) \to N$ is a mapping. Then G is f-paintable if and only if

1. either $E(G) = \emptyset$ and $f(v) \geq 1$ for each $v \in V$,
2. or $\forall M \subseteq V(G)$, there exists an independent set $I \subseteq M$, such that $G - I$ is $(f - \delta_M)$-paintable.

It follows easily from the definition [9] that if G is f-paintable, then G is f-choosable. The converse is not true. The f-painting game on G is also called an *online list colouring* of G, as each vertex v eventually have $f(v)$ chances to be coloured, where each chance can be viewed as a permissible colour for v, and the goal of Painter to colour all the vertices of G with their permissible colours. However, Painter needs to colour the vertices online, i.e., before knowing the full list assignment.

Given a graph G and a mapping $f : V(G) \to N$, it is rather difficult to determine whether G is f-choosable or not (respectively, f-paintable or not), even if G has very simple structure.

Nevertheless, for a complete bipartite graph $K_{n,m}$, there is one type of functions f for which there is a simple characterization of functions f for which $K_{n,m}$ is f-choosable and f-paintable.

Theorem 2 ([1]). *Assume $G = K_{n,m}$ is a complete bipartite graph with partite sets $A = \{v_1, v_2, \dots, v_n\}$ and $B = \{u_1, u_2, \dots, u_m\}$. If $f : V(G) \to N$ is a mapping such that $f(u_i) = n$ for each vertex $u_i \in B$, then the following are equivalent:*

1. *G is f-choosable.*
2. *G is f-paintable.*
3. *$m < \prod_{i=1}^{n} f(v_i)$.*

For graphs G and H, the *join* of G and H, denoted by $G \oplus H$, is the graph obtained from the disjoint union of G and H by adding edges connecting every vertex of G to every vertex of H. Let $\overline{K_n}$ be the edgeless graph on n vertices. Then $K_{n,m} = \overline{K_n} \oplus \overline{K_m}$. The following result is a consequence of Theorem 2:

Corollary 3. *For any graph G and any mapping $f : V(G) \to N$, there is an integer m_0 such that if $m \geq m_0$ and f is extended to $G' = G \oplus \overline{K_m}$ with $f(v) = |V(G)|$ for every vertex v of $\overline{K_m}$, then G' is not f-choosable.*

Indeed, Theorem 2 implies that $m_0 = \prod_{v \in V(G)} f(v)$ is enough. However, for some graphs, we can choose much smaller m_0. For example, if G is not f-choosable, then we can simply let $m_0 = 0$. This motivates the following definition.

Definition 4. Assume G is a graph and $f : V(G) \to N$ is a mapping. Given an integer m, let $f^{(m)}$ be the extension of f to the graph $G \oplus \overline{K_m}$ for which $f^{(m)}(v) = |V(G)|$ for each vertex v of $\overline{K_m}$. We define $m_c(G, f)$ and $m_p(G, f)$ as follows:

$$m_c(G, f) = \min\{m : G \oplus \overline{K_m} \text{ is not } f^{(m)}\text{-choosable}\}$$

$$m_p(G, f) = \min\{m : G \oplus \overline{K_m} \text{ is not } f^{(m)}\text{-paintable}\}.$$

The following observation follows directly from the definition.

Observation 5. *For any graph G and mapping $f : V(G) \to N$, $m_c(G, f) = 0$ if and only if G is not f-choosable, and $m_p(G, f) = 0$ if and only if G is not f-paintable.*

Theorem 2 is equivalent to say that if G has no edges, then for any mapping $f : V(G) \to N$, $m_c(G, f) = m_p(G, f) = \prod_{v \in V(G)} f(v)$. In this paper, we first study $m_c(G, f)$ and $m_p(G, f)$ for the case that G is a complete graph. This problem turns out to be related to the number of generalized Dyck paths and the Catalan numbers. The *lattice graph* $Z \times Z$ has vertex set $\{(i, j) : i, j \in Z\}$ and in which $(i, j) \sim (i', j')$ if either $i = i'$ and $j' = j + 1$ or $i' = i + 1$ and $j = j'$, i.e., $(a', b') = (a, b) + (0, 1)$ or $(a', b') = (a, b) + (1, 0)$. By a *lattice path* we mean a path in the grid graph in which each edge is either a *vertical edge* from (i, j) to $(i, j + 1)$ or a *horizontal edge* from (i, j) to $(i + 1, j)$. Note that by this definition, a lattice path is a directed path, where each edge either goes vertically up or goes horizontally to the right.

A *Dyck path* of semi-length n is a lattice path P from $(0, 0)$ to (n, n) in which each vertex $(i, j) \in P$ satisfies $i \leq j$. The number of Dyck paths of semi-length n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Assume $\vec{x} = (x_1, x_2, \dots, x_n)$, where each x_i is a non-negative integer. An $\vec{x}$-dominated lattice path ending at (a, b) is a directed path P in S from $(0, 0)$ to (a, b) such that each vertex $(i, j) \in P$ satisfies $i \leq x_{j+1}$. So if $\vec{x} = (0, 1, \dots, n - 1)$, then $\vec{x}$-dominated lattice paths ending at (n, n) are exactly the original Dyck paths.

We denote by

$$\mathcal{P}(\vec{x})$$

the set of all $\vec{x}$-dominated lattice paths ending at (x_n, n), and let

$$\psi(\vec{x}) = |\mathcal{P}(\vec{x})|.$$

Definition 6. Assume K_n has vertex set $\{v_1, v_2, \ldots, v_n\}$ and $f : V(K_n) \to Z$ is a mapping with $f(v_1) \le f(v_2) \le \ldots \le f(v_n)$. Let

$$\vec{x}(f) = (x_1, x_2, \ldots, x_n),$$

where $x_i = f(v_i) - i$ for $i = 1, 2, \ldots, n$.

Theorem 7. *Assume K_n has vertex set $\{v_1, v_2, \ldots, v_n\}$ and $f : V(K_n) \to Z$ is a mapping with $f(v_1) \le f(v_2) \le \ldots \le f(v_n)$. Then*

$$m_c(K_n, f) = m_p(K_n, f) = \psi(\vec{x}(f)).$$

In other words, for an integer $m \ge 0$, for $G = K_n \oplus \overline{K_m}$, the following are equivalent:

1. *G is $f^{(m)}$-choosable.*
2. *G is $f^{(m)}$-paintable.*
3. *$m < \psi(\vec{x}(f))$.*

Observe that if $m = 0$, then $G = K_n$. In this case, G is f-paintable if and only if $f(v_i) \ge i$, which is equivalent to $x_i \ge 0$ for each $1 \le i \le n$, which in turn is equivalent to $\psi(\vec{x}) \ge 1$.

Next we consider the case that G is the disjoint union of graphs. Assume for $i = 1, 2, \ldots, p$, G_i is a graph and $f_i : V(G_i) \to N$ is a mapping. Denote by $G_1 \cup G_2 \cup \cdots \cup G_p$ the vertex disjoint union of $G_1, G_2, \ldots, G_p$, and denote by $f_1 \cup f_2 \cup \cdots \cup f_p : \cup_{i=1}^{p} V(G_i) \to N$ the mapping defined as $(f_1 \cup f_2 \cup \cdots \cup f_p)(v) = f_i(v)$ if $v \in V(G_i)$. The following result is a generalization of Theorem 2.

Theorem 8. *For $i = 1, 2, \ldots, p$, assume G_i is a graph and $f_i : V(G_i) \to N$ is a mapping which assigns to each vertex v of G_i a positive integer $f_i(v)$. Then we have*

$$m_c(G_1 \cup G_2 \cup \cdots \cup G_p, f_1 \cup f_2 \cup \cdots \cup f_p) = \prod_{i=1}^{p} m_c(G_i, f_i),$$

$$m_p(G_1 \cup G_2 \cup \cdots \cup G_p, f_1 \cup f_2 \cup \cdots \cup f_p) = \prod_{i=1}^{p} m_p(G_i, f_i).$$

2. Proof of Theorem 7

In this section, we assume

- $G = K_n \oplus \overline{K_m}$ and the vertices of K_n are $v_1, v_2, \ldots, v_n$.
- $f : V(K_n) \to \{1, 2, \ldots\}$ is a mapping such that $f(v_1) \le f(v_2) \le \ldots \le f(v_n)$.
- $f^{(m)}$ is an extension of f to G with $f^{(m)}(v) = n$ for each vertex v of $\overline{K_m}$.

Since $f^{(m)}$-paintable implies $f^{(m)}$-choosable, to prove Theorem 7, it suffices to show that if $m = \psi(\vec{x}(f))$, then G is not $f^{(m)}$-choosable, and if $m < \psi(\vec{x}(f))$, then G is $f^{(m)}$-paintable. We prove these as two lemmas.

Lemma 9. *If $m = \psi(\vec{x}(f))$, then G is not $f^{(m)}$-choosable.*

Proof. Each path $P \in \mathcal{P}(\vec{x}(f))$ can be encoded as a set $s(P)$ of n positive integers, where $i \in s(P)$ if and only if the ith edge of the path P is a vertical edge going up. Assume $s(P) = \{i_0, i_1, \ldots, i_{n-1}\}$, where $i_0 < i_1 < \cdots < i_{n-1}$. Then $P \in \mathcal{P}(\vec{x})$ if and only if for each $j \in \{0, 1, \ldots, n-1\}$, there are at most x_{j+1} horizontal edges before the $(j+1)$th vertical edge. In other words, $P \in \mathcal{P}(\vec{x})$ and if and only if for each index $j \in \{0, 1, \ldots, n-1\}$, $i_j \le x_{j+1} + j$.

Let L be the $f^{(m)}$-list assignment of G defined as follows:

- For each v_i of K_n, let $L(v_i) = \{1, 2, \ldots, f(v_i)\}$.
- Since $m = \psi(\vec{x})$, there is a bijection $\pi : V(\overline{K_m}) \to \mathcal{P}(\vec{x})$. For each vertex v of $\overline{K_m}$, let $L(v) = s(\pi(v))$.

We shall show that G is not L-colourable.

Assume to the contrary that c is an L-colouring of G. Assume $c(K_n) = \{i_0, i_1, \ldots, i_{n-1}\}$, where $i_0 < i_1 < \cdots < i_{n-1}$. Since $f(v_i) \le f(v_{i+1})$ for each i and $L(v_i) = \{1, 2, \ldots, f(v_i)\}$, we must have $i_j \in L(v_{j+1})$ for $j = 0, 1, 2, \ldots, n-1$. So $c(K_n) = s(P)$ for some $P \in \mathcal{P}(\vec{x})$. However, there is a vertex $v \in V(\overline{K_m})$ with $L(v) = s(P)$. Hence there is no legal colour for v, contrary to the assumption that c is an L-colouring of G. This completes the proof of Lemma 9. ∎

Lemma 10. *If $m < \psi(\vec{x}(f))$, then G is $f^{(m)}$-paintable.*

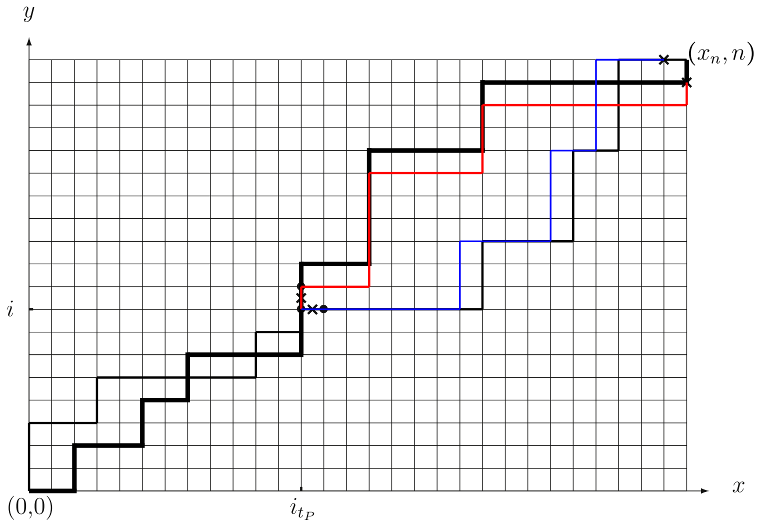

Fig. 1. For the proof of Lemma 10.

Proof. Assume $m < \psi(\vec{x}(f))$. We shall give a winning strategy for Painter in the $f^{(m)}$-painting game of G.

The proof is by induction on the number of vertices of G. If $n = m = 1$, then $m < \psi(\vec{x}(f))$ implies that $f(v_1) \geq 2$. So G is $f^{(m)}$-paintable. Assume $n + m \geq 3$ and Lemma 10 holds for $K_{n'} \oplus \overline{K_{m'}}$ when $n' + m' < n + m$.

For $\vec{x} = (x_1, x_2, \dots, x_n)$ and $\vec{y} = (y_1, y_2, \dots, y_n)$, we write $\vec{x} \leq \vec{y}$ if $x_i \leq y_i$ for $i = 1, 2, \dots, n$. It follows from the definition that if $\vec{x} \leq \vec{y}$, then $\mathcal{P}(\vec{x}) \subseteq \mathcal{P}(\vec{y})$ and hence $\psi(\vec{x}) \leq \psi(\vec{y})$.

For $1 \leq i \leq n$, let

$$\vec{x} \to i = (x_1, x_2, \dots, x_{i-1}, x_i - 1, \dots, x_n - 1),$$
$$\vec{x} \uparrow i = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, x_{i+2}, \dots, x_n).$$

Lemma 11. *For any $\vec{x} = (x_1, x_2, \dots, x_n)$ with $x_i \geq 1$ for each i, and for any $1 \leq i \leq n$, we have*

$$\psi(\vec{x}) = \psi(\vec{x} \to i) + \psi(\vec{x} \uparrow i).$$

Proof. Let i be a fixed index such that $1 \leq i \leq n$. For each path

$$P = ((i_0, j_0), (i_1, j_1), \dots, (i_{n+x_n}, j_{n+x_n})) \in \mathcal{P}(\vec{x}),$$

let t_P be the smallest index such that $j_{t_P} = i$. We say P is of Type I (respectively, of Type II) if the edge following the vertex (i_{t_P}, j_{t_P}) is a horizontal edge (respectively, a vertical edge).

Let $\mathcal{P}_1$ (respectively, $\mathcal{P}_2$) be the set of Type I (respectively, Type II) paths in $\mathcal{P}(\vec{x})$. Then

$$\mathcal{P}(\vec{x}) = \mathcal{P}_1 \cup \mathcal{P}_2$$

and

$$\psi(\vec{x}) = |\mathcal{P}_1| + |\mathcal{P}_2|.$$

For $P \in \mathcal{P}(\vec{x})$, let P' be the lattice path obtained from P by contracting the edge of P following the vertex (i_{t_P}, j_{t_P}). It is straightforward to verify that $P' \in \mathcal{P}(\vec{x} \to i)$ if and only if P is of Type I and $P' \in \mathcal{P}(\vec{x} \uparrow i)$ if and only if P is of Type II. Therefore

$$\psi(\vec{x}) = |\mathcal{P}_1| + |\mathcal{P}_2| = |\mathcal{P}(\vec{x} \to i)| + |\mathcal{P}(\vec{x} \uparrow i)|. \quad \blacksquare$$

In Fig. 1, the thin black path is of Type I, the blue path is obtained from the thin black path by contracting the edge following the vertex (i_{t_P}, j_{t_P}). The thick black path is of Type II, the red path is obtained from the thick black path by contracting the edge following the vertex (i_{t_P}, j_{t_P}).

Lemma 12. *Assume M is a subset of $V(K_n)$ and i is the smallest index such that $v_i \in M$. Then $\vec{x}((f - \delta_M)) \geq \vec{x}(f) \to i$ and $\vec{x}((f - \delta_M)|_{K_n - v_i}) \geq \vec{x}(f) \uparrow i$.*

Proof. First recall that

$$\vec{x} \to i = (x_1, x_2, \ldots, x_{i-1}, x_i - 1, \ldots, x_n - 1),$$
$$\vec{x} \uparrow i = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_n).$$

We may assume that

$$\vec{x}((f - \delta_M)|_{K_n}) = (y_1, y_2, \ldots, y_n),$$
$$\vec{x}((f - \delta_M)|_{K_n - v_i}) = (z_1, z_2, \ldots, z_{n-1}).$$

Since i is the smallest index such that $v_i \in M$, by Definition 6, we have that

- For any $1 \le j \le i - 1$, $y_j = z_j = x_j$.
- For any $i \le j \le n$,

$$\begin{aligned} y_j &= (f - \delta_M)(v_j) - j \\ &= f(v_j) - j - \delta_M(v_j) \\ &= x_j - \delta_M(v_j) \\ &\ge x_j - 1. \end{aligned}$$

- For any $i \le j \le n - 1$,

$$\begin{aligned} z_j &= (f - \delta_M)(v_{j+1}) - j \\ &= f(v_{j+1}) - j - \delta_M(v_{j+1}) \\ &= x_{j+1} + 1 - \delta_M(v_{j+1}) \\ &\ge x_{j+1}. \end{aligned}$$

Hence

$$\vec{x}((f - \delta_M)|_{K_n}) \ge \vec{x}(f) \to i,$$

and

$$\vec{x}((f - \delta_M)|_{K_n - v_i}) \ge \vec{x}(f) \uparrow i. \quad \blacksquare$$

Lemma 13 follows easily from the definitions and is well-known (cf [9]).

Lemma 13. *For any graph and mapping $f : V(G) \to N$, if $v \in V(G)$ and $f(v) > d_G(v)$, then G is f-paintable if and only if $G - v$ is f-paintable.*

To prove that G is $f^{(m)}$-paintable, it suffices to show that for any subset $M = U + m'$ of $V(G)$ (i.e., $U = M \cap V(K_n)$ and M contains m' vertices of $\overline{K_m}$), there is an independent set I of G contained in M such that $G - I$ is $(f^{(m)} - \delta_M)$-paintable. If $U = \emptyset$, then let $I = M$, and by induction hypothesis, $G - M$ is $f^{(m-m')}$-paintable.

Assume $U \ne \emptyset$. Let i be the smallest index such that $v_i \in U$. By Lemma 11, $\psi(\vec{x}(f)) = \psi(\vec{x}(f) \to i) + \psi(\vec{x}(f) \uparrow i)$. Since $m < \psi(\vec{x}(f))$, we have either $m' < \psi(\vec{x}(f) \uparrow i)$ or $m - m' < \psi(\vec{x}(f) \to i)$.

If $m' < \psi(\vec{x}(f) \uparrow i)$, then Painter colours v_i. The remaining game is the $(f - \delta_M)^{(m')}$-painting game on $(K_n - v_i) \oplus \overline{K_{m'}}$. As $\vec{x}((f - \delta_M)|_{K_n - v_i}) \ge \vec{x}(f) \uparrow i$, by induction hypothesis, $m_p(K_n - v_i, (f - \delta_M)|_{K_n - v_i}) > m'$ and hence Painter has a winning strategy on the $(f - \delta_M)^{(m')}$-painting game on $(K_n - v_i) \oplus \overline{K_{m'}}$.

If $m - m' < \psi(\vec{x}(f) \to i)$, then Painter colours $M \cap V(\overline{K_m})$. The remaining game is the $(f - \delta_M)^{(m-m')}$-painting game on $K_n \oplus \overline{K_{m-m'}}$. As $\vec{x}((f - \delta_M)|_{K_n}) \ge \vec{x}(f) \to i$, by induction hypothesis, $m_p(K_n, f - \delta_M) > m - m'$ and hence Painter has a winning strategy on the $(f - \delta_M)^{(m-m')}$-painting game on $K_n \oplus \overline{K_{m-m'}}$. $\blacksquare$

The number of $\vec{x}$-dominated paths is thoroughly studied in the literature and $\psi(\vec{x})$ is known to be the determinant of a matrix whose entries are determined by $\vec{x}$. First we have the following observation.

Observation 14. *Given a vector $\vec{x} = (x_1, x_2, \ldots, x_n)$, if $x_i > x_{i+1}$, then let*

$$\vec{x}' = (x_1, \ldots, x_{i-1}, x_{i+1}, x_{i+1}, \ldots, x_n),$$

we have $\mathcal{P}(\vec{x}) = \mathcal{P}(\vec{x}')$.

Proof. It is obvious that $\mathcal{P}(\vec{x}') \subseteq \mathcal{P}(\vec{x})$. On the other hand, for any $P \in \mathcal{P}(\vec{x})$, if $(a, i - 1) \in P$, then we must have $a \le x_{i+1}$, for otherwise, after arriving at vertex $(a, i - 1)$, P cannot have any up edge and hence cannot reach the vertex (x_n, n). So $P \in \mathcal{P}(\vec{x}')$. $\blacksquare$

We say $\vec{x}$ is *reduced* if $x_1 \le x_2 \le \ldots \le x_n$. For any vector $\vec{x}$, let $\vec{x}^*$ be the maximum reduced vector such that $\vec{x}^* \le \vec{x}$.

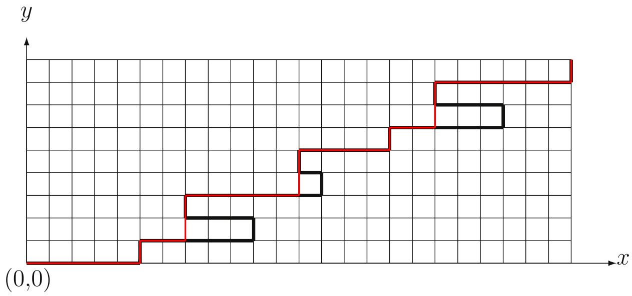

Fig. 2. The vector $\vec{x}$ (black) and its reduced form $\vec{x}^*$ (red). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Example 15. In the Fig. 2,

$$\vec{x} = (5, 10, 7, 13, 12, 16, 21, 18, 24),$$

while

$$\vec{x}^* = (5, 7, 7, 12, 12, 16, 18, 18, 24).$$

Corollary 16. *For any vector $\vec{x}$, $\psi(\vec{x}) = \psi(\vec{x}^*)$.*

Thus to obtain a formula for $\psi(\vec{x})$, we can restrict to the case that $\vec{x}$ is of reduced form.

The following theorem gives an explicit formula for $\psi(\vec{x})$.

Theorem 17 ([4]). *Assume $\vec{x} = (x_1, x_2, \ldots, x_n)$ is of reduced form. For $1 \le i, j \le n$, let*

$$a_{ij} = \binom{x_i + 1}{j - i + 1}_+,$$

where

$$\binom{y}{z}_+ = \begin{cases} \binom{y}{z}, & when\ y \ge z, \\ 0, & when\ z < 0\ or\ y < z, \\ 1, & when\ z = 0\ and\ y \ge 0. \end{cases}$$

Then

$$\psi(\vec{x}(f)) = \det_{n\times n}(a_{ij}).$$

Indeed, a more general formula for the number of families of lattice paths is given in [4]. The formula stated in Theorem 17 is a special case of the more general formula.

Example 18. For K_4, if $f(v_1) = 3, f(v_2) = f(v_3) = 6, f(v_4) = 9$, then we can conclude that $\vec{x} = (2, 4, 3, 5)$, and $\vec{x}^* = (2, 3, 3, 5)$, thus we have

$$\psi(\vec{x}) = \psi(\vec{x}^*) = \begin{vmatrix} \binom{3}{1}_+ & \binom{3}{2}_+ & \binom{3}{3}_+ & \binom{3}{4}_+ \\ \binom{4}{0}_+ & \binom{4}{1}_+ & \binom{4}{2}_+ & \binom{4}{3}_+ \\ \binom{4}{-1}_+ & \binom{4}{0}_+ & \binom{4}{1}_+ & \binom{4}{2}_+ \\ \binom{6}{-2}_+ & \binom{6}{-1}_+ & \binom{6}{0}_+ & \binom{6}{1}_+ \end{vmatrix} = \begin{vmatrix} 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 6 \end{vmatrix} = 72.$$

Alternatively, the number of $\vec{x}$-dominated lattice paths can also be calculated recursively, as depicted in the Fig. 3, where the number in each lattice point (a, b) is the number of $\vec{x}$-dominated paths ending at (a, b). It follows from the definition that the number at (a, b) is the summation of the numbers at $(a - 1, b)$ and $(a, b - 1)$, with obvious boundary values.

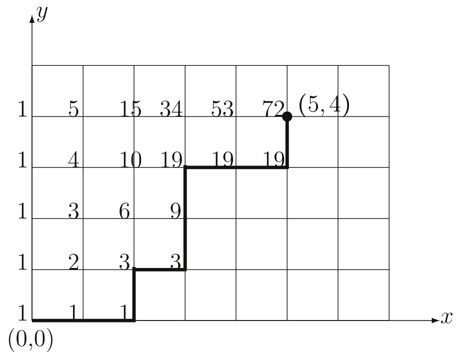

Fig. 3. $\psi((2,3,3,5))$.

3. Proof of Theorem 8

Definition 19. Assume G is a graph and $f : V(G) \to N$ is a mapping and L is an f-list assignment of G. An extension of L to $G \oplus \overline{K_m}$ is a list assignment L' of $G \oplus \overline{K_m}$ such that for each vertex v of G, $L'(v) = L(v)$ and for each vertex v of $\overline{K_m}$, $|L(v)| = |V(G)|$. We say L is *m-extendable* if for any extension L' of L to $G \oplus \overline{K_m}$, there exists an L'-colouring of $G \oplus \overline{K_m}$.

The following is an equivalent definition of $m_c(G, f)$:

$$m_c(G, f) = \min\{m : \text{ there is an } f\text{-list assignment } L \text{ of } G \text{ which is not } m\text{-extendable}\}.$$

Lemma 20. *Assume G is a graph and $f : V(G) \to N$ is a mapping and L is an f-list assignment of G.*

(1) *If there is an L-colouring ϕ of G such that $|\phi(V(G))| < |V(G)|$, then L is m-extendable for any m.*
(2) *Otherwise, for any L-colouring ϕ of G, $|\phi(V(G))| = |V(G)|$. Then L is m-extendable if and only if*

$$m < |\{\phi(V(G)) : \phi \text{ is an } L\text{-colouring of } G\}|.$$

Proof. (1) Assume there is an L-colouring ϕ of G such that $|\phi(V(G))| < |V(G)|$. For an arbitrary integer m and an extension L' of L to $G \oplus \overline{K_m}$ with $|L(v)| = |V(G)|$ for each vertex v of $\overline{K_m}$, we can extend ϕ to an L'-colouring of $G \oplus \overline{K_m}$ by assigning a colour $\phi(v) \in L'(v) - \phi(V(G))$ for each vertex v of $\overline{K_m}$.

(2) We first prove that if $m = |\{\phi(V(G)) : \phi$ is an L-colouring of $\}|$, then L is not m-extendable.

Let π be a one-to-one correspondence between $V(\overline{K_m})$ and $\{\phi(V(G)) : \phi$ is an L-colouring of $G\}$. Let L' be the extension of L to $G \oplus \overline{K_m}$ such that for each vertex v of $\overline{K_m}$, $L'(v) = \pi(v)$. Then any L-colouring ϕ of G cannot be extended to an L'-colouring of $G \oplus \overline{K_m}$.

Next we prove that if $m < |\{\phi(V(G)) : \phi$ is an L-colouring of $G\}|$, then L is m-extendable. Since $m < |\{\phi(V(G))\}|$, there exists an L-colouring ϕ of G such that $\phi(V(G)) \neq L'(v)$ for any vertex v of $\overline{K_m}$. Therefore ϕ can be extended to an L'-colouring of $G \oplus \overline{K_m}$, by assigning a colour $c \in L'(v) - \phi(V(G))$ for each vertex v of $\overline{K_m}$. ■

Assume L is an f-list assignment of G.

If G has no L-colouring ϕ in which two vertices are coloured by the same colour, then let

$$\Phi(G, L) = \{\phi(V(G)) : \phi \text{ is an } L\text{-colouring of } G\}$$

and let

$$\kappa(G, L) = \begin{cases} \infty, & \text{if } G \text{ has an } L\text{-colouring } \phi \text{ with } |\phi(V(G))| < |V(G)|, \\ |\Phi(G, L)|, & \text{otherwise.} \end{cases}$$

Note that if G is not L-colourable, then $\Phi(G, L) = \emptyset$ and $\kappa(G, L) = 0$.

Corollary 21. *Assume G is a graph and $f : V(G) \to N$ is a mapping. Then*

$$m_c(G, f) = \min\{\kappa(G, L) : L \text{ is an } f\text{-list assignment of } G\}.$$

Now we are ready to prove the first equality in Theorem 8.

Lemma 22. *For any graphs G_i and mappings $f_i : V(G_i) \to N$ $(i = 1, 2)$,*

$$m_c(G_1 \cup G_2, f_1 \cup f_2) = m_c(G_1, f_1) m_c(G_2, f_2).$$

Proof. Assume L_i is an f_i-list assignment of G_i, $L_1(v) \cap L_2(u) = \emptyset$ for any $v \in V(G_1)$, $u \in V(G_2)$ and G_i has no L_i-colouring ϕ_i in which two vertices are coloured by the same colour. Then $L = L_1 \cup L_2$ is an $(f_1 \cup f_2)$-list assignment of $G_1 \cup G_2$ and for any L-colouring ϕ of G, $|\phi(V(G))| = |V(G)|$. Furthermore,

$$\Phi(G, L) = \{\phi_1(V(G_1)) \cup \phi_2(V(G_2)) : \phi_i \in \Phi(G_i, L_i)\}.$$

Hence by Corollary 21,

$$m_c(G, f) \le |\Phi(G, L)| = |\Phi(G_1, L_1)| \times |\Phi(G_2, L_2)| = m_c(G_1, f_1)m_c(G_2, f_2).$$

If $m_c(G_i, f_i) = 0$ for some $i = 1, 2$, then $m_c(G, f) = 0$. Assume $m_c(G_i, f_i) \ne 0$ for $i = 1, 2$. By Corollary 21, there is an f-list assignment L of G such that $m_c(G, f) = \kappa(G, L)$. As $m_c(G_i, f_i) \ne 0$, we know that G_i is L-colourable, which implies that G is L-colourable. Hence $m_c(G, f) > 0$. If $m_c(G_i, f_i) = \infty$ for some i, then by Lemma 20(2), there is an L-colouring ϕ_i of G_i with $|\phi(V(G_i))| < |V(G_i)|$. This implies that there is an L-colouring ϕ of G with $|\phi(V(G))| < |V(G)|$, and hence $m_c(G, f) = \infty$. Otherwise, $m_c(G, f) \le m_c(G_1, f_1)m_c(G_2, f_2) < \infty$ implies that $|\phi(V(G))| = |V(G)|$ for any L-colouring ϕ of G. For $i = 1, 2$, let L_i be the restriction of L to G_i. For any L-colouring ϕ of G, let ϕ_i be the restriction of ϕ to G_i, then ϕ_i is an L_i-colouring of G_i. Conversely, if for $i = 1, 2$, ϕ_i is an L_i-colouring of G_i, then $\phi_1 \cup \phi_2$ is an L-colouring of G. Since $|\phi(V(G))| = |V(G)|$ for any L-colouring of G, we conclude that for $i = 1, 2$, $|\phi_i(V(G_i))| = |V(G_i)|$ for any L_i-colouring of G_i. Therefore by Lemma 20(2),

$$m_c(G, f) = |\Phi(G, L)| = |\Phi(G_1, L_1)| \times |\Phi(G_2, L_2)| \ge m_c(G_1, f_1)m_c(G_2, f_2).$$

This completes the proof. ■

Corollary 23. *Assume $G_1, G_2, \ldots, G_p$ are vertex disjoint graphs and $f_i : V(G_i) \to N$ are mappings. Let $G = G_1 \cup G_2 \cup \cdots \cup G_p$ and $f = f_1 \cup f_2 \cup \cdots \cup f_p$. Then*

$$m_c(G, f) = \prod_{i=1}^{p} m_c(G_i, f_i).$$

Before proving the second equality of Theorem 8, we first study the parameter $m_p(G, f)$.

The following well-known lemma follows easily from the definition.

For a subset U of $V(G)$, we denote by $U + m'$ a subset M of $V(G \oplus \overline{K_m})$ such that $m' = |M \cap V(\overline{K_m})|$ and $U = M \cap V(G)$.

As observed before, $m_p(G, f) = 0$ if and only if G is not f-paintable.

Lemma 24. *Assume G is f-paintable.*

(1) *We say (G, f, m) satisfies (1) if there is a subset U of $V(G)$ such that for any vertex $v \in U$, $m - m_p(G, (f - \delta_U)) \ge m_p(G - v, (f - \delta_U))$, and for any independent set I of G contained in U with $|I| \ge 2$, then $G - I$ is not $(f - \delta_U)$-paintable.*

(2) *We say (G, f, m) satisfies (2) if for any subset U of $V(G)$, either there is a vertex $v \in U$ such that $m - m_p(G, (f - \delta_U)) \le m_p(G - v, (f - \delta_U))$, or there is an independent set I of G contained in U with $|I| \ge 2$, such that $G - I$ is $(f - \delta_U)$-paintable.*

If (G, f, m) satisfies (1), then $m_p(G, f) \le m$; If (G, f, m) satisfies (2), then $m_p(G, f) \ge m$.

Proof. Assume (G, f, m) satisfies (1). Let U be a subset of $V(G)$ such that for any vertex $v \in U$, $m - m_p(G, (f - \delta_U)) \ge m_p(G - v, (f - \delta_U))$, and for any independent set I of G contained in U with $|I| \ge 2$, then $G - I$ is not $(f - \delta_U)$-paintable.

Let $m' = m - m_p(G, (f - \delta_U))$. We shall prove that $U + m'$ is a winning move for Lister in the $f^{(m)}$-painting game on $G \oplus \overline{K_m}$, and hence $m_p(G, f) \le m$.

If Painter colours the m' vertices of $\overline{K_m}$, then the remaining game is $(f - \delta_U)^{(m-m')}$-painting game on $G \oplus \overline{K_{m-m'}}$. As $m - m' = m_p(G, (f - \delta_U))$, we conclude that Lister has a winning strategy for the remaining game.

If Painter colours a vertex $v \in U$, then by applying Lemma 13 and deleting those vertices in $\overline{K_m}$ whose number of tokens is more than the number of their neighbours, the remaining game is the $(f - \delta_U)^{(m')}$-painting game on $(G - v) \oplus \overline{K_{m'}}$. As $m' = m - m_p(G, (f - \delta_U)) \ge m_p(G - v, (f - \delta_U))$, Lister has a winning strategy for the remaining game.

If Painter colours an independent set I of G contained in U with $|I| \ge 2$, then by applying Lemma 13 again, the remaining game is the $(f - \delta_U)$-painting game on $G - I$. As $G - I$ is not $(f - \delta_U)$-paintable, Lister has a winning strategy for the remaining game.

Assume (G, f, m) satisfies (2). We prove Painter has a winning strategy for the $f^{(m-1)}$-painting game on $G \oplus \overline{K_{m-1}}$.

Let $U + m'$ be Lister's first move.

If there is an independent set I of G contained in U with $|I| \ge 2$ for which $G - I$ is $(f - \delta_U)$-paintable, then Painter will colour I in the first round. By Lemma 13, Painter has a winning strategy for the remaining game.

Assume for any independent set I of G contained in U with $|I| \ge 2$ for which $G - I$ is not $(f - \delta_U)$-paintable. As (2) holds, there is a vertex $v \in U$ such that $m - m_p(G, (f - \delta_U)) \le m_p(G - v, (f - \delta_U))$.

If $m' \ge m - m_p(G, (f - \delta_U))$, then Painter colours the m' vertices of $\overline{K_{m-1}}$. The remaining game is an $(f - \delta_U)^{m-1-m'}$-painting game on $G \oplus \overline{K_{m-1-m'}}$. As $m - 1 - m' < m_p(G, (f - \delta_U))$, by definition of $m_p(G, (f - \delta_U))$, Painter has a winning strategy in the remaining game.

Assume $m' < m - m_p(G, (f - \delta_U))$. Let $v \in U$ be the vertex for which $m - m_p(G, (f - \delta_U)) \le m_p(G - v, (f - \delta_U))$. Painter colours v. Applying Lemma 13, the remaining game is the $(f - \delta_U)^{m'}$-painting game on $(G - v) \oplus \overline{K_{m'}}$. As $m' < m - m_p(G, (f - \delta_U)) \le m_p(G - v, (f - \delta_U))$, Painter has a winning strategy for the remaining game.

This completes the proof of Lemma 24. ■

It is easy to check that if (G, f, m) does not satisfy (2), then $(G, f, m-1)$ satisfies (1), and hence $m_p(G, f) \le m-1$. If (G, f, m) does not satisfy (1), then $(G, f, m + 1)$ satisfies (2), and hence $m_p(G, f) \ge m + 1$. Thus we have the following corollary.

Corollary 25. *Assume G is f-paintable. For Conditions (1) and (2) defined as in Lemma 24, (1) holds if and only if $m_p(G, f) \le m$ and (2) holds if and only if $m_p(G, f) \ge m$.*

Lemma 26. *For any graphs G_i and mappings $f_i : V(G_i) \to N$ ($i = 1, 2$),*

$$m_p(G_1 \cup G_2, f_1 \cup f_2) = m_p(G_1, f_1)m_p(G_2, f_2).$$

Proof. Let $m_i = m_p(G_i, f_i)$ for $i = 1, 2$ and let $m = m_1m_2$. Let $G = G_1 \cup G_2$ and $f = f_1 \cup f_2$. We shall prove that $m = m_p(G, f)$.

The proof is by induction on m.

If $m = 0$, then one of m_1, m_2, say m_1 is 0. Then G_1 is not f_1-paintable, implying that G is not f-paintable, and hence $m_p(G, f) = 0$.

Assume $m > 0$. To prove that $m_p(G, f) = m$, by Lemma 24, we need to show that (G, f, m) satisfies (1) and (2).

By Corollary 25, (G_1, f_1, m_1) satisfies (1). We shall prove that (G, f, m) satisfies (1).

By Corollary 25, there is a subset U of $V(G_1 \oplus \overline{K_{m_1}})$ such that for any vertex $v \in U$, $m_1 - m_p(G_1, (f_1 - \delta_U)) \ge m_p(G_1 - v, (f_1 - \delta_U))$ and for any independent set I of G contained in U with $|I| \ge 2$, $G_1 - I$ is not $(f_1 - \delta_U)$-paintable. Hence $m_1m_2 - m_p(G_1, (f_1 - \delta_U))m_2 \ge m_p(G_1 - v, (f_1 - \delta_U))m_2$. By induction hypothesis, $m_p(G_1, (f_1 - \delta_U))m_2 = m_p(G, (f - \delta_U))$ and $m_p(G_1 - v, (f_1 - \delta_U))m_2 = m_p(G - v, f - \delta_U)$. Therefore $m - m_p(G, (f - \delta_U)) \ge m_p(G - v, f - \delta_U)$.

If I is an independent set of G contained in U with $|I| \ge 2$, then I is an independent set of G_1. Hence $G_1 - I$ is not $(f_1 - \delta_U)$-paintable, which implies that $G - I$ is not $(f - \delta_U)$-paintable.

Now we prove that (G, f, m) satisfies (2).

Assume U is a subset of $V(G)$. Assume first that $U \cap V(G_1) = U_1 \ne \emptyset$ and $U \cap V(G_2) = U_2 \ne \emptyset$. Since for $i = 1, 2$, G_i is f_i-paintable, there exists an non-empty independent I_i of G_i contained in U_i such that $G_i - I_i$ is $(f_i - \delta_{U_i})$-paintable. Let $I = I_1 \cup I_2$. Then $G - I$ is $(f - \delta_U)$-paintable. As $|I| \ge 2$, so (2) holds.

By symmetry, we may assume that $U \cap V(G_2) = \emptyset$. Then $U = U \cap V(G_1)$.

By Corollary 25,

- either there is a vertex $v \in U$ such that $m_1 - m_p(G_1, (f_1 - \delta_U)) \le m_p(G_1 - v, (f_1 - \delta_U))$.
- or there is an independent set I of G_1 contained in U with $|I| \ge 2$, and $G_1 - I$ is $(f_1 - \delta_U)$-paintable.

In the former case, by induction hypothesis, we have

$$m_p(G - v, (f - \delta_U)) = m_p(G_1 - v, (f_1 - \delta_U))m_2 \ge m_1m_2 - m_p(G_1, (f_1 - \delta_U))m_2 = m_1m_2 - m_p(G, (f - \delta_U)).$$

So (2) holds.

In the later case, I is also an independent of G and $G - I$ is $(f - \delta_U)$-paintable. So (2) holds.

This completes the proof of Lemma 26. ■

Corollary 27. *Assume $G_1, G_2, \ldots, G_p$ are vertex disjoint graphs and $f_i : V(G_i) \to N$ are mappings. Let $G = G_1 \cup G_2 \cup \cdots \cup G_p$ and $f = f_1 \cup f_2 \cup \cdots \cup f_p$. Then*

$$m_p(G, f) = \prod_{i=1}^{p} m_p(G_i, f_i).$$

To determine $m_c(G, f)$ and/or $m_p(G, f)$ is difficult for even very simple graphs. Indeed, to determine whether or not $m_c(G, f) = 0$ (respectively, $m_p(G, f) = 0$) is equivalent to determine if G is not f-choosable (respectively, f-paintable). By using Corollaries 23 and 27, we can determine $m_c(G, f)$ and $m_p(G, f)$ for the case that G is the disjoint union of complete graphs. Currently, we do not know $m_c(G, f)$ and $m_p(G, f)$ for any other graph G, if the mapping f is arbitrary. The simplest unknown case is that G is a path on three vertices.

It would be interesting to determine $m_c(P_3, f)$ and $m_p(P_3, f)$ for arbitrary mappings f.

Question 28. *Let P_3 be the path on three vertices. What is $m_c(P_3, f)$ and $m_p(P_3, f)$ for an arbitrary mapping f?*

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References

[1] J. Carraher, S. Loeb, T. Mahoney, G.J. Puleo, M. Tsai, D.B. West, Three topics in online list coloring, J. Combin. 5 (1) (2014) 115–130.
[2] P. Erdős, A.L. Rubin, H. Taylor, Choosability in graph, in: West Coast Conf. Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), in: Congr. Numer., vol. 26, 1980, pp. 125–157.
[3] J. Kozik, P. Micek, X. Zhu, Towards an on-line version of Ohbas conjecture, European J. Combin. 36 (2014) 110–121.
[4] S.G. Mohanty, Lattice Path Counting and Applications, Academic Press, 1979, p. 018025.
[5] U. Schauz, Mr. Paint and Mrs. Correct, Electron. J. Combin. 16 (1) (2009) Research Paper 77, 18. MR 2515754 (2010i:91064).
[6] R.P. Stanley, Enumerative Combinatorics, Cambridge University Press, 2012.
[7] Z. Tuza, Graph colorings with local constraints–a survey, Discuss. Math. Graph Theory 17 (2) (1997) 161–228.
[8] V.G. Vizing, Vertex colorings with given colors, Diskret. Anal. 29 (1976) 3–10 (in Russian).
[9] X. Zhu, Online list colouring of graphs, Electron. J. Combin. 16 (2009) pp, Research Paper 127, 16.