

# Context-free grammars for several polynomials associated with Eulerian polynomials

Shi-Mei Ma<sup>\*</sup>

School of Mathematics and Statistics  
Northeastern University at Qinhuangdao  
Hebei 066000, P.R. China  
shimeimapapers@163.com

Jun Ma<sup>†</sup>

Department of mathematics  
Shanghai jiao tong university  
Shanghai, P.R. China  
majun904@sjtu.edu.cn

Yeong-Nan Yeh<sup>‡</sup>

Institute of Mathematics  
Academia Sinica  
Taipei, Taiwan  
mayeh@math.sinica.edu.tw

Bao-Xuan Zhu<sup>§</sup>

School of Mathematical Sciences  
Jiangsu Normal University  
Xuzhou 221116, P.R. China  
bxzhu@jsnu.edu.cn

Submitted: Apr 19, 2017; Accepted: Jan 23, 2018; Published: Feb 16, 2018  
Mathematics Subject Classifications: 05A05, 05A15

## Abstract

In this paper, we present grammatical descriptions of several polynomials associated with Eulerian polynomials, including  $q$ -Eulerian polynomials, alternating run polynomials and derangement polynomials. As applications, we get several convolution formulas involving these polynomials.

**Keywords:** Eulerian polynomials; Alternating runs; Derangement polynomials; Context-free grammars

## 1 Introduction

For an alphabet  $A$ , let  $\mathbb{Q}[[A]]$  be the rational commutative ring of formal power series in monomials formed from letters in  $A$ . Following [6], a context-free grammar over  $A$  is a function  $G : A \rightarrow \mathbb{Q}[[A]]$  that replaces a letter in  $A$  by a formal function over  $A$ . The

---

<sup>\*</sup>Supported by NSFC (11401083), Natural Science Foundation of Hebei Province (A2017501007) and the Fundamental Research Funds for the Central Universities (N152304006).

<sup>†</sup>Supported by NSFC 11571235.

<sup>‡</sup>Supported partially by NSC under the Grant No. 104-2115-M-001-010.

<sup>§</sup>Supported by NSFC 11571150.

formal derivative  $D$  is a linear operator defined with respect to a context-free grammar  $G$ . More precisely, the derivative  $D = D_G: \mathbb{Q}[[A]] \rightarrow \mathbb{Q}[[A]]$  is defined as follows: for  $x \in A$ , we have  $D(x) = G(x)$ ; for a monomial  $u$  in  $\mathbb{Q}[[A]]$ ,  $D(u)$  is defined so that  $D$  is a derivation, and for a general element  $q \in \mathbb{Q}[[A]]$ ,  $D(q)$  is defined by linearity.

Many combinatorial structures can be generated by using context-free grammars, such as set partitions [6], permutations [12, 16], Stirling permutations [8, 19], increasing trees [8, 12], rooted trees [13] and perfect matchings [15]. In this paper, using a grammatical labeling introduced by Chen and Fu [8], we study several polynomials associated with the Eulerian polynomials.

Let  $\mathfrak{S}_n$  be the symmetric group on the set  $[n] = \{1, 2, \dots, n\}$ , and let  $\pi = \pi(1) \cdots \pi(n) \in \mathfrak{S}_n$ . We say that the index  $i \in [n-1]$  is an *excedance* of  $\pi$  if  $\pi(i) > i$ . Denote by  $\text{exc}(\pi)$  the number of excedances of  $\pi$ . The *Eulerian polynomials* are defined by

$$A_0(x) = 1, \quad A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} = \sum_{k=0}^{n-1} \left\langle n \atop k \right\rangle x^k \text{ for } n \geq 1,$$

where  $\left\langle n \atop k \right\rangle$  are the *Eulerian numbers*. The numbers  $\left\langle n \atop k \right\rangle$  satisfy the recurrence relation

$$\left\langle n \atop k \right\rangle = (k+1) \left\langle n-1 \atop k \right\rangle + (n-k) \left\langle n-1 \atop k-1 \right\rangle,$$

with the initial conditions  $\left\langle 0 \atop 0 \right\rangle = 1$  and  $\left\langle 0 \atop k \right\rangle = 0$  for  $k \geq 1$ .

The *hyperoctahedral group*  $B_n$  is the group of signed permutations of the set  $\pm[n]$  such that  $\pi(-i) = -\pi(i)$  for all  $i$ , where  $\pm[n] = \{\pm 1, \pm 2, \dots, \pm n\}$ . Throughout this paper, we always identify a signed permutation  $\pi = \pi(1) \cdots \pi(n)$  with the word  $\pi(0)\pi(1) \cdots \pi(n)$ , where  $\pi(0) = 0$ . Let

$$\text{des}_B(\pi) = \#\{i \in \{0, 1, 2, \dots, n-1\} \mid \pi(i) > \pi(i+1)\}.$$

The *Eulerian polynomials of type B* are defined by

$$B_n(x) = \sum_{\pi \in B_n} x^{\text{des}_B(\pi)} = \sum_{k=0}^n B(n, k) x^k,$$

where  $B(n, k)$  are called the *Eulerian numbers of type B*. The numbers  $B(n, k)$  satisfy the recurrence relation

$$B(n+1, k) = (2k+1)B(n, k) + (2n-2k+3)B(n, k-1),$$

with the initial conditions  $B(0, 0) = 1$  and  $B(0, k) = 0$  for  $k \geq 1$ . Let us now recall two results on context-free grammars.

**Proposition 1** ([12, Section 2.1]). *If  $A = \{x, y\}$  and  $G = \{x \rightarrow xy, y \rightarrow xy\}$ , then for  $n \geq 1$*

$$D^n(x) = x \sum_{k=0}^{n-1} \left\langle n \atop k \right\rangle x^k y^{n-k}.$$

**Proposition 2** ([15, Theorem 10]). *If  $A = \{x, y\}$  and  $G = \{x \rightarrow xy^2, y \rightarrow x^2y\}$ , then for  $n \geq 1$ ,*

$$D^n(x^2) = 2^n \sum_{k=0}^{n-1} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle x^{2n-2k} y^{2k+2},$$

$$D^n(xy) = \sum_{k=0}^n B(n, k) x^{2n-2k+1} y^{2k+1}.$$

The  $q$ -Eulerian polynomials of types  $A$  and  $B$  are respectively defined by

$$A_n(x; q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)},$$

$$B_n(x; q) = \sum_{\pi \in B_n} x^{\text{des}_B(\pi)} q^{N(\pi)},$$

where  $\text{cyc}(\pi)$  is the number of cycles in  $\pi$  and  $N(\pi) = \#\{i \in [n] : \pi(i) < 0\}$ . Following [11, 157-162] and [24, Theorem 4.3.3], the *alternating run polynomials* of types  $A$  and  $B$  can be respectively defined by

$$R_n(x) = (1-w) \left( \frac{1+x}{2} \right)^{n-1} (1+w)^n A_n \left( \frac{1-w}{1+w} \right),$$

$$T_n(x) = \frac{x}{2} \left( \frac{1+x}{2} \right)^{n-1} (1+w)^n B_n \left( \frac{1-w}{1+w} \right)$$

for  $n \geq 2$ , where  $w = \sqrt{\frac{1-x}{1+x}}$ . Following [3, Proposition 5] and [9, Theorem 3.2], the *derangement polynomials* of types  $A$  and  $B$  can be respectively defined by

$$d_n(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} A_k(x),$$

$$d_n^B(x) = \sum_{k=0}^n (-1)^{n-k} x^{n-k} \binom{n}{k} B_k(x).$$

The purpose of this paper is to explore grammatical descriptions of the  $q$ -Eulerian polynomials, the alternating run polynomials and the derangement polynomials.

## 2 $q$ -Eulerian polynomials

According to [5, Proposition 7.2], the polynomials  $A_n(x; q)$  satisfy the recurrence relation

$$A_{n+1}(x; q) = (nx + q)A_n(x; q) + x(1-x) \frac{\partial}{\partial x} A_n(x; q), \quad (1)$$

with the initial condition  $A_0(x; q) = 1$ . In [4, Theorem 3.4], Brenti showed that the polynomials  $B_n(x; q)$  satisfy the recurrence relation

$$B_{n+1}(x; q) = ((n + nq + q)x + 1)B_n(x; q) + (1 + q)x(1 - x)\frac{\partial}{\partial x}B_n(x; q), \quad (2)$$

with the initial condition  $B_0(x; q) = 1$ . The polynomials  $A_n(x; q)$  and  $B_n(x; q)$  have been extensively studied (see [2, 17] for instance).

## 2.1 Context-free grammar for $A_n(x; q)$

A *standard cycle decomposition* of  $\pi \in \mathfrak{S}_n$  is defined by requiring that each cycle is written with its smallest element first, and the cycles are written in increasing order of their smallest element. In the following discussion, we always write  $\pi$  in standard cycle decomposition. We say that the index  $i \in [n]$  is an *anti-excedance* of  $\pi$  if  $\pi(i) \leq i$ . Denote by  $\text{aexc}(\pi)$  the number of anti-excedances of  $\pi$ . It is clear that  $\text{exc}(\pi) + \text{aexc}(\pi) = n$ . We now give a labeling of  $\pi$  as follows:

- (i) Put a superscript label  $z$  right after each excedance;
- (ii) Put a superscript label  $y$  right after each anti-excedance;
- (iii) Put a superscript label  $q$  before each cycle of  $\pi$ ;
- (iv) Put a superscript label  $x$  at the end of  $\pi$ .

For example, the permutation  $(1, 3, 4)(2)(5, 6)$  can be labeled as  ${}^q(1^z3^z4^y)q(2^y)q(5^z6^y)^x$ . The weight of  $\pi$  is the product of its labels.

Let  $\mathfrak{S}_n(i, j, k) = \{\pi \in \mathfrak{S}_n : \text{aexc}(\pi) = i, \text{exc}(\pi) = j, \text{cyc}(\pi) = k\}$ . When  $n = 1$ , we have  $\mathfrak{S}_1(1, 0, 1) = \{q(1^y)^x\}$ . When  $n = 2$ , we have  $\mathfrak{S}_2(2, 0, 2) = \{q(1^y)q(2^y)^x\}$  and  $\mathfrak{S}_2(1, 1, 1) = \{q(1^z2^y)^x\}$ . Suppose we get all labeled permutations in  $\mathfrak{S}_n(i, j, k)$  for all  $i, j, k$ . Let  $\pi' \in \mathfrak{S}_{n+1}$  be obtained from  $\pi \in \mathfrak{S}_n(i, j, k)$  by inserting the entry  $n + 1$  into  $\pi$ . We distinguish the following three cases:

- (c<sub>1</sub>) If the entry  $n + 1$  is inserted as a new cycle  $(n + 1)$ , then  $\pi' \in \mathfrak{S}_{n+1}(i + 1, j, k + 1)$ . In this case, the insertion of  $n + 1$  corresponds to the operation  $x \rightarrow qxy$ ;
- (c<sub>2</sub>) If the entry  $n + 1$  is inserted right after an excedance, then  $\pi' \in \mathfrak{S}_{n+1}(i + 1, j, k)$ . In this case, the insertion of  $n + 1$  corresponds to the operation  $z \rightarrow yz$ ;
- (c<sub>3</sub>) If the entry  $n + 1$  is inserted right after an anti-excedance, then  $\pi' \in \mathfrak{S}_{n+1}(i, j + 1, k)$ . In this case, the insertion of  $n + 1$  corresponds to the operation  $y \rightarrow yz$ .

By induction, we get the following result.

**Theorem 3.** *If  $A = \{x, y, z\}$  and  $G = \{x \rightarrow qxy, y \rightarrow yz, z \rightarrow yz\}$ , then*

$$D^n(x) = x \sum_{\pi \in \mathfrak{S}_n} y^{\text{aexc}(\pi)} z^{\text{exc}(\pi)} q^{\text{cyc}(\pi)}. \quad (3)$$

Setting  $y = 1$  in (3), we get

$$D^n(x)|_{y=1} = xA_n(z; q).$$

Furthermore, setting  $y = z = 1$  in (3), we get

$$D^n(x)|_{y=z=1} = x \sum_{\pi \in \mathfrak{S}_n} q^{\text{cyc}(\pi)} = xq(q+1)(q+2)\cdots(q+n-1).$$

**Corollary 4.** For  $n \geq 1$ , we have

$$A_{n+1}(x; q) = qA_n(x; q) + qx \sum_{k=0}^{n-1} \binom{n}{k} A_k(x; q) A_{n-k}(x).$$

*Proof.* Using the *Leibniz's formula*, we get

$$D^{n+1}(x) = D^n(qxy) = qD^n(xy) = q \sum_{k=0}^n \binom{n}{k} D^k(x) D^{n-k}(y).$$

Combining Proposition 1 and Theorem 3, we obtain

$$A_{n+1}(x; q) = q \sum_{k=0}^n \binom{n}{k} A_k(x; q) x^{n-k} A_{n-k} \left( \frac{1}{x} \right).$$

Recall that  $A_n(x)$  are symmetric, i.e.,  $x^n A_n \left( \frac{1}{x} \right) = x A_n(x)$  for  $n \geq 1$ . Thus we get the desired result.  $\square$

## 2.2 Context-free grammar for $B_n(x; q)$

For a permutation  $\pi \in B_n$ , we define an *ascent* (resp. a *descent*) to be a position  $i \in \{0, 1, 2, \dots, n-1\}$  such that  $\pi(i) < \pi(i+1)$  (resp.  $\pi(i) > \pi(i+1)$ ). Let  $\text{asc}_B(\pi)$  be the number of ascents of  $\pi \in B_n$ . As usual, denote by  $\bar{i}$  the negative element  $-i$ . Let  $A = \{x, y, z, u\}$ . We shall show that the following grammar

$$G = \{x \rightarrow qxyu, y \rightarrow xyz, z \rightarrow yzu, u \rightarrow qxz\} \tag{4}$$

can be used to generate permutations of  $B_n$ . Now we give a labeling of  $\pi \in B_n$  as follows:

- (L<sub>1</sub>) If  $i$  is an ascent and  $\pi(i+1) > 0$ , then put a superscript label  $z$  and a subscript  $x$  right after  $\pi(i)$ ;
- (L<sub>2</sub>) If  $i$  is a descent and  $\pi(i+1) > 0$ , then put a superscript label  $y$  and a subscript  $u$  right after  $\pi(i)$ ;
- (L<sub>3</sub>) If  $i$  is an ascent and  $\pi(i+1) < 0$ , then put a superscript label  $z$  and a subscript  $qx$  right after  $\pi(i)$ ;

(L<sub>4</sub>) If  $i$  is a descent and  $\pi(i + 1) < 0$ , then put a superscript label  $y$  and a subscript  $qu$  right after  $\pi(i)$ ;

(L<sub>5</sub>) Put a superscript label  $y$  and a subscript  $x$  at the end of  $\pi$ .

Note that the weight of  $\pi$  is given by

$$w(\pi) = xy(xz)^{\text{asc}_B(\pi)}(yu)^{\text{des}_B(\pi)}q^{N(\pi)}.$$

Let

$$B_n(i, j, k) = \{\pi \in B_n : \text{asc}_B(\pi) = i, \text{des}_B(\pi) = j, N(\pi) = k\}.$$

When  $n = 1$ , we have  $B_1(1, 0, 0) = \{0_x^z 1_x^y\}$  and  $B_1(0, 1, 1) = \{0_{qu}^y \overline{1}_x^y\}$ . Note that

$$D(xy) = x^2yz + qxy^2u.$$

Thus the sum of weights of the elements of  $B_1$  is given by  $D(xy)$ .

Suppose we get all labeled permutations in  $B_n(i, j, k)$  for all  $i, j, k$ , where  $n \geq 1$ . Let  $\pi' \in B_{n+1}$  be obtained from  $\pi \in B_n(i, j, k)$  by inserting the entry  $n + 1$  or  $\overline{n + 1}$ . We distinguish the following six cases:

- (c<sub>1</sub>) If  $i$  is an ascent and we insert  $n + 1$  right after  $\pi(i)$ , then  $\pi' \in B_{n+1}(i, j + 1, k)$ . In this case, the insertion of  $n + 1$  corresponds to applying the rule  $z \rightarrow yzu$  to the label  $z$  associated with  $\pi(i)$ ;
- (c<sub>2</sub>) If  $i$  is an ascent and we insert  $\overline{n + 1}$  right after  $\pi(i)$ , then  $\pi' \in B_{n+1}(i, j + 1, k + 1)$ . In this case, the insertion of  $\overline{n + 1}$  corresponds to applying the rule  $x \rightarrow qxyu$  to the label  $x$  associated with  $\pi(i)$ ;
- (c<sub>3</sub>) If  $i$  is a descent and we insert  $n + 1$  right after  $\pi(i)$ , then  $\pi' \in B_{n+1}(i + 1, j, k)$ . In this case, the insertion of  $n + 1$  corresponds to applying the rule  $y \rightarrow xyz$  to the label  $y$  associated with  $\pi(i)$ ;
- (c<sub>4</sub>) If  $i$  is a descent and we insert  $\overline{n + 1}$  right after  $\pi(i)$ , then  $\pi' \in B_{n+1}(i + 1, j, k + 1)$ . In this case, the insertion of  $\overline{n + 1}$  corresponds to applying the rule  $u \rightarrow qxz u$  to the label  $u$  associated with  $\pi(i)$ ;
- (c<sub>5</sub>) If we insert  $n + 1$  at the end of  $\pi$ , then  $\pi' \in B_{n+1}(i + 1, j, k)$ . In this case, the insertion of  $n + 1$  corresponds to applying the rule  $y \rightarrow xyz$  to the label  $y$  at the end of  $\pi$ ;
- (c<sub>6</sub>) If we insert  $\overline{n + 1}$  at the end of  $\pi$ , then  $\pi' \in B_{n+1}(i, j + 1, k + 1)$ . In this case, the insertion of  $\overline{n + 1}$  corresponds to applying the rule  $x \rightarrow qxyu$  to the label  $x$  at the end of  $\pi$ .

In general, the insertion of  $n + 1$  (resp.  $\overline{n + 1}$ ) into  $\pi$  corresponds to the action of the formal derivative  $D$  on a superscript label (resp. subscript label). By induction, we get the following result.

**Theorem 5.** *If  $D$  is the formal derivative with respect to the grammar (4), then*

$$D^n(xy) = xy \sum_{\pi \in B_n} (xz)^{\text{asc}_B(\pi)} (yu)^{\text{des}_B(\pi)} q^{N(\pi)}. \quad (5)$$

Setting  $x = y = z = 1$  in (5), we get

$$D^n(xy)|_{x=y=z=1} = B_n(u; q).$$

A *perfect matching* of  $[2n]$  is a partition of  $[2n]$  into  $n$  blocks of size 2. Denote by  $N(n, k)$  the number of perfect matchings of  $[2n]$  with the restriction that only  $k$  matching pairs have even larger entries. The numbers  $N(n, k)$  satisfy the recurrence relation

$$N(n+1, k) = 2kN(n, k) + (2n - 2k + 3)N(n, k-1)$$

for  $n, k \geq 1$ , where  $N(1, 1) = 1$  and  $N(1, k) = 0$  for  $k \geq 2$  or  $k \leq 0$  (see [18, Proposition 1]). Let  $N_n(x) = \sum_{k=1}^n N(n, k)x^k$ . Set  $N_0(x) = 1$  and

$$\tilde{N}_n(x) = x^n N_n\left(\frac{1}{x}\right) \text{ for } n \geq 0.$$

From [15, Corollary 11], we have

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} N_k(x) \tilde{N}_{n-k}(x). \quad (6)$$

We end this section by giving a dual of (6). By the same way as in the proof of Theorem 5, it is routine to check that if  $A = \{x, y, z, u\}$  and  $G = \{x \rightarrow xyu, y \rightarrow xyz, z \rightarrow yzu, u \rightarrow xzu\}$ , then

$$\begin{aligned} D^n(x) &= u \sum_{k=0}^n N(n, k) (yu)^k (xz)^{n-k}, \\ D^n(u) &= u \sum_{k=0}^n N(n, k) (xz)^k (yu)^{n-k}, \\ D^n(xy) &= xy \sum_{k=0}^n B(n, k) (yu)^k (xz)^{n-k}. \end{aligned}$$

Note that

$$D^{n+1}(x) = D^n(xyu) = \sum_{k=0}^n \binom{n}{k} D^k(u) D^{n-k}(xy).$$

Therefore, we get the following result.

**Proposition 6.** *For  $n \geq 0$ , we have*

$$N_{n+1}(x) = x \sum_{k=0}^n \binom{n}{k} \tilde{N}_k(x) B_{n-k}(x). \quad (7)$$

It should be noted that the convolution formulas (6) and (7) reveal that there exist some connections between Brauer algebra and Coxeter groups.

### 3 Alternating run polynomials

For  $\pi \in \mathfrak{S}_n$ , a *left peak index* is an index  $i \in [n - 1]$  such that  $\pi(i - 1) < \pi(i) > \pi(i + 1)$ , where we take  $\pi(0) = 0$ , and the entry  $\pi(i)$  is called a *left peak*. Let  $P(n, k)$  be the number of permutations of  $\mathfrak{S}_n$  with  $k$  left peaks. It is well known that the numbers  $P(n, k)$  satisfy the recurrence relation

$$P(n, k) = (2k + 1)P(n - 1, k) + (n - 2k + 1)P(n - 1, k - 1),$$

with the initial conditions  $P(1, 0) = 1$  and  $P(1, k) = 0$  for  $k \geq 1$  (see [16] for instance). We say that  $\pi \in \mathfrak{S}_n$  changes direction at position  $i$  if either  $\pi(i - 1) < \pi(i) > \pi(i + 1)$ , or  $\pi(i - 1) > \pi(i) < \pi(i + 1)$ , where  $i \in \{2, 3, \dots, n - 1\}$ . We say that  $\pi$  has  $k$  *alternating runs* if there are  $k - 1$  indices  $i$  such that  $\pi$  changes direction at these positions. The *up-down runs* of a permutation  $\pi$  are the alternating runs of  $\pi$  endowed with a 0 in the front. Let  $R(n, k)$  (resp.  $M(n, k)$ ) be the number of permutations of  $\mathfrak{S}_n$  with  $k$  alternating runs (resp. up-down runs). The numbers  $R(n, k)$  and  $M(n, k)$  respectively satisfy the recurrence relations

$$R(n, k) = kR(n - 1, k) + 2R(n - 1, k - 1) + (n - k)R(n - 1, k - 2), \quad (8)$$

$$M(n, k) = kM(n - 1, k) + M(n - 1, k - 1) + (n - k + 1)M(n - 1, k - 2) \quad (9)$$

for  $n, k \geq 1$ , where  $R(1, 0) = M(0, 0) = M(1, 1) = 1$  and  $R(1, k) = M(n, 0) = M(0, k) = 0$  for  $n, k \geq 1$  (see [16, 21]). Let  $R_n(x) = \sum_{k=1}^{n-1} R(n, k)x^k$  and  $M_n(x) = \sum_{k=1}^n M(n, k)x^k$ . There is a unified grammatical descriptions of  $R_n(x)$  and  $M_n(x)$ .

**Proposition 7** ([16, Theorem 6]). *If  $A = \{x, y, z\}$  and  $G = \{x \rightarrow xy, y \rightarrow yz, z \rightarrow y^2\}$ , then*

$$D^n(x^2) = x^2 \sum_{k=0}^n R(n + 1, k)y^k z^{n-k},$$

$$D^n(x) = x \sum_{k=1}^n M(n, k)y^k z^{n-k}.$$

*In particular,  $D^n(x^2)|_{z=1} = x^2 R_{n+1}(y)$ ,  $D^n(x)|_{z=1} = x M_n(y)$ .*

A *run* of a signed permutation  $\pi \in B_n$  is defined as a maximal interval of consecutive elements on which the elements of  $\pi$  are monotonic in the order  $\dots < \bar{2} < \bar{1} < 0 < 1 < 2 < \dots$ . The *up signed permutations* are signed permutations with  $\pi(1) > 0$ . For example, the up signed permutation  $03\bar{1}24\bar{5} \in B_5$  has four runs, i.e.,  $03, 3\bar{1}, \bar{1}24$  and  $4\bar{5}$ . Let  $T(n, k)$  denote the number of up signed permutations in  $B_n$  with  $k$  alternating runs. Zhao [24, Theorem 4.2.1] showed that the numbers  $T(n, k)$  satisfy the the following recurrence relation

$$T(n, k) = (2k - 1)T(n - 1, k) + 3T(n - 1, k - 1) + (2n - 2k + 2)T(n - 1, k - 2) \quad (10)$$



for  $n \geq 2$  and  $1 \leq k \leq n$ , where  $T(1, 1) = 1$  and  $T(1, k) = 0$  for  $k > 1$ . The alternating run polynomials of type  $B$  are defined by  $T_n(x) = \sum_{k=1}^n T(n, k)x^k$ . The first few of the polynomials  $T_n(x)$  are given as follows:

$$T_1(x) = x, T_2(x) = x + 3x^2, T_3(x) = x + 12x^2 + 11x^3, T_4(x) = x + 39x^2 + 95x^3 + 57x^4.$$

The similarity of the recurrence relations (8), (9) and (10) suggests the existence of a context-free grammar for  $T_n(x)$ . We now present the third main result of this paper.

**Theorem 8.** *If  $A = \{x, y, z\}$  and  $G = \{x \mapsto xy^2, y \mapsto yz^2, z \mapsto y^4z^{-1}\}$ , then for  $n \geq 1$ ,*

$$\begin{aligned} D^n(x^3y) &= x^3y \sum_{k=1}^{n+1} T(n+1, k)y^{2k-2}z^{2n-2k+2}, \\ D^n(xy) &= xy(y^2 + z^2) \sum_{k=1}^n T(n, k)y^{2k-2}z^{2n-2k}, \\ D^n(x^2) &= 2^n x^2 \sum_{k=0}^n M(n, k)y^{2k}z^{2n-2k}, \\ D^n(x^2y^2) &= 2^{n-1}x^2(y^2 + z^2) \sum_{k=1}^n R(n+1, k)y^{2k}z^{2n-2k}, \\ D^n(y^2) &= 2^n y^2 \sum_{k=0}^{\lfloor n/2 \rfloor} P(n, k)y^{4k}z^{2n-4k}. \end{aligned}$$

*Proof.* We only prove the assertion for  $D^n(x^3y)$  and the others can be proved in a similar way. Note that  $D(x^3y) = x^3y(z^2 + 3y^2)$  and  $D^2(x^3y) = x^3y(z^4 + 12y^2z^2 + 11y^4)$ . we define  $\tilde{T}(n, k)$  by

$$D^n(x^3y) = x^3y \sum_{k=0}^{n+1} \tilde{T}(n+1, k)y^{2k-2}z^{2n-2k+2}. \quad (11)$$

Then  $\tilde{T}(2, 0) = 0, \tilde{T}(2, 1) = T(2, 1) = 1$  and  $\tilde{T}(2, 2) = T(2, 2) = 3$ . It follows from (11) that

$$\begin{aligned} D^{n+1}(x^3y) &= D(D^n(x^3y)) \\ &= x^3y \sum_{k=1}^{n+1} (2k-1)\tilde{T}(n+1, k)y^{2k-2}z^{2n-2k+4} + x^3y \sum_{k=1}^{n+1} 3\tilde{T}(n+1, k)y^{2k}z^{2n-2k+2} \\ &\quad + x^3y \sum_{k=1}^{n+1} (2n-2k+2)\tilde{T}(n+1, k)y^{2k+2}z^{2n-2k}. \end{aligned}$$

Hence

$$\tilde{T}(n+2, k) = (2k-1)\tilde{T}(n+1, k) + 3\tilde{T}(n+1, k-1) + (2n-2k+6)\tilde{T}(n+1, k-2). \quad (12)$$

By comparing (12) with (10), we see that the numbers  $\tilde{T}(n, k)$  satisfy the same recurrence relation and initial conditions as  $T(n, k)$ , so they agree.  $\square$

It follows from the *Leibniz's formula* that

$$\begin{aligned} D^n(x^3y) &= \sum_{k=0}^n \binom{n}{k} D^k(xy) D^{n-k}(x^2) \\ &= xyD^n(x^2) + \sum_{k=1}^n \binom{n}{k} D^k(xy) D^{n-k}(x^2). \end{aligned}$$

So the following corollary is immediate.

**Corollary 9.** *We have*

$$T_{n+1}(x) = 2^n x M_n(x) + (1+x) \sum_{k=1}^n 2^{n-k} \binom{n}{k} T_k(x) M_{n-k}(x).$$

By using the fact that

$$\begin{aligned} D^n(x^2y^2) &= \sum_{k=0}^n \binom{n}{k} D^k(xy) D^{n-k}(xy) \\ &= 2xyD^n(xy) + \sum_{k=1}^{n-1} \binom{n}{k} D^k(xy) D^{n-k}(xy), \end{aligned}$$

we get a grammatical proof of the following result.

**Proposition 10** ([10, Theorem 13]). *For  $n \geq 2$ , we have*

$$2^{n-1} R_{n+1}(x) = 2T_n(x) + \frac{1+x}{x} \sum_{k=1}^{n-1} \binom{n}{k} T_k(x) T_{n-k}(x).$$

Let  $P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} P(n, k) x^k$ . From

$$D^n(x^2y^2) = \sum_{k=0}^n \binom{n}{k} D^k(x^2) D^{n-k}(y^2),$$

we obtain

$$(1+x)R_{n+1}(x) = 2x \sum_{k=0}^n \binom{n}{k} M_k(x) P_{n-k}(x^2).$$

Recall that Bóna [1, Section 1.3.2] obtained the following identity:

$$M_n(x) = \frac{1}{2}(1+x)R_n(x) \text{ for } n \geq 2.$$

Therefore,

$$M_{n+1}(x) = x \sum_{k=0}^n \binom{n}{k} M_k(x) P_{n-k}(x^2).$$

## 4 Derangement polynomials

There is a larger literature devoted to derangement polynomials (see [3, 7, 9, 14] for instance). We now recall some basic definitions, notation and results. We say that a permutation  $\pi$  is a *derangement* if  $\pi(i) \neq i$  for any  $i \in [n]$ . Let  $\mathcal{D}_n$  be the set of derangements of  $\mathfrak{S}_n$ . The *derangement polynomials* are defined by

$$d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\text{exc}(\pi)}.$$

The polynomials  $d_n(x)$  satisfy the following recurrence relation

$$d_{n+1}(x) = nx(d_n(x) + d_{n-1}(x)) + x(1-x)d'_n(x)$$

for  $n \geq 1$ , with the initial conditions  $d_0(x) = 1, d_1(x) = 0$  (see [23]). Brenti [3, Proposition 5] showed that

$$\sum_{n \geq 0} d_n(x) \frac{z^n}{n!} = \frac{1-x}{e^{xz} - xe^z}.$$

For  $\pi \in \mathfrak{S}_n$ , we define

$$\text{fix}(\pi) = \#\{i \in [n] : \pi(i) = i\},$$

$$\text{dc}(\pi) = \#\{i \in [n] : \pi(i) < i\}.$$

Clearly,  $\text{exc}(\pi) + \text{fix}(\pi) + \text{dc}(\pi) = n$ . In [12, Section 2.2], Dumont proved that if  $A = \{x, y, z, e\}$  and  $G = \{x \rightarrow xy, y \rightarrow xy, z \rightarrow xy, e \rightarrow ez\}$ , then

$$D^n(e) = e \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} y^{\text{dc}(\pi)} z^{\text{fix}(\pi)}. \quad (13)$$

In particular, setting  $y = e = 1, z = 0$  in (13), we get  $D^n(e)|_{y=e=1, z=0} = d_n(x)$ .

In [4, p. 431], Brenti introduced a definition of type  $B$  weak excedance. Let  $\pi \in B_n$ . We say that  $i \in [n]$  is a *type  $B$  weak excedance* of  $\pi$  if  $\pi(i) = i$  or  $\pi(|\pi(i)|) > \pi(i)$ . Let  $\text{wexc}(\pi)$  be the number of weak excedances of  $\pi$ . It follows from [4, Theorem 3.15] that

$$B_n(x) = \sum_{\pi \in B_n} x^{\text{wexc}(\pi)}.$$

A *fixed point* of  $\pi \in B_n$  is an index  $i \in [n]$  such that  $\pi(i) = i$ . A *derangement of type  $B$*  is a signed permutation  $\pi \in B_n$  with no fixed points. Let  $\mathcal{D}_n^B$  be the set of derangements of  $B_n$ . Following [9], the *type  $B$  derangement polynomials*  $d_n^B(x)$  are defined by

$$d_0^B(x) = 1, \quad d_n^B(x) = \sum_{\pi \in \mathcal{D}_n^B} x^{\text{wexc}(\pi)} \quad \text{for } n \geq 1.$$

The first few of the polynomials  $d_n^B(x)$  are given as follows:

$$d_1^B(x) = 1, d_2^B(x) = 1 + 4x, d_3^B(x) = 1 + 20x + 8x^2, d_4^B(x) = 1 + 72x + 144x^2 + 16x^3.$$

Chow [9, Theorem 3.2] proved that

$$\sum_{n \geq 0} d_n^B(x) \frac{z^n}{n!} = \frac{1-x}{e^{(2x-1)z} - xe^z}.$$

Moreover, the polynomials  $d_n^B(x)$  satisfy the following recurrence relation

$$d_{n+1}^B(x) = 2nx(d_n^B(x) + d_{n-1}^B(x)) + d_n^B(x) + 2x(1-x) \frac{d}{dx} d_n^B(x).$$

for  $n \geq 2$ , with the initial conditions  $d_0(x) = 1, d_1(x) = 0$  (see [9, Proposition 3.1]). Chen et al. [7] showed that the limiting distribution of the coefficients of  $d_n^B(x)$  is normal.

Given  $\pi \in \mathcal{D}_n^B$ . Clearly,

$$\text{wexc}(\pi) = \#\{i \in [n] : \pi(|\pi(i)|) > \pi(i)\}.$$

We say that  $i$  is an *anti-excedance* of  $\pi$  if  $\pi(|\pi(i)|) < \pi(i)$ . Let  $\text{aexc}(\pi)$  be the number of anti-excedances of  $\pi$ . We say that  $i$  is a *singleton* if  $(i)$  is a cycle of  $\pi$ . Let  $\text{single}(\pi)$  be the number of singletons of  $\pi$ . Then

$$\text{wexc}(\pi) + \text{aexc}(\pi) + \text{single}(\pi) = n. \tag{14}$$

In the following discussion, we always write  $\pi$  by using its *standard cycle decomposition*, in which each cycle is written with its largest entry last and the cycles are written in ascending order of their last entry. For example,  $\bar{3}51\bar{7}2\bar{6} \bar{4} \in \mathcal{D}_7^B$  can be written as  $(\bar{6})(\bar{7}, 4)(\bar{3}, 1)(2, 5)$ . Let  $(c_1, c_2, \dots, c_i)$  be a cycle in standard cycle decomposition of  $\pi$ . We say that  $c_j$  is an *ascent* in the cycle if  $c_j < c_{j+1}$ , where  $1 \leq j < i$ . We say that  $c_j$  is a *descent* in the cycle if  $c_j > c_{j+1}$ , where  $1 \leq j \leq i$  and we set  $c_{i+1} = c_1$ . As pointed out by Chow [9, p. 819], if  $\pi \in \mathcal{D}_n^B$  with no singletons, then  $\text{wexc}(\pi)$  equals the sum of the number of ascents in each cycle and  $\text{aexc}(\pi)$  equals the sum of the number of descents in each cycle. Let  $d(n, i, j)$  be the number of derangements of type  $B$  with  $i$  weak excedances and  $j$  anti-excedances.

We can now conclude the fourth main result of this paper from the discussion above.

**Theorem 11.** *Let  $A = \{x, y, z, e\}$  and*

$$G = \{x \rightarrow xy^2, y \rightarrow x^2y, z \rightarrow x^2y^2z^{-3}, e \rightarrow ez^4\}. \tag{15}$$

*Then*

$$D^n(e) = e \sum_{\pi \in \mathcal{D}_n^B} x^{2\text{wexc}(\pi)} y^{2\text{aexc}(\pi)} z^{4\text{single}(\pi)}.$$

*Equivalently,*

$$D^n(e) = e \sum_{i, j \geq 0} d(n, i, j) x^{2i} y^{2j} z^{4(n-i-j)}. \tag{16}$$

*Setting  $y = z = 1$  in (16), we get*

$$D^n(e)|_{y=z=1} = ed_n^B(x^2).$$

*Proof.* Now we give a labeling of  $\pi \in \mathcal{D}_n^B$  as follows:

- (L<sub>1</sub>) If  $i$  is a singleton, then put a superscript label  $z^4$  right after  $\bar{i}$ ;
- (L<sub>2</sub>) If  $c_i$  is an ascent in a cycle, then put a superscript label  $x^2$  right after  $c_i$ ;
- (L<sub>3</sub>) If  $c_i$  is a descent in a cycle, then put a superscript label  $y^2$  right after  $c_i$ ;
- (L<sub>4</sub>) Put a superscript label  $e$  in the front of  $\pi$ .

The weight of  $\pi$  is defined by

$$w(\pi) = ex^{2\text{wexc}(\pi)}y^{2\text{aexc}(\pi)}z^{4\text{single}(\pi)}.$$

Note that  $\mathcal{D}_1^B = \{e(\bar{1}^{z^4})\}$  and  $\mathcal{D}_2^B = \{e(\bar{2}^{z^4})(\bar{1}^{z^4}), e(\bar{1}^{x^2}2^{y^2}), e(1^{x^2}2^{y^2}), e(\bar{2}^{x^2}\bar{1}^{y^2}), e(\bar{2}^{x^2}1^{y^2})\}$ . Thus the weight of  $e(\bar{1}^{z^4})$  is given by  $D(e)$  and the sum of weights of the permutations in  $\mathcal{D}_2^B$  is given by  $D^2(e)$ , since  $D(e) = ez^4$  and  $D^2(e) = e(z^8 + 4x^2y^2)$ .

To illustrate the relation between the action of the formal derivative  $D$  of the grammar (15) and the insertion of  $n + 1$  or  $\bar{n} + \bar{1}$  into a permutation  $\pi \in \mathcal{D}_n^B$ , we give the following example. Let  $\pi = (\bar{6})(\bar{7}, \bar{4})(\bar{3}, 1)(2, 5) \in \mathcal{D}_7^B$ . Then  $\pi$  can be labeled as

$$e(\bar{6}^{z^4})(\bar{7}^{x^2}\bar{4}^{y^2})(\bar{3}^{x^2}1^{y^2})(2^{x^2}5^{y^2}).$$

We distinguish the following four cases:

- (c<sub>1</sub>) If we insert  $\bar{8}$  as a new cycle, then the resulting permutation is given below,

$$e(\bar{8}^{z^4})(\bar{6}^{z^4})(\bar{7}^{x^2}\bar{4}^{y^2})(\bar{3}^{x^2}1^{y^2})(2^{x^2}5^{y^2}).$$

This case corresponds to applying the rule  $e \rightarrow ez^4$  to the label  $e$ .

- (c<sub>2</sub>) If we insert  $\bar{8}$  or  $8$  into the cycle  $(\bar{6})$ , then the resulting permutations are given below,

$$\begin{aligned} &e(\bar{8}^{x^2}\bar{6}^{y^2})(\bar{7}^{x^2}\bar{4}^{y^2})(\bar{3}^{x^2}1^{y^2})(2^{x^2}5^{y^2}), \\ &e(\bar{6}^{x^2}8^{y^2})(\bar{7}^{x^2}\bar{4}^{y^2})(\bar{3}^{x^2}1^{y^2})(2^{x^2}5^{y^2}), \\ &e(6^{x^2}8^{y^2})(\bar{7}^{x^2}\bar{4}^{y^2})(\bar{3}^{x^2}1^{y^2})(2^{x^2}5^{y^2}), \\ &e(\bar{8}^{x^2}6^{y^2})(\bar{7}^{x^2}\bar{4}^{y^2})(\bar{3}^{x^2}1^{y^2})(2^{x^2}5^{y^2}). \end{aligned}$$

It should be noted that in the latter two permutations, we need to replace  $\bar{6}$  by  $6$ . This case corresponds to applying the rule  $z \rightarrow x^2y^2z^{-3}$  to the label  $z^4$ , since  $D(z^4) = 4x^2y^2$ .

(c<sub>3</sub>) If we insert  $\bar{8}$  or 8 right after  $\bar{7}$ , then the resulting permutations are given below,

$$\begin{aligned} e(\bar{6}^{z^4})(\bar{7}^{y^2} \bar{8}^{x^2} \bar{4}^{y^2})(\bar{3}^{x^2} 1^{y^2})(2^{x^2} 5^{y^2}), \\ e(\bar{6}^{z^4})(\bar{7}^{x^2} 8^{y^2} \bar{4}^{y^2})(\bar{3}^{x^2} 1^{y^2})(2^{x^2} 5^{y^2}). \end{aligned}$$

This case corresponds to applying the rule  $x \rightarrow xy^2$  to the label  $x^2$ , since  $D(x^2) = 2x^2y^2$ .

(c<sub>4</sub>) If we insert  $\bar{8}$  or 8 right after  $\bar{4}$ , then the resulting permutations are given below,

$$\begin{aligned} e(\bar{6}^{z^4})(\bar{7}^{x^2} \bar{4}^{y^2} \bar{8}^{x^2})(\bar{3}^{x^2} 1^{y^2})(2^{x^2} 5^{y^2}), \\ e(\bar{6}^{z^4})(\bar{7}^{x^2} \bar{4}^{x^2} 8^{y^2})(\bar{3}^{x^2} 1^{y^2})(2^{x^2} 5^{y^2}). \end{aligned}$$

This case corresponds to applying the rule  $y \rightarrow x^2y$  to the label  $y^2$ , since  $D(y^2) = 2x^2y^2$ .

In general, the insertion of  $n+1$  or  $\overline{n+1}$  into  $\pi \in \mathcal{D}_n^B$  corresponds to the action of the formal derivative  $D$  on a superscript label. By induction, we see that grammar (15) generates all of the derangements of type  $B$ .  $\square$

Let  $D$  be the formal derivative with respect to the grammar (15). Clearly,  $D(z^4) = 4x^2y^2$ . It is easy to verify that

$$D^n(x^2y^2) = 2^n \sum_{k=0}^n \left\langle \begin{matrix} n+1 \\ k \end{matrix} \right\rangle x^{2k+2} y^{2n-2k+2}.$$

Hence

$$D^n(z^4) = 4D^{n-1}(x^2y^2) = 2^{n+1} \sum_{k=0}^{n-1} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle x^{2k+2} y^{2n-2k} \text{ for } n \geq 1. \quad (17)$$

Note that

$$\begin{aligned} D^{n+1}(e) &= D^n(ez^4) \\ &= \sum_{k=0}^n \binom{n}{k} D^k(e) D^{n-k}(z^4) \\ &= z^4 D^n(e) + \sum_{k=0}^{n-1} \binom{n}{k} D^k(e) D^{n-k}(z^4). \end{aligned}$$

Combining (17) and Theorem 11, we get the following corollary.

**Corollary 12.** *For  $n \geq 1$ , we have*

$$d_{n+1}^B(x) = d_n^B(x) + x \sum_{k=0}^{n-1} 2^{n-k+1} \binom{n}{k} d_k^B(x) A_{n-k}(x).$$

Set  $d_n^B = d_n^B(1)$ . We have

$$d_{n+1}^B = d_n^B + n! \sum_{k=0}^{n-1} 2^{n-k+1} \frac{d_k^B}{k!}.$$

Moreover, from (16), the following result is immediate.

**Corollary 13.** For  $n \geq 0$ , we have

$$d(n+1, i, j) = d(n, i, j) + 2id(n, i, j-1) + 2jd(n, i-1, j) + 4(n-i-j+2)d(n, i-1, j-1), \quad (18)$$

with initial conditions  $d(1, 0, 0) = 1$  and  $d(1, i, j) = 0$  for  $i \neq 0$  or  $j \neq 0$ .

Let

$$G(x, y; t) = \sum_{n \geq 0} \sum_{\pi \in \mathcal{D}_n^B} x^{\text{wexc}(\pi)} y^{\text{aexc}(\pi)} \frac{t^n}{n!}.$$

**Proposition 14.** We have

$$G(x, y; t) = \frac{e^{(1-2x)t}}{1 - \frac{x}{y-x}(e^{2(y-x)t} - 1)}. \quad (19)$$

*Proof.* Let  $d_n(x, y) = \sum_{i, j \geq 0} d(n, i, j)x^i y^j$ . It follows from (18) that the polynomials  $d_n(x, y)$  satisfy the recurrence relation

$$d_{n+1}(x, y) = (1 + 4nxy)d_n(x, y) + (2xy - 4x^2y) \frac{\partial}{\partial x} d_n(x, y) + (2xy - 4xy^2) \frac{\partial}{\partial y} d_n(x, y), \quad (20)$$

with the initial conditions  $d_0(x, y) = d_1(x, y) = 1$ . By rewriting (20) in terms of the generating function  $G$ , we get

$$(1 - 4xyt)G_t = G + (2xy - 4x^2y)G_x + (2xy - 4xy^2)G_y. \quad (21)$$

It is routine to check that

$$\tilde{G}(x, y; t) = \frac{e^{(1-2x)t}}{1 - \frac{x}{y-x}(e^{2(y-x)t} - 1)}$$

satisfies (21). Also,  $\tilde{G}(x, y; 0) = 1$  and  $\tilde{G}(0, y; t) = \tilde{G}(x, 0; t) = e^t$ . Hence  $G = \tilde{G}$ .  $\square$

Let  $P(\pi) = \#\{i \in [n] : \pi(i) > 0\}$ . Then  $N(\pi) + P(\pi) = n$ . It is not hard to check that

$$\sum_{n \geq 0} \sum_{\pi \in \mathcal{D}_n^B} x^{\text{wexc}(\pi)} y^{\text{aexc}(\pi)} z^{\text{single}(\pi)} u^{P(\pi)} v^{N(\pi)} \frac{t^n}{n!} = \frac{e^{(vz - (u+v)x)t}}{1 - \frac{x}{y-x}(e^{(u+v)(y-x)t} - 1)}.$$

In the proof of Theorem 11, if we further put a superscript label  $q$  before each cycle of  $\pi \in \mathcal{D}_n^B$ , then we get the following result.

**Proposition 15.** Let  $A = \{x, y, z, e\}$  and

$$G = \{x \rightarrow xy^2, y \rightarrow x^2y, z \rightarrow x^2y^2z^{-3}, e \rightarrow qez^4\}.$$

Then

$$D^n(e) = e \sum_{\pi \in \mathcal{D}_n^B} x^{2\text{wexc}(\pi)} y^{2\text{aexc}(\pi)} z^{4\text{single}(\pi)} q^{\text{cyc}(\pi)}.$$

## 5 Concluding remarks

In this paper, we present grammatical descriptions of several polynomials associated with the Eulerian polynomials. There are various  $q$ -analogue of the Eulerian polynomials. A challenging work is to study grammatical descriptions of several enumerative polynomials of Euler-Mahonian statistics (see [20] for instance). The Robinson-Schensted-Knuth correspondence associates with any permutation a pair of paths in a Young graph. It would be interesting to encode Young tableaux by introducing a grammatical labeling. One of the possible ways is based on the work of Stembridge [22, Section 4], which established a connection between permutations and pairs of a marked tableau and a standard tableau of the same shape.

## Acknowledgements

This work was finished while Y.-N. Yeh was visiting the School of Mathematical Sciences, Dalian University of Technology, Dalian, P.R. China. The authors thank the referee for valuable suggestions which led to a substantial improvement of the paper.

## References

- [1] M. Bóna, *Combinatorics of Permutations*, second edition, CRC Press, Boca Raton, FL, 2012.
- [2] P. Brändén, On linear transformations preserving the Pólya frequency property, *Trans. Amer. Math. Soc.*, 358 (2006), 3697–3716.
- [3] F. Brenti, Unimodal polynomials arising from symmetric functions, *Proc. Amer. Math. Soc.*, 108 (1990), 1133–1141.
- [4] F. Brenti,  $q$ -Eulerian polynomials arising from Coxeter groups, *European J. Combin.*, 15 (1994), 417–441.
- [5] F. Brenti, A class of  $q$ -symmetric functions arising from plethysm, *J. Combin. Theory Ser. A*, 91 (2000), 137–170.
- [6] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, *Theoret. Comput. Sci.*, 117 (1993), 113–129.
- [7] W.Y. C. Chen, R.L. Tang and A.F. Y. Zhao, Derangement polynomials and excedances of type  $B$ , *Electron. J. Combin.*, 16(2) (2009), #R15.
- [8] W.Y.C. Chen, A.M. Fu, Context-free grammars for permutations and increasing trees, *Adv. in Appl. Math.*, 82 (2017), 58–82.
- [9] C.-O. Chow, On derangement polynomials of type  $B$ . II, *J. Combin. Theory Ser. A*, 116 (2009), 816–830.
- [10] C.-O. Chow, S-M. Ma, Counting signed permutations by their alternating runs, *Discrete Math.*, 323 (2014), 49–57.



- [11] F. N. David and D. E. Barton, *Combinatorial Chance*, Charles Griffin and Company, Ltd. London, UK, 1962.
- [12] D. Dumont, Grammaires de William Chen et dérivations dans les arbres et arborescences, *Sém. Lothar. Combin.*, 37, Art. B37a (1996), 1–21.
- [13] D. Dumont, A. Ramamonjisoa, Grammaire de Ramanujan et Arbres de Cayley, *Electron. J. Combin.*, 3(2) (1996), #R17.
- [14] Z. Lin, J. Zeng, The  $\gamma$ -positivity of basic Eulerian polynomials via group actions, *J. Combin. Theory Ser. A*, 135 (2015), 112–129.
- [15] S.-M. Ma, Some combinatorial arrays generated by context-free grammars, *European J. Combin.*, 34 (2013), 1081–1091.
- [16] S.-M. Ma, Enumeration of permutations by number of alternating runs, *Discrete Math.*, 313 (2013), 1816–1822.
- [17] S.-M. Ma, Y. Wang,  $q$ -Eulerian polynomials and polynomials with only real zeros, *Electron. J. Combin.*, 15 (2008), R17.
- [18] S.-M. Ma, Y.-N. Yeh, Stirling permutations, cycle structure of permutations and perfect matchings, *Electron. J. Combin.* 22(4) (2015), #P4.42.
- [19] S.-M. Ma, Y.-N. Yeh, Eulerian polynomials, Stirling permutations of the second kind and perfect matchings, *Electron. J. Combin.* 24(4) (2017), #P4.27.
- [20] J. Shareshian, M.L. Wachs,  $q$ -Eulerian polynomials: excedance number and major index, *Electron. Res. Announc. Amer. Math. Soc.*, 13 (2007), 33–45.
- [21] R.P. Stanley, Longest alternating subsequences of permutations, *Michigan Math. J.*, 57 (2008), 675–687.
- [22] J.R. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety, *Discrete Math.*, 99 (1992), 307–320.
- [23] X.D. Zhang, On  $q$ -derangement polynomials, *Combinatorics and Graph Theory 95*, Vol. 1 (Hefei), World Sci. Publishing, River Edge, NJ, 1995, pp. 462–465.
- [24] A.F.Y. Zhao, The combinatorics on permutations and derangements of type  $B$ , Ph.D. dissertation, Nankai University, 2011.