Note

Some statistics on Stirling permutations and Stirling derangements

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ABSTRACT

A permutation of the multiset \(\{1, 1, 2, 2, \ldots, n, n\}\) is called a Stirling permutation of order \(n\) if every entry between the two occurrences of \(i\) is greater than \(i\) for each \(i \in \{1, 2, \ldots, n\}\).

In this paper, we introduce the definitions of block, even indexed entry, odd indexed entry, Stirling derangement, marked permutation and bicolored increasing binary tree. We first study the joint distribution of ascent plateaux, even indexed entries and left-to-right minima over the set of Stirling permutations of order \(n\). We then present an involution on Stirling derangements.

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1. Introduction

Stirling permutations were introduced by Gessel and Stanley [6]. Let \([n]_2\) denote the multiset \(\{1, 1, 2, 2, \ldots, n, n\}\). A Stirling permutation of order \(n\) is a permutation of \([n]_2\) such that every entry between the two occurrences of \(i\) is greater than \(i\) for each \(i \in \{1, 2, \ldots, n\}\). Various statistics on Stirling permutations have been extensively studied in the past decades, including descents [3,4,6–8], plateaux [1,3,7,8], block patterns [12], ascent plateaux [9,10] and cycle ascent plateaux [11]. In the following, we collect some definitions, notation, and results that will be used throughout the rest of this paper.

Let \(\sigma_n\) denote the symmetric group of all permutations of \([n]\), where \([n] = \{1, 2, \ldots, n\}\). Let \(\pi = \pi_1 \pi_2 \cdots \pi_n \in \sigma_n\). An ascent of \(\pi\) is an entry \(\pi_i, i \in \{2, 3, \ldots, n\}\), such that \(\pi_{i-1} < \pi_i\). Denote by \(\text{ASC}(\pi)\) the set of all ascents of \(\pi\). Let \(|S|\) denote the cardinality of a set \(S\). Let \(\text{asc}(\pi) = |\text{ASC}(\pi)|\). For example, \(\text{asc}(33412) = |\{2, 4\}| = 2\). The classical Eulerian polynomials are defined by

\[
A_n(x) = \sum_{\pi \in \sigma_n} x^{\text{asc}(\pi)}.
\]

Set \(A_0(x) = 1\). The exponential generating function for \(A_n(x)\) is

\[
A(x, t) = \sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{1 - x}{e^{(x-1) t} - x}.
\]

Let \(Q_n\) be the set of all Stirling permutations of order \(n\), and let \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in Q_n\). An occurrence of an ascent plateau is an entry \(\sigma_i, i \in \{2, 3, \ldots, 2n-1\}\), such that \(\sigma_{i-1} < \sigma_i = \sigma_{i+1}\) (see [9]). Let \(\text{AP}(\sigma)\) be the set of all ascent plateaux of \(\sigma\), and

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Definition 2. Let \( \text{ap}(\sigma) = |\text{AP}(\sigma)| \). As an example, \( \text{ap}(221133) = |\{3\}| = 1 \). Let

\[
N_n(x) = \sum_{\pi \in \mathcal{Q}_n} x^{\text{lrmin}(\pi)}. 
\]

Set \( N_0(x) = 1 \). It follows from [9, Theorem 2] that

\[
N(x, t) = \sum_{n \geq 0} N_n(x) \frac{t^n}{n!} = \sqrt{A(x, 2t)}. 
\]

In [11], Ma and Yeh introduced Stirling permutations of the second kind and counted them by their cycle ascent plateaus, fixed points and cycles. As a continuation of [11], in this paper we introduce the definitions of block, even indexed entry, odd indexed entry, Stirling derangement, marked permutation and bicolored increasing binary tree. This paper is motivated by the following problem.

**Problem 1.** Is there a unified framework for (1) and (2)?

Let \( w = w_1 w_2 \cdots w_n \) be a word of length \( n \), where \( w_i \) are all integers. A left-to-right minimum of \( w \) is an element \( w_i \) such that \( w_i < w_{i+1} \) for every \( j \in [i-1] \) or \( i = 1 \); a right-to-left minimum of \( w \) is an element \( w_i \) such that \( w_i < w_{i+1} \) for every \( j \in [i+1, i+2, \ldots, n] \) or \( i = n \). Let LRMIN(\( w \)) and RLMIN(\( w \)) denote the set of entries of left-to-right minima and right-to-left minima of \( w \), respectively. Let lrlmin(w) = |LRMIN(w)|, and let rlmin(w) = |RLMIN(w)|. For example, lrmin(223311) = |{1, 2}| = 2 and rlmin(223311) = |{1}| = 1.

The following definition will be used repeatedly.

**Definition 2.** A block of the word \( w = w_1 w_2 \cdots w_n \) is a substring which begins with a left-to-right minimum, and contains exactly this one left-to-right minimum; moreover, the substring is maximal, i.e., not contained in any larger such substring.

It is easily derived by induction that any Stirling permutation has a unique decomposition as a sequence of blocks. For example, the block decomposition of 34664325527711 is given by [34664325527711] [11].

An entry \( k \) of a Stirling permutation \( \sigma \) is called an even indexed entry (resp. odd indexed entry) if the first appearance of \( k \) occurs at an even (resp. odd) position of \( \sigma \). Let EVEN(\( \sigma \)) (resp. ODD(\( \sigma \)) ) denote the set of even (resp. odd) indexed entries of \( \sigma \). Let even(\( \sigma \)) = |EVEN(\( \sigma \))|, and let odd(\( \sigma \)) = |ODD(\( \sigma \))|. For example, even(221331) = |{3}| = 1 and odd(221331) = |{1, 2}| = 2. Clearly, even(\( \sigma \)) + odd(\( \sigma \)) = \( n \) for \( \sigma \in \mathcal{Q}_n \).

We define

\[
E_n(x, y) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{odd}(\sigma)} y^{\text{even}(\sigma)}, \\
R_n(x, y) = \sum_{\sigma \in \mathcal{Q}_n} q^{\text{lrmin}(\sigma)} x^{\text{ap}(\sigma)} y^{\text{even}(\sigma)}, \\
R(q, x, y; t) = 1 + \sum_{n \geq 1} R_n(q, x, y) \frac{t^n}{n!}. 
\]

We can now conclude the main result of this paper from the discussion above.

**Theorem 3.** For \( n \geq 1 \), we have

\[
E_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} x^{k} (x + y)^{n-k}, 
\]

where \( \binom{n}{k} \) is the (signless) Stirling number of the first kind, i.e., the number of permutations of \( \mathcal{S}_n \) with \( i \) cycles. In particular, we have \( E_n(1, 0) = n! \), \( E_n(1, 1) = (2n - 1)!! \). Moreover,

\[
R(q, x, y; t) = A(x, t(1 + y)) ^{q}, 
\]

where \( A(x, t) \) is the exponential generating function given by (1). In particular, we have

\[
R(1, x, 0; t) = A(x, t), \hspace{1em} R(1, 1; t) = N(x, t). 
\]

A perfect matching of \([2n]\) is a set partition of \([2n]\) with blocks (disjoint nonempty subsets) of size exactly 2. Let \( \mathcal{M}_{2n} \) be the set of matchings of \([2n]\), and let \( M \in \mathcal{M}_{2n} \). The standard form of \( M \) is a list of blocks \([ (i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n) ] \) such that \( i_r < j_r \) for all \( 1 \leq r \leq n \) and \( 1 = i_1 < i_2 < \cdots < i_n \). Throughout this paper we always write \( M \) in the standard form. If \( (i, j) \) is a block of \( M \), then we say that \( i \) (resp. \( j \)) is an ascending (resp. a descending) entry of \( M \).

**Definition 4.** Let \( h_k \) be the number of perfect matchings of \([4k]\) such that \( 2i - 1 \) and \( 2i \) are either both ascending or both descending for every \( i \in [2k] \).
Let $r^2 = -1$. Following [14], we have
\[
\sqrt{\sec(2iz)} = \sum_{n \geq 0} (-1)^n h_n \frac{z^n}{(2n)!}.
\]
It should be noted that the numbers $h_n$ appear as A186491 in [13]. In the following, we shall recall a connection between the numbers $h_n$ and Stirling derangements.

Let $[k]^n$ denote the set of words of length $n$ in the alphabet $[k]$. For $\omega = \omega_1\omega_2 \cdots \omega_n \in [k]^n$, the reduction of $\omega$, denoted by $\text{red}(\omega)$, is the unique word of length $n$ obtained by replacing the $i$th smallest entry by $i$. For example, $\text{red}(33224547) = 2213435$. Very recently, Ma and Yeh [11] introduced the definition of Stirling permutations of the second kind. A permutation $\sigma$ of the multiset $[n]_2$ is a Stirling permutation of the second kind of order $n$ whenever $\sigma$ can be written as a nonempty disjoint union of its distinct cycles and $\sigma$ has a standard cycle form that satisfies the following conditions:

(i) For each $i \in [n]$, the two copies of $i$ appear in exactly one cycle;
(ii) Each cycle is written with one of its smallest entry first and the cycles are written in increasing order of their smallest entries;
(iii) The reduction of the word formed by all entries of each cycle is a Stirling permutation. In other words, if $(c_1, c_2, \ldots, c_{2k})$ is a cycle of $\sigma$, then $\text{red}(c_1c_2 \cdots c_{2k}) \in Q_k$.

Let $Q_n^2$ denote the set of all Stirling permutations of the second kind of order $n$. In the following discussion, we always write $\sigma \in Q_n^2$ in the standard cycle form. If $(c_1, c_2, \ldots, c_{2k})$ is a cycle of $\sigma$, then an entry $c_i$ is called a cycle ascent plateau if $c_{i-1} < c_i = c_{i+1}$, where $2 \leq i \leq 2k - 1$. Denote by $\text{cap}(\sigma)$ (resp. $\text{cyc}(\sigma)$) the number of cycle ascent plateaux (resp. cycles) of $\sigma$. An entry $k \in [n]$ is called a fixed point of $\sigma$ if $(kk)$ is a cycle of $\sigma$. Let $\text{fix}(\sigma)$ denote the number of fixed points of $\sigma$. Using the fundamental transformation of Foata and Schützenberger [5], we have
\[
\sum_{\sigma \in Q_n^2} q^{\text{cap}(\sigma)} x^{\text{cyc}(\sigma)} y^{\text{bk}_2(\sigma)} = \sum_{\tau \in \Omega_n^2} q^{\text{cyc}(\tau)} x^{\text{cap}(\tau)} y^{\text{fix}(\tau)},
\]
where $\text{bk}_2(\sigma)$ is the number of blocks of $\sigma$ with length (number of terms) 2.

Following [11], we say that $\sigma \in Q_n^2$ is a Stirling derangement if $\text{fix}(\sigma) = 0$. According to (5), in this paper we use the following definition on word structure.

**Definition 5.** A Stirling derangement is a Stirling permutation without blocks of length 2. Let $DQ_n$ be the set of Stirling derangements of order $n$.

For example, $DQ_1 = \emptyset$, $DQ_2 = \{1122, 2211\}$ and
\[
DQ_3 = \{112233, 112323, 113232, 133122, 122133, 122331, 123231, 133221\}.
\]

Let
\[
L_n(q, x, y) = \sum_{\tau \in \Omega_n^2} q^{\text{cyc}(\tau)} x^{\text{cap}(\tau)} y^{\text{fix}(\tau)}.
\]

The following result has been algebraically obtained by Ma and Yeh [11, Section 3]:
\[
L_n(1, -1, 0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^n h_k, & \text{if } n = 2k \text{ is even.} \end{cases} \quad (6)
\]

As an equivalent form of (6), we get the following result.

**Proposition 6.** We have
\[
\sum_{\sigma \in DQ_n} (-1)^{\text{ap}(\sigma)} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^n h_k, & \text{if } n = 2k \text{ is even.} \end{cases}
\]

This paper is organized as follows. In Section 2, we prove Theorem 3 by introducing marked permutations. In Section 3, we prove Proposition 6 by using bicolored increasing binary trees.

**2. A proof of Theorem 3**

**2.1. A proof of (3)**

Let $E(n, k) = \# \{ \sigma \in Q_n : \text{even}(\sigma) = k \}$. There are two ways in which a Stirling permutation $\sigma' \in Q_n$ with even$(\sigma') = k$ can be obtained from a Stirling permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n-2} \in Q_{n-1}$.
(i) If $\text{even}(\sigma) = k - 1$, then we can insert the two copies of $n$ right after $\sigma_{2i-1}$, where $i \in [n - 1]$.
(ii) If $\text{even}(\sigma) = k$, then we can insert the two copies of $n$ in the front of $\sigma$ or right after $\sigma_{2i}$, where $i \in [n - 1]$.

Clearly, $E(1, 0) = 1$, which corresponds to $11 \in Q_1$. Therefore, the numbers $E(n, k)$ satisfy the recurrence relation

$$E(n, k) = (n - 1)E(n - 1, k - 1) + nE(n - 1, k),$$

with the initial conditions $E(1, 0) = 1$ and $E(1, k) = 0$ for $k \geq 1$. Let $E_n(y) = \sum_{k=0}^{n} E(n, k)y^k$. It follows from (7) that $E_n(y) = (n + (n - 1)y)E_{n-1}(y)$ for $n \geq 1$, with the initial value $E_0(y) = 1$. Thus

$$E_n(y) = \prod_{k=1}^{n} (k + (k - 1)y).$$

Recall that

$$\sum_{k=0}^{n} \binom{n}{k} x^k = x(x + 1) \cdots (x + n - 1).$$

Therefore, $E_n(y) = \sum_{k=0}^{n} \binom{n}{k} (1 + y)^{n-k}$, which leads to (3), since

$$E_n(x, y) = x^nE_n\left(\frac{y}{x}\right).$$

2.2. A proof of (4)

We first introduce a definition of marked permutations.

**Definition 7.** A marked permutation is a permutation with marks on some of its non-left-to-right minima.

An element $i$ is denoted by $i$ when it is marked. Let $\mathcal{M}_n$ denote the set of marked permutations of $[n]$. For example, $\mathcal{M}_1 = \{1\}$, $\mathcal{M}_2 = \{12, 1^2, 21\}$ and

$$\mathcal{M}_3 = \{123, 132, 213, 231, 312, 321, 12\bar{3}, 1\bar{2}3, 1\bar{3}2, 1\bar{3}2, 21\bar{3}, 2\bar{3}1, 31\bar{2}\}.$$}

Given $\pi \in \mathcal{M}_n$, let $\text{MARK}(\pi)$ be the set of marked entries of $\pi$, and let $\text{mark}(\pi) = |\text{MARK}(\pi)|$. The sets $\text{ASC}(\pi)$ and $\text{LRMIN}(\pi)$ are defined by forgetting the marks of $\pi$. Let $\text{asc}(\pi) = |\text{ASC}(\pi)|$ and $\text{lrmin}(\pi) = |\text{LRMIN}(\pi)|$. For example, $\text{MARK}(3521467) = \{4, 5\}$, $\text{mark}(3521467) = 2$, $\text{ASC}(3521467) = \{4, 5, 6, 7\}$, $\text{asc}(3521467) = 4$, $\text{LRMIN}(3521467) = \{1, 2, 3\}$ and $\text{lrmin}(3521467) = 3$. Moreover, it is the same as Stirling permutation, any marked permutation has a unique decomposition as a sequence of blocks. For example, the block decomposition of $3521467$ is given by $[35][2][1467]$.

The following lemma is fundamental.

**Lemma 8.** There is a one-to-one correspondence between the triple statistics $(\text{LRMIN}, \text{AP}, \text{EVEN})$ on Stirling permutations and the triple statistics $(\text{LRMIN}, \text{ASC}, \text{MARK})$ on marked permutations.

**Proof.** Let $X, Y, Z \subseteq [n]$. We define

$$Q(n; X, Y, Z) = \{\sigma \in \mathcal{Q}_n \mid \text{LRMIN}(\sigma) = X, \text{AP}(\sigma) = Y, \text{EVEN}(\sigma) = Z\},$$

and

$$S(n; X, Y, Z) = \{\pi \in \mathcal{M}_n \mid \text{LRMIN}(\pi) = X, \text{ASC}(\pi) = Y, \text{MARK}(\pi) = Z\}.$$
(ii) If $m m$ is inserted immediately before $s$, $s \neq r$, then $\Phi(\sigma') = \pi'$ is obtained by inserting $m$ or $\overline{m}$ into the $p$th block of $\pi$ such that $m$ or $\overline{m}$ is immediately before $s$. The inserted entry $m$ is marked if and only if $m$ is an even indexed entry of $\sigma'$. When $s \in Y$, let $Y' = (Y \cup \{m\}) \setminus \{s\}$. Otherwise, let $Y' = Y \cup \{m\}$. When $m$ is an even indexed entry, let $Z' = Z \cup \{m\}$. Otherwise, let $Z' = Z$. In each possible case, we see that $\sigma' \in Q(m; X, Y', Z')$ and $\Phi(\sigma') \in S(m; X, Y', Z')$.

(iii) If $m m$ is inserted at the end of the $p$th block, then $\Phi(\sigma') = \pi'$ is obtained by inserting an unmarked $m$ at the end of the $p$th block of $\pi$. Note that $\sigma$ does not gain any additional even indexed entry after inserting $m m$, but gain the ascent plateau $m$. On the other hand, $m$ is a new ascent of $\pi'$ after inserting $m$ into $\pi$. Hence $\sigma' \in Q(m; X, Y \cup \{m\}, Z)$ and $\Phi(\sigma') \in S(m; X, Y \cup \{m\}, Z)$.

The above argument shows that $\Phi(Q_n) \subseteq \overline{S}_n$, and that $\Phi$ is one-to-one correspondence between $Q_n$ and $\Phi(Q_n)$. Since the cardinality of $Q_n$ is the same as that of $\overline{S}_n$, $\Phi$ must be a bijection between $Q_n$ and $\overline{S}_n$. By induction, we see that $\Phi$ is the desired bijection between Stirling permutations and marked permutations. □

**Example 9.** Consider $\sigma = 266255133441 \in Q_6$. The correspondence between $\sigma$ and $\Phi(\sigma) = 256143$ is built up as follows (in order to illustrate the bijection $\Phi$, we write the objects in their block decompositions):

\[
\begin{align*}
[22][11] & \iff [2][1] \\
[22][1331] & \iff [2][1\overline{3}] \\
[22][134431] & \iff [2][14\overline{3}] \\
[2255][134431] & \iff [25][14\overline{3}] \\
[266255][134431] & \iff [25\overline{6}][14\overline{3}] \\
\end{align*}
\]

**Lemma 10.** We have

\[
\sum_{n \geq 0} \sum_{\pi \in \overline{S}_n} q^{\text{Irmin}(\pi)} x^{\text{asc}(\pi)} y^{\text{mark}(\pi)} \frac{t^n}{n!} = A(x, t(1 + y)) \frac{t^n}{n!},
\]

where $A(x, t)$ is the exponential generating function given by (1).

**Proof.** Combining [2, Proposition 7.3] and the fundamental transformation introduced by Foata and Schützenberger [5], we have

\[
\sum_{n \geq 0} \sum_{\pi \in \overline{S}_n} q^{\text{Irmin}(\pi)} x^{\text{asc}(\pi)} y^{\text{mark}(\pi)} \frac{t^n}{n!} = \sum_{n \geq 0} \sum_{\pi \in \overline{S}_n} q^{\text{cyc}(\pi)} x^{\text{exc}(\pi)} \frac{t^n}{n!} = A^q(x, t)
\]

For a permutation $\pi$ in $\overline{S}_n$ with $\text{Irmin}(\pi) = \ell$, there are $n - \ell$ entries that could be either marked or not. Therefore, we have

\[
\sum_{n \geq 0} \sum_{\pi \in \overline{S}_n} q^{\text{Irmin}(\pi)} x^{\text{asc}(\pi)} y^{\text{mark}(\pi)} \frac{t^n}{n!} = \sum_{n \geq 0} \sum_{\pi \in \overline{S}_n} q^{\text{Irmin}(\pi)} x^{\text{asc}(\pi)} (1 + y)^{n - \text{Irmin}(\pi)} \frac{t^n}{n!}
\]

\[
= \sum_{n \geq 0} \sum_{\pi \in \overline{S}_n} \left( \frac{q}{1 + y} \right)^{\text{Irmin}(\pi)} x^{\text{asc}(\pi)} \frac{(1 + y)^n}{n!}
\]

\[
= A(x, t(1 + y)) \frac{q^n}{n!}. \quad \Box
\]

**A proof of (4).**

**Proof.** It follows from Lemma 8 that

\[
\sum_{\sigma \in Q_n} q^{\text{Irmin}(\sigma)} x^{\text{ap}(\sigma)} y^{\text{even}(\sigma)} = \sum_{\pi \in \overline{S}_n} q^{\text{Irmin}(\pi)} x^{\text{asc}(\pi)} y^{\text{mark}(\pi)}.
\]

Combining (8) and Lemma 10, we get (4). This completes the proof. □

It should be noted that exploiting the bijection $\Phi$ used in the proof of Lemma 8, we can also get the following result.

**Proposition 11.** For $n \geq 1$, we have

\[
\sum_{\sigma \in Q_n} x^{\text{ap}(\sigma)} (-1)^{\text{even}(\sigma)} = \sum_{\pi \in \overline{S}_n} x^{\text{asc}(\pi)} (-1)^{\text{mark}(\pi)} = 1.
\]
Fig. 1. The bicolored tree corresponded to 231546.

Fig. 2. Involution on trees in $T_n$; the move is made on the node with label 5.

\[ \pi = 46387512 \quad \pi' = 46358712 \]

**Proof.** From (8), we get the first equality of (9). For the last equality of (9), we will consider an involution $\iota$ on $\overline{S}_n$ as follows. For a marked permutation $\pi$ that has non-left-to-right minima, define $\iota(\pi)$ to be the marked permutation obtained from $\pi$ by adding or deleting the mark of the smallest non-left-to-right minimum of $\pi$. For example, when $\pi = 25314$, then $\iota(\pi) = 25314$. This map $\iota$ is clearly a sign-reversing involution because the number of marks is either plus 1 or minus 1. It is also clear that $\iota$ preserves the ascent statistic since no entry leaves its position. The only marked permutation in $\overline{S}_n$ which cannot be mapped by $\iota$ is the one in which every entry is a left-to-right minimum, i.e., the marked permutation $n(n-1)\ldots21$, which does not have any ascent. This completes the proof. \(\square\)

3. A proof of Proposition 6

In this section, we first state the definition of bicolored increasing binary trees, and then we construct an involution on it.

Let $\varphi$ be the bijection between permutations and increasing binary trees defined by the inorder traversal in depth-first search. The left-to-right minima of $\sigma \in Q_n$ will be the labels in the leftmost path of the corresponding trees. Let $T_n$ denote the set of bicolored increasing binary trees on $n$ nodes such that all the nodes in the leftmost path are white and the other nodes are white or black. Then $\varphi$ is a bijection from $\overline{Q}_n$ to $T_n$ if we match white nodes (resp. black nodes) with those unmarked letters (resp. marked letters.) Note that all of the left-to-right minima of a marked permutation are mapped to the nodes of the leftmost path of the corresponding increasing binary tree. For convenience, the nodes of the leftmost path are called special nodes.

For example, the Stirling permutation $\sigma = 223315544166$ corresponds to the marked permutation $\pi = 231546$, which in turn corresponds to the bicolored increasing tree with 6 nodes in Fig. 1. Note that the special nodes are labeled 1 and 2.

We consider the composition $\varphi \circ \Phi$ from $Q_n$ to $T_n$, where $\Phi$ is defined in the proof of Lemma 8. It is easily observed that $k$ is an ascent plateau of $\sigma \in Q_n$ if and only if the node $k$ in the bicolored increasing binary tree $\varphi(\Phi(\sigma))$ is non-special and does not have a left child. Also the Stirling derangements are mapped to those trees each of whose special nodes has a right child.

For trees in $T_n$, there are some trees that have a non-special node with only one child. An involution can be introduced on these trees: among the non-special nodes with only one child, find the one with the minimal label, then move its whole subtree to the other branch. An example on this involution is shown in Fig. 2.
Following the discussion in the previous paragraph, we immediately see that this involution is sign-reversing if we attach the sign \((-1)^{ap(\pi)}\) to the tree \(\Phi(\pi)\).

For those trees to which the above involution cannot be applied, all of its nodes have either zero or two children; these trees are called complete, whose number of nodes \(n\) must be even, and it has the sign \((-1)^{ap(\pi)} = (-1)^{n/2}\). From here we conclude that when \(n\) is odd, these trees are sign-balanced. Our remaining task is to give a proper count on those complete bicolored increasing binary trees on \(n = 2k\) nodes.

Let \(\pi \in D_{Q_n}\) such that \(\varphi(\Phi(\pi))\) is complete. Then the intermediate marked permutation

\[\pi = \Phi(\pi) = \pi_1\pi_2 \cdots \pi_n\]

is reverse alternating, i.e., \(\pi_1 < \pi_2 > \pi_3 < \cdots\). Note that in these situations the left-to-right minima of \(\pi\) always occur at odd-indexed positions. We now apply the Foata transformation on \(\pi\) by making its left-to-right minima as the head of each cycle; say \(\pi \mapsto C_1C_2 \cdots C_\ell\). Each cycle determines some pairings on \(\pm[n]\); for any \(C = (c_1c_2 \cdots c_\ell)\), we pair \(c_i\) with \(c_{i+1}\) for \(1 \leq i < \ell\), and pair \(c_\ell\) with \(c_1\). Since the number at the beginning of a cycle must be unmarked, we have a bijection between complete bicolored increasing binary trees on \(n\) nodes and those special perfect matchings on \([4k]\) \(= [2n] \simeq \pm[n]\). Using the definition of the numbers \(h_{k}\), we get the desired result. \(\square\)

**Example 12.** We use the ordering \(\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{n} < n \in \pm[n] \cong [2n] = [4k]\). The reverse alternating marked permutation \(\pi = 3 \bar{5} 1 \bar{6} 2 4\) corresponds to the following perfect matching on \([6]\): \(35/53/16/62/24/41\). Because \(\pi\) is reverse alternating, the numbers \(i\) and \(i\) are both ascending or both descending for all \(i \in [n]\) in the corresponding perfect matching.

4. Concluding remarks

We call a permutation of the multiset \(\{1^k, 2^k, \ldots, n^k\}\) a \(k\)-Stirling permutation of order \(n\) if for each \(i, 1 \leq i \leq n\), all entries between the two occurrences of \(i\) are at least \(i\). When \(k = 2\), the \(k\)-Stirling permutations become the ordinary Stirling permutations. The notions introduced in this paper, including block, even-indexed and odd indexed entries, left-to-right and right-to-left minima, can also be defined over \(k\)-Stirling permutations.

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