Catalan and Motzkin numbers modulo 4 and 8

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Dedicated to Professor Zhe-Xian Wan on the occasion of his 80th birthday.

Abstract

In this paper, we compute the congruences of Catalan and Motzkin numbers modulo 4 and 8. in particular, we prove the conjecture proposed by Deutsch and Sagan that no Motzkin number is a multiple of 8.

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1 Introduction

Congruences of many well-known combinatorial numbers pulled research interest. The most famous and also age-old is the Pascal's fractal which is formed by the parities of binomial coefficients $\binom{n}{k}$ [14]. As a pioneer of this problem, Kummer formulated the

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maximum power of 2 dividing $\binom{n}{k}$ treated as a special case of Kummer's Theorem [9] (or see Theorem 5.12 in [4]). Lucas, another pioneer, developed a very useful tool that $\binom{n}{k} \equiv_p \prod_i \binom{n_i}{r_i}$, where p is a prime, "$\equiv_p$" denotes congruence modulo p, and $[n]_p = \langle n_r \ldots n_1 n_0 \rangle_p$ denotes the sequence of digits representing n in base p [11]. A generalization of Lucus' Theorem for prime powers was established by Davis and Webb [2]. The classical problem on Pascal's triangle also has modulo 4 and modulo 8 versions [3, 8]. Several other combinatorial numbers have been studied on their congruences, too. Like Apéry numbers [7, 12] and Central Delannoy numbers [6], not to say Catalan numbers.

The sequence of the Catalan numbers, $\langle C_n \rangle_{n=0}^{\infty} = \langle 1, 1, 2, 5, 14, 42, 132, \ldots \rangle$, defined by

$$C_n := \frac{1}{n+1}\binom{2n}{n},$$

is one of the most important sequences in combinatorics for its ubiquitous appearances in numerous problems and areas. Close related and also well known is the sequence of the Motzkin numbers, $\langle M_n \rangle_{n=0}^{\infty} = \langle 1, 1, 2, 4, 9, 21, 51, \ldots \rangle$, which can be defined in terms of the Catalan numbers by

$$M_n := \sum_{k \geq 0} \binom{n}{2k} C_k.$$

There are many ways to define M_n, but in order to calculate the congruences, we choose the above definition. Readers may refer to [5, 13] for further information on the Catalan numbers and the Motzkin numbers.

It is well known that C_n is odd if and only if $n = 2^k - 1$ for a nonnegative integer k. The rare appearance of odd Catalan numbers partitions even Catalan numbers into consecutive runs of length $b_i = 2^i - 1$. This fact was generalized by Alter and Kubota [1]. They investigated the congruence of C_n modulo any prime p and the length of runs corresponding to congruence p. They also studied the divisibility of Catalan numbers with respect to primes and prime powers. Deutsch and Sagan [4] took one step further and derived the formula for the highest power of 2 dividing C_n. However, there is lack of studies on the nonzero congruences for C_n. This paper is devoted to this.

On the other hand, the studies of the congruences of the Motzkin numbers M_n are few and emerged very recently. Luca and Klazar proved that the Motzkin numbers will never be periodic modulo any prime [10]. It seems that Deutsch and Sagan started the first systematical study on the congruences for the Motzkin numbers [4]. The congruences of M_n modulo 2, 3 and 5 are computed exactly in their paper.

However, even in light of [1], there are few *exact* results concerning the nonzero congruences of C_n and M_n modulo a prime power. This paper is our first attempt to compensate this situation. In this paper the congruences of C_n modulo 4 and 8 are fully investigated. As for M_n, all even congruences modulo 4 and 8 are studied and this result proves a conjecture stated as the following theorem.

Theorem 1.1 *We have $M_n \equiv_4 0$ if and only if*

$$n = (4i+1)4^{j+1} - 1 \quad \text{or} \quad n = (4i+3)4^{j+1} - 2,$$

where i and j are nonnegative integers. Furthermore we never have $M_n \equiv_8 0$.

The part of modulo 4 was conjectured by Amdeberhan and the part of modulo 8 was recently conjectured by Deutsch and Sagan.

Literally, C_n is constructed by factorials $2n!$ and $n!$; so solving the congruences of factorials is the fountainhead for dealing with the congruences of C_n. Furthermore, via the defining-equality of Motzkin-number $M_n = \sum_k \binom{n}{2k} C_k$ and our result on C_k, we settle the even congruences of M_n by computing $\binom{n}{2k}$ and dealing with the summation.

The paper is organized as follows. In Section 2, we develop the main tool $E_4(3, \cdot)$ and compute the congruences of Catalan numbers modulo 4. Section 3, we prove the first part of the conjecture about the Motzkin numbers modulo 4. In Section 4 is devoted to Catalan numbers modulo 8. The similar tool $E_8(t, \cdot)$ is designed for $t = 3, 5, 7$. Finally, we prove the second part of the conjecture by showing all even congruences of Motzkin numbers modulo 8 in Section 5.

2 Catalan numbers modulo 4

Define $[a, b] := \{a, a+1, \ldots, b\}$ for two positive integers a and b with $a \leq b$. Given positive integers n and p, let $[n]_p := \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_p$ denote the sequence of digits representing n in base p, *i.e.*, $n = n_r p^r + n_{r-1} p^{r-1} + \cdots + n_1 p + n_0$ with $n_i \in [0, p-1]$ and $n_r \neq 0$ for some integer r. For convenience, we let $n_{r+1} = n_{r+2} = \cdots = 0$, but these digits do not belong to the sequence $[n]_p$. We can even define $[0]_p$ as an empty sequence, while $0_0 = 0_1 = \cdots = 0$. Reversely, given a sequence of nonnegative integers $\langle n_r n_{r-1} \ldots n_1 n_0 \rangle$ with $0 \leq n_i \leq p-1$ for $0 \leq i \leq r$, we define $|\langle n_r n_{r-1} \ldots n_1 n_0 \rangle|_p := n_r p^r + n_{r-1} p^{r-1} + \cdots + n_1 p + n_0$. We will use $p = 2$ in the whole paper.

Now let $[n]_2 = \langle n_r n_{r-1} \dots n_1 n_0 \rangle_2$. Define $d_k(n) := \sum_{i \geq k} n_i$, which counts the number of the digit 1's from n_k to n_r. We also let $d(n) = d_0(n)$ for it will be used frequently. For a statement S, we set $\chi(S) = 1$ if S is true, otherwise $\chi(S) = 0$. Let us define $c_2(n) := \sum_{i=0}^{r-1} \chi(n_i = n_{i+1} = 1)$ be the number of the consecutive pairs of 1's in the sequence $[n]_2$, and $r(n)$ the number of runs of digit 1's in $\langle n_r n_{r-1} \dots n_1 n_0 \rangle_2$. Clearly, $c_2(n) = d(n) - r(n)$.

Given a positive integer n, let $\alpha(n)$ be the highest power index of base 2 such that $2^{\alpha(n)}$ divides n. Let $m_1, m_2, \dots, m_k$ be positive integers. For the formal product $\Pi_{i=1}^k m_i$, we define $E_4(3, \prod_{i=1}^k m_i) := \sum_{i=1}^k \chi(m_i/2^{\alpha(m_i)} \equiv_4 3)$. For instance, $E_4(3, 3 \times 4 \times 6) = 2$ and $E_4(3, 72) = 0$ even though $3 \times 4 \times 6 = 144$. However, $E_4(3, \prod_{i=1}^k m_i) \equiv_2 E_4(3, n)$, where n is the value of the product $\prod_{i=1}^k m_i$, because $3^2 \equiv_4 1$. Furthermore, for a formal quotient of products, $\Pi_{i=1}^k m_i / \Pi_{j=1}^l q_j$, we define $E_4(3, \Pi_{i=1}^k m_i / \Pi_{j=1}^l q_j) := E_4(3, \Pi_{i=1}^k m_i) - E_4(3, \Pi_{j=1}^l q_j)$. Usually the value of the formal quotient applied in $E_4(3, \cdot)$ is an integer; so $E_4(3, \cdot) \geq 0$. In the following two lemmas, we compute $\alpha(n!)$ and the parity of $E_4(3, n!)$.

Lemma 2.1 *Let $[n]_2 = \langle n_r n_{r-1} \dots n_1 n_0 \rangle_2$ and $2^{\alpha(n!)}$ be the highest power of 2 which divides $n!$. The power index $\alpha(n!)$ equals $\sum_{k=1}^r (2^k - 1) n_k$, and also equals $n - d(n)$.*

Proof. Notice that $\alpha(n!) = \sum_{k=1}^r \lfloor n/2^k \rfloor = |\langle n_r n_{r-1} \dots n_2 n_1 \rangle|_2 + |\langle n_r n_{r-1} \dots n_2 \rangle|_2 + \cdots + |\langle n_r \rangle|_2$, for $\lfloor n/2^k \rfloor$ counts the number of integers in $[1, n]$ that are multiples of 2^k. From this equation, the total contribution of n_k to $\alpha(n!)$ is $(2^{k-1} + 2^{k-2} + \cdots + 1) n_k$; thus $\alpha(n!) = \sum_{k=1}^r (2^k - 1) n_k = n - d(n)$ and the proof follows. ■

Lemma 2.2 *We have $E_4(3, n!) \equiv_2 d_2(n) + c_2(n)$; also $E_4(3, n!) \equiv_2 r(n) + n_0 + n_1$.*

Proof. Suppose $[n]_2 = \langle n_r n_{r-1} \dots n_1 n_0 \rangle_2$. For those $m \in [1, n]$ with the same value of $\alpha(m)$, say i, the sum of their $\chi(m/2^{\alpha(m)} \equiv_4 3)$ equals $\lfloor (\lfloor \frac{n}{2^i} \rfloor + 1)/4 \rfloor$. Therefore, we have

$$\begin{aligned} E_4(3, n!) &= \sum_{i \geq 0} \left\lfloor \frac{\lfloor \frac{n}{2^i} \rfloor + 1}{4} \right\rfloor & (1) \\ &= \sum_{i \geq 2} (2^{i-1} - 1) n_i + \sum_{i=0}^{r-1} \chi(n_i = n_{i+1} = 1) & (2) \\ &\equiv_2 \sum_{i \geq 2} n_i + \sum_{i=0}^{r-1} \chi(n_i = n_{i+1} = 1) \\ &= d_2(n) + c_2(n), \end{aligned}$$

where the second summation in (2) counts the effect caused by 1 in the right hand of (1), while the first summation is obtained by ignoring 1.

Since $d_2(n) = d(n) - n_0 - n_1$ and $c_2(n) = d(n) - r(n)$, we get $d_2(n) + c_2(n) \equiv_2 r(n) + n_0 + n_1$, and then the second statement is proved. ■

We are ready to compute the congruences of the Catalan numbers modulo 4.

Theorem 2.3 *Let C_n be the n-th Catalan number. First of all, $C_n \not\equiv_4 3$ for any n. As for other congruences, we have*

$$C_n \equiv_4 \begin{cases} 1 & \text{if } n = 2^a - 1 \text{ for some } a \geq 0; \\ 2 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b > a \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We shall first consider the case that $C_n = \frac{1}{n+1}\binom{2n}{n}$ is an odd integer. This case occurs if and only if $\alpha(n+1) = \alpha(\binom{2n}{n})$. By Lemma 2.1, we get

$$\begin{aligned} \alpha(\binom{2n}{n}) &= \alpha(2n) - 2\alpha(n) \\ &= 2n - d(n) - 2[n - d(n)] \\ &= d(n). \end{aligned} \tag{3}$$

Noticing that $\alpha(n+1)$ equals the length of 1's run in $[n]_2$ starting at n_0, we then derive that $\alpha(n+1) = d(n)$ if and only if $n = 2^a - 1$ for some integer $a \geq 0$. The congruence of this odd C_n satisfies

$$C_n \equiv_4 (-1)^{E_4(3,(2n)!) - 2E_4(3,n!)} = (-1)^{E_4(3,(2^{a+1}-2)!)} = 1,$$

where the last equality is due to Lemma 2.2 given that both $r(2^{a+1} - 2)$ and $(2^{a+1} - 2)_1$ are the same (both are 0 or 1) and $(2^{a+1} - 2)_0 = 0$. Now proof of (a) and $C_n \not\equiv_4 3$ follows.

Now we consider the case that $C_n \equiv_4 2$. Residue 2 happens if and only if $\alpha(n+1) = \alpha(\binom{2n}{n}) - 1 = d(n) - 1$, which is independent on the parity of $E_4(3, C_n)$ for $2 \times 3 \equiv_4 2$. Equation $\alpha(n+1) = d(n) - 1$ holds if and only if $n = 2^a + 2^b - 1$ for some $b > a \geq 0$, *i.e.*, $[n]_2$ is of form $\langle 1, 0, 0, \ldots, 0, 1, 1, \ldots, 1 \rangle$. The proof of (b) and (c) follows. ■

In Theorem 2.3, we set the second condition as "$n = 2^a + 2^b - 1$, for some $b > a \geq 0$". This unusual form ($b > a \geq 0$ not $a > b \geq 0$) is more convenient for nitational purposes.

Remark. We use Lemma 2.1 to find $\alpha(\binom{n}{k})$ (see Eq. (3)). Actually there is a well-known formula for $\alpha(\binom{n}{k})$ due to Kummer [9], namely it is the number of caries used when adding k to $n-k$. In the following, Eqs. (5) and (8) can be derived by this way. Part of Theorem 2.3 (as well as Theorem 4.2) is also a previous work of Deutsch and Sagan (see Theorem 2.1 in [4]). They showed a neat formula

$$\alpha(C_n) = d(n+1) - 1.$$

3 Motzkin numbers module 4

Let us define $S_4(i, C) = \{k \in \mathbb{N} \mid C_k \equiv_4 i\}$ for $i = 0, 1, 2, 3$, where $\mathbb{N}$ is the set of nonnegative integers. By Theorem 2.3, $S_4(1, C) = \{2^a - 1 \mid a \geq 0\}$, $S_4(2, C) = \{2^a + 2^b - 1 \mid b > a \geq 0\}$, and $S_4(3, C) = \emptyset$. Using the defining-equality of the Motzkin-number, $M_n = \sum_k \binom{n}{2k} C_k$, we derive that

$$M_n \equiv_4 \sum_{k \in S_4(1,C)} \binom{n}{2k} + 2 \sum_{k \in S_4(2,C)} \binom{n}{2k}. \tag{4}$$

We will analyze the above two summations to verify the conjecture that $M_n \equiv_4 0$ if and only if $n = (4i+1)4^{j+1} - 1$ or $n = (4i+3)4^{j+1} - 2$ for integers $i, j \geq 0$. For short, let us define

$$f(n) := \sum_{k \in S_4(1,C)} \binom{n}{2k}.$$

Since the second summation of (4) is even, $M_n \equiv_4 0$ happens only if $f(n)$ is even. Now we claim the first lemma as follows. This lemma is our stepping-stone for solving the even congruences of M_n.

Lemma 3.1 *The summation $f(n) := \sum_{k \in S_4(1,C)} \binom{n}{2k}$ is even if and only if $n = X \cdot 4^{j+1} - \delta$ for $X, j \in \mathbb{N}$ with X being odd and $\delta = 1$ or 2.*

Proof. We shall apply the Lucas' Theorem [11] that claims $\binom{n}{m} \equiv_p \prod_i \binom{n_i}{m_i}$, where $[n]_p = \langle \ldots n_1 n_0 \rangle_p$ and $[m]_p = \langle \ldots m_1 m_0 \rangle_p$. Here we take $p = 2$. Clearly, $f(n) \equiv_2 |\{k \in \mathbb{N} \mid \binom{n}{2k} \equiv_2 1\}|$. By the Lucas' Theorem, the equivalence in the above set means

$$n_{i+1} \geq k_i \text{ for all } i \geq 0.$$

Also notice that $k \in S_4(1, C)$; so either $k = 0$ or $[k]_2$ is a sequence of all 1's. Therefore, no matter n_0 is 0 or 1, we have

$$|\{k \in \mathbb{N} \mid \binom{n}{2k} \equiv_2 1\}| = lor(\langle \ldots n_2 n_1 \rangle_2) + 1,$$

where $lor(\langle \ldots n_2 n_1 \rangle_2)$ is the length of 1's run starting at n_1 (the rightest digit). Thus, $f(n)$ is even if and only if $lor(\langle \ldots n_2 n_1 \rangle_2)$ is odd. Suppose that this odd number is $2j+1$ for $j \geq 0$. Finally, we conclude that $n = X \cdot 4^{j+1} - \delta$ for $X, j \in \mathbb{N}$ with X being odd and $\delta = 1$ or 2, where δ depends on n_0. ■

As X is odd, we say $X = 4i + \varepsilon$ with $\varepsilon = 1$ or 3. Now we narrow down to only four types of n depending on ε and δ. The layout of the sequence $[n]_2 = [(4i+\varepsilon)4^{j+1} - \delta]_2$ is important for the rest of the paper. From left to right, the sequence $[n]_2$ has four parts (four subsequences): $A := [i]_2$, $B := \langle \frac{\varepsilon-1}{2} 0 \rangle_2$, $C := \langle 11..1 \rangle_2$, and the single digit $D := \langle 2 - \delta \rangle$, where $\langle 11..1 \rangle_2$ is of length $2j+1$.

Furthermore, we investigate the congruence of this even $f(n)$ modulo 4. But first we shall see the detail of $\binom{n}{2k}$ for those $n = (4i+\varepsilon)4^{j+1} - \delta$ and $k = 2^a - 1$ in the following lemmas and corollary.

Lemma 3.2 *Given $n = (4i+\varepsilon)4^{j+1} - \delta$ and $k = 2^a - 1$ for $a, i, j \in \mathbb{N}$, $\varepsilon = 1, 3$ and $\delta = 1, 2$, we have*

$$\alpha(\binom{n}{2k}) = \begin{cases} 0 & \text{if } a \leq 2j+1; \\ 1 & \text{if } a = 2j+2 \text{ and } \varepsilon = 3; \\ 2 & \begin{array}{l}\text{if } a = 2j+2,\ \varepsilon = 1, \text{ and } i \equiv_2 1, \text{ or} \\ \text{if } a = 2j+3 \text{ and } i \equiv_2 1,\end{array} \end{cases}$$

otherwise $\alpha(\binom{n}{2k}) \geq 3$.

Proof. By Lemma 2.1, we have

$$\begin{aligned} \alpha(\binom{n}{2k}) &= [n - d(n)] - [2k - d(2k)] - [(n-2k) - d(n-2k)] \\ &= -d(n) + d(k) + d(n-2k) \\ &= -d(n) + a + d(n-2k). \end{aligned} \tag{5}$$

We take more effort on the value of $d(n-2k)$. Let us observe the change of A, B, and C as we subtract $2k$ from n, while the unchanged D is ignored because $n_0 = (n-2k)_0$. The discussion is listed as four cases below. (This discussion can be translated to Kummer's method.)

(a) When $a \leq 2j+1$, only a digit 1's in C are eliminated by subtracting $2k$; so $d(n-2k) = d(n) - a$ and then $\alpha(\binom{n}{2k}) = 0$.

(b) When $a = 2j+2$ and $\varepsilon = 3$, not only all 1's in C are eliminated, but also $B = \langle 10 \rangle_2$ turns to be $\langle 01 \rangle_2$. Thus, $d(n-2k) = d(n) - a + 1$ and then $\alpha(\binom{n}{2k}) = 1$.

(c) As for $a = 2j+2$ and $\varepsilon = 1$, we must have $i \geq 1$, otherwise $n < 2k$. After subtracting $2k$, all 1's in C are eliminated, $B = \langle 00 \rangle_2$ turns to be $\langle 11 \rangle_2$. Also A becomes at least $d(i) - 1$ digit 1's, while this minimum happens if and only if i is odd. Thus, $d(n-2k) = d(n) - a + 2$ and $\alpha(\binom{n}{2k}) = 2$ if i is odd, and $d(n-2k) \geq d(n) - a + 3$ and $\alpha(\binom{n}{2k}) \geq 3$ otherwise.

(d) Finally, let us check the remained case $a \geq 2j+3$. After subtracting $2k$, all 1's in C are eliminated and $B = \langle \frac{\varepsilon-1}{2} 0 \rangle_2$ turns to be $\langle \frac{\varepsilon-1}{2} 1 \rangle_2$. Also A become $[i-(2^{a-2j-3}-1)-1]_2 = [i - 2^{a-2j-3}]_2$. Notice that $d(i - 2^{a-2j-3}) \geq d(i) - 1$ while the inequality holds if and only if $i_{a-2j-3} = 1$. Therefore, we have $\alpha(\binom{n}{2k}) \geq a - 2j - 1$ in this case. We conclude that if $a = 2j+3$ and $i_0 = 1$ (*i.e.*, i is odd) then $d(n-2k) = d(n) - a + 2$ and $\alpha(\binom{n}{2k}) = 2$; otherwise $d(n-2k) \geq d(n) - a + 3$ and $\alpha(\binom{n}{2k}) \geq 3$. ∎

Lemma 3.3 *Given* $n = (4i+\varepsilon)4^{j+1} - \delta$ *and* $k = 2^a - 1$ *for* $a, i, j \in \mathbb{N}$, $\varepsilon = 1, 3$ *and* $\delta = 1, 2$, *we have*

$$\binom{n}{2k} \equiv_4 \begin{cases} (-1)^{\chi(a\geq 1)\chi(\delta=2)+\chi(a=2j+1)} & \text{if } a \leq 2j+1, \\ 2 & \text{if } a = 2j+2 \text{ and } \varepsilon = 3; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{k \in S_4(1,C)} \binom{n}{2k} \equiv_4 2(j + \chi(\varepsilon = 3)) + (-1)^{\chi(\delta=1)} + 1.$$

Proof. For the first equation, the congruences 2 and 0 are direct consequences of Lemma 3.2; so we need only calculate the nontrivial case when $a \leq 2j+1$. In this condition, $\binom{n}{2k} \equiv_4 (-1)^{E_4(3,\binom{n}{2k})}$. We have

$$\begin{aligned}
E_4(3, \binom{n}{2k}) &\equiv_2 E_4(3, n!) + E_4(3, (2k)!) + E_4(3, (n-2k)!) \\
&\equiv_2 [r(n) + (2-\delta) + 1] + [r(2k) + 0 + k_0] \\
&\qquad +[r(n-2k) + (2-\delta) + (1-k_0)] \qquad (6) \\
&\equiv_2 r(n) + r(2k) + r(n-2k) \\
&\equiv_2 r(n) + \chi(a \geq 1) + [r(n) + \chi(a \geq 1)(2-\delta) - \chi(a = 2j+1)] \qquad (7) \\
&\equiv_2 \chi(a \geq 1) + \chi(a \geq 1)(2-\delta) + \chi(a = 2j+1) \\
&= \chi(a \geq 1)(3-\delta) + \chi(a = 2j+1) \\
&\equiv_2 \chi(a \geq 1)\chi(\delta = 2) + \chi(a = 2j+1).
\end{aligned}$$

In the above equalities, (6) is obtained by Lemma 2.2. To derive the three terms in the brackets of (7), we shall refer to case (a) in the last proof. In that case, a digit 1's are eliminated from C. Of course, nothing changes when $a = 0$. If $1 \leq a \leq 2j$ and $\delta = 1$, then the rightest run of $[n]_2$ is bisected. The number of runs keeps the same if $a = 2j+1$ and $\delta = 1$ or if $1 \leq a \leq 2j$ and $\delta = 2$. As for the last case $a = 2j+1$ and $\delta = 2$, the new number of runs becomes $r(n) - 1$.

Applying the first equation, we obtain

$$\begin{aligned}
\sum_{k \in S_4(1,C)} \binom{n}{2k} &\equiv_4 \sum_{a=0}^{2j+2} \binom{n}{2k} \\
&\equiv_4 1 + 2j(-1)^{\chi(\delta=2)} + (-1)^{\chi(\delta=2)+1} + 2\chi(\varepsilon = 3) \\
&\equiv_4 1 + (-1)^{\chi(\delta=1)} + 2(j + \chi(\varepsilon = 3)),
\end{aligned}$$

where the second equivalence is obtained by considering $a = 0$, $1 \leq a \leq 2j$, $a = 2j+1$, and $a = 2j+2$. The proof is complete now. ∎

Now we shall turn our attention to the necessary and sufficient condition for $\sum_{k \in S_4(2,C)} \binom{n}{2k} \equiv_2 1$ because Eq. (4) has the term $2\sum_{k \in S_4(2,C)} \binom{n}{2k}$.

Lemma 3.4 *Given $n = (4i+\varepsilon)4^{j+1} - \delta$ and $k = 2^a + 2^b - 1$ for $a, b, i, j \in \mathbb{N}$ with $b > a$, $\varepsilon = 1, 3$ and $\delta = 1, 2$, the value $\binom{n}{2k}$ is odd if only if $a \leq 2j+1$ and $n_{b+1} = 1$. Furthermore, $\sum_{k \in S_4(2,C)} \binom{n}{2k} \equiv_2 j$.*

Proof. By Lemma 2.1, we have

$$\begin{aligned}\alpha(\binom{n}{2k}) &= [n-d(n)]-[2k-d(2k)]-[(n-2k)-d(n-2k)] \\ &= -d(n)+d(k)+d(n-2k) \\ &= -d(n)+(a+1)+d(n-2k). \end{aligned} \tag{8}$$

For $d(n-2k)$, we can consider $n-2k$ as n subtracted by $2(a^2-1)$ and then subtracted by $2(2^b)$. Before the second subtraction, the evaluation of $\alpha(\binom{n}{2k})$ shall be the same as the one in the proof of Lemma 3.2. We refer to cases (a) to (d) in that proof and also compare (8) with (5); The value of α here is 1 greater than it is in each case there. For the value of d(n-2k), the second subtraction is independent on the first. Subtracting $2(2^b)$ can at most reduce the value of d by 1 (sometimes, it can even increase d). This extreme case happens if and only if $n_{b+1}=1$. Now this case combined with case (a), which requires $a \le 2j+1$, forms the necessary and sufficient condition for $\alpha(\binom{n}{2k})=0$, also for $\binom{n}{2k} \equiv_2 1$.

To evaluate the parity of $\sum_{k\in S_4(2,C)}\binom{n}{2k}$, we refer to the proof of Lemma 3.2 again for the four parts of the sequence $[n]_2=[(4i+\varepsilon)4^{j+1}-\delta]_2$. Given a fixed a, we count how many b's that make $\binom{n}{2k}$ odd. For $a=2j+1$, there are $d(i)+\chi(\varepsilon=3)$ such b's. For $0\le a\le 2j$, there are $d(i)+\chi(\varepsilon=3)+(2j-a)$ such b's. Therefore

$$\begin{aligned}\sum_{k\in S_4(2,C)}\binom{n}{2k} &\equiv_2 d(i)+\chi(\varepsilon=3)+\sum_{a=0}^{2j}[d(i)+\chi(\varepsilon=3)+(2j-a)] \\ &\equiv_2 (2j+2)(d(i)+\chi(\varepsilon=3))+\sum_{a=0}^{2j}a \\ &\equiv_2 j(2j+1) \\ &\equiv_2 j. \end{aligned}$$

■

Proof for the first part of Theorem 1.1. Lemmas 3.3 and 3.4, we can simplify Eq. (4) as $M_n \equiv_4 [2(j+\chi(\varepsilon=3))+(-1)^{\chi(\delta=1)}+1]+2j$. We simply plug in the four types of $n=(4i+\varepsilon)4^{j+1}-\delta$ with $\varepsilon=1,3$ and $\delta=1,2$. We find

$$\begin{aligned} M_n &\equiv_4 0 \quad \text{if } (\varepsilon,\delta)=(1,1) \text{ or } (3,2); \\ M_n &\equiv_4 2 \quad \text{if } (\varepsilon,\delta)=(1,2) \text{ or } (3,1). \end{aligned} \tag{9}$$

Notice that these four types of n is the necessary condition for M_n to be even; so the proof is complete. ■

Eq. (9) in the above proof offers an auxiliary property as follows.

Theorem 3.5 *We have $M_n \equiv_4 2$ if and only if*

$$n = (4i+1)4^{j+1} - 2 \quad or \quad n = (4i+3)4^{j+3} - 1, \quad where\ i, j \in \mathbb{N}.$$

4 Factorials and Catalan numbers modulo 8

The process is almost the same. To verify the conjecture that $M_n \equiv_8 0$ never happens, we need first to take care of the congruences of factorials and Catalan numbers modulo 8. For a formal product $\Pi_{i=1}^k m_i$ of positive integers, we define $E_8(t, \prod_{i=1}^k m_i) := \sum_{i=1}^k \chi(m_i/2^{\alpha(m_i)} \equiv_8 t)$ for $t = 3, 5, 7$. Similarly, we define $E_8(t, \Pi_{i=1}^k m_i/\Pi_{j=1}^l q_j) := E_8(t, \Pi_{i=1}^k m_i) - E_8(t, \Pi_{j=1}^l q_j)$. For example, $E_8(3, 7!) = 2$ and $E_8(5, 7!) = E_8(7, 7!) = 1$; however, $E_8(3, 5040) = 1$ and $E_8(5, 5040) = E_8(7, 5040) = 0$. The difference between $E_8(t, 7!)$ and $E_8(t, 5040)$ is due to $5 \times 7 \equiv_8 3$. The table for the product rules of $\mathbb{Z}_8$ as follows is useful.

	3	5	7	2	4	6
3	1	7	5	6	4	2
5		1	3	2	4	6
7			1	6	4	2

Again, we are interested in the parity of $E_8(t, \prod_{i=1}^k m_i)$ because $3^2 \equiv_8 5^2 \equiv_8 7^2 \equiv_8 1$. Several new notations help us to evaluate these parities. Suppose $[n]_2 = \langle n_r \dots n_1 n_0 \rangle_2$. In the bit sequence $[n]_2$, let r_1 be the number of isolated 1's, $zr(n)$ the number of runs made by 0's, $zr_1(n)$ the number of isolated 0's. We compute the parity of $E_8(t, n!)$ in the following lemma. Note that $[0]_2$ is an empty sequence, while $0_0 = 0_1 = \cdots = 0$.

Lemma 4.1 *We have*

1. $E_8(3, n!) \equiv_2 r_1(n) + zr(n) + n_0 + n_2$,
2. $E_8(5, n!) \equiv_2 r(n) + zr_1(n) + n_0 + n_2$, *and*
3. $E_8(7, n!) \equiv_2 r_1(n) + n_0 + n_1 + n_2$.

Proof. Suppose $[n]_2 = \langle n_r n_{r-1} \dots n_2 n_1 n_0 \rangle_2$. For $t = 3, 5, 7$, we define $A_t := \{[x]_2 \mid t \leq x \leq 7\}$, for here $[3]_2 = \langle 011 \rangle_2$ not $\langle 11 \rangle_2$ to make all elements in A_t of length 3. The argument for Eqs. (1) and (2) still works here. So we obtain

$$\begin{aligned} E_8(t,n!) &= \sum_{i\ge 0}\left\lfloor \frac{\lfloor \frac{n}{2^i}\rfloor + (8-t)}{8}\right\rfloor \\ &= \sum_{i\ge 3}(2^{i-2}-1)n_i + \sum_{i=0}^{r-1}\chi(|\langle n_{i+2}n_{i+1}n_i\rangle|_2 \ge t) \qquad (10) \\ &\equiv_2 d_3(n) + \sum_{i=0}^{r-1}\chi(\langle n_{i+2}n_{i+1}n_i\rangle \in A_t) \qquad (11) \end{aligned}$$

for $t = 3, 5, 7$. Notice that in the second summation of (10) the index i has upper bound $r-1$ because we treat n_{r+1} as 0 and this setting only occurs when $t = 3$; for other t, we always have $\chi = 0$ when $i = r-1$. For the further precise evaluation we need the following equations which are easy to check. We remind that $d(n)$, $r(n)$, $r_1(n)$, $zr(n)$, $zr_1(n)$ are irrelevant to n_{r+1} because it does no belong to $[n]_2$.

(a) $\sum_{i=0}^{r-1}\chi(\langle n_{i+2}n_{i+1}n_i\rangle = \langle 011\rangle) = r(n) - r_1(n)$.

(b) $\sum_{i=0}^{r-1}\chi(\langle n_{i+2}n_{i+1}n_i\rangle = \langle 100\rangle) = zr(n) - zr_1(n)$.

(c) $\sum_{i=0}^{r-1}\chi(\langle n_{i+2}n_{i+1}n_i\rangle = \langle 101\rangle) = zr_1(n) - n_1(1-n_0)$.

(d) $\sum_{i=0}^{r-1}\chi(\langle n_{i+2}n_{i+1}n_i\rangle = \langle 110\rangle) = r(n) - r_1(n) - n_0n_1$.

(e) $\sum_{i=0}^{r-1}\chi(\langle n_{i+2}n_{i+1}n_i\rangle = \langle 111\rangle) = c_3(n) = d(n) - 2r(n) + r_1(n)$.

Therefore, we obtain

$$\begin{aligned} E_8(5,n!) &\equiv_2 d_3(n) + [zr_1(n) - n_1(1-n_0)] + [r(n) - r_1(n) - n_0n_1] \\ &\quad + [d(n) - 2r(n) + r_1(n)] \\ &\equiv_2 r(n) + zr_1(n) + n_0 + n_2. \end{aligned}$$

The check of the other two is omitted. ■

With the help of Lemma 4.1, we can bisect each condition in Theorem 2.3 to form a new conditional equation as a refinement to evaluate C_n modulo 8.

Theorem 4.2 *Let C_n be the n-th Catalan number. First of all, $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$ for any n. As for other congruences, we have*

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0 \text{ or } 1; \\ 2 & \text{if } n = 2^a + 2^{a+1} - 1 \text{ for some } a \geq 0; \\ 4 & \text{if } n = 2^a + 2^b + 2^c - 1 \text{ for some } c > b > a \geq 0; \\ 5 & \text{if } n = 2^a - 1 \text{ for some } a \geq 2; \\ 6 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b - 2 \geq a \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.3, $C_n \equiv_4 1$ if and only if $n = 2^a - 1$ for some $a \geq 0$. Since $C_0 = C_1 = 1$, we need only tocheck that if $n = 2^a - 1$ for some $a \geq 2$ then $C_n \equiv_8 5$. We apply $E_8(t, \cdot)$ to the formal quotient $C_n = \binom{2n}{n}/(n+1)$ with $n = 2^a - 1$ for $a \geq 2$, and obtain $E_8(t, C_n) = E_8(t, (2n)!) - 2E_8(t, n!) - E_8(t, n+1) \equiv_8 E_8(t, (2n)!)$. After deriving $E_8(3, C_n) \equiv_2 E_8(7, C_n) \equiv_2 0$ and $E_8(5, C_n) \equiv_2 1$ by Lemma 4.1, the cases of congruences 1 and 5 are done.

Now for the cases of congruences 2 and 6. By Theorem 2.3, we shall suppose $n = 2^a + 2^b - 1$ for $b > a \geq 0$. We need to find out the necessary and sufficient condition for congruence 2. In the ring $\mathbb{Z}_8$, we have four possible products to create congruence 2, namely $2, 2\times3\times5\times7, 2\times5$ and $2\times3\times7$. For $\alpha(C_{2^a+2^b-1}) = 1$ we shall consider the possible combination of the parities of $E_8(t, n!)$ for $t = 3, 5, 7$. Let $W_{35} := E_8(3, C_n) + E_8(5, C_n)$ and $W_{37} := E_8(3, C_n) + E_8(7, C_n)$. Clearly, $(W_{35} + 1)(W_{37} + 1) \equiv_2 1$ is the necessary and sufficient condition for one of the first two products occurs (*i.e.*, 2 and $2 \times 3 \times 5 \times 7$). Also $W_{35}(W_{37} + 1) \equiv_2 1$ is the one for either of the last two products appears. We conclude that

$$C_{2^a+2^b-1} \equiv_8 2 \quad \Longleftrightarrow \quad W_{37} \equiv_2 0. \tag{12}$$

By Lemma 4.1 and the layout of $[2^a + 2^b - 1]_2$, we have

$$\begin{aligned} W_{37} &\equiv_2 [E_8(3, (2n)!) + E_8(7, (2n)!)] + [E_8(3, n+1) + E_8(7, n+1)] \\ &\equiv_2 [zr(2n) + (2n)_1] + [E_8(3, 2^{b-a} + 1) + E_8(7, 2^{b-a} + 1)] \\ &\equiv_2 [1 + \chi(a \geq 1) + \chi(a + 2 \leq b) + \chi(a \geq 1)] + [\chi(a + 1 = b)\chi(a + 2 \leq b)] \\ &\equiv_2 1 + \chi(a + 1 = b). \end{aligned}$$

Therefore, $W_{37} \equiv_2 0$ if and only if $a+1=b$, and the proof of this case follows.

It requires $\alpha(C_n)=2$ to get congruence 4. Referring to the proof of Theorem 2.3, it is the same to required $\alpha(n+1)=d(n)-2$. The check for the necessary and sufficient condition $n=2^a+2^b+2^c-1$ is left to the reader. ■

5 Motzkin numbers modulo 8

Now we are ready to prove that $M_n \equiv_8 0$ never happens. Actually, we focus on even congruences 0, 2, 4, and 6 modulo 8, and we shall suppose $n=(4i+\varepsilon)4^{j+1}-\delta$ for $i,j\in\mathbb{N}$, $\varepsilon=1,3$ and $\delta=1,2$.

We remind that the layout of $[(4i+\varepsilon)4^{j+1}-\delta]_2$, from left to right, consists of four subsequences: $A:=[i]_2$, $B:=\langle\frac{\varepsilon-1}{2}0\rangle_2$, $C:=\langle 11..1\rangle_2$, and the single digit $D:=\langle 2-\delta\rangle$, where C is of length $2j+1$. In addition, we define $Y:=4i+\varepsilon-1$ and $y:=d(Y)$. The sequence $[Y]_2$ consists of the first two parts of $[n]_2$, namely A and B.

Let $S_8(i,C):=\{k\in\mathbb{N}\mid C_k\equiv_8 i\}$ for $i=0,\ldots,7$. By Theorem 4.2, we know $S_8(3,C)=S_8(7,C)=\emptyset$ and

$$M_n \equiv_8 A_1(n)+2A_2(n)+4A_4(n)+5A_5(n)+6A_6(n),$$

where $A_t(n)=\sum_{k\in S_8(t,C)}\binom{n}{2k}$. Evaluating $4A_4(n)$ is easy. We need only check the parity of A_4 to prove the following lemma.

Lemma 5.1 *Let $n=(4i+\varepsilon)4^{j+1}-\delta$ for $i,j\in\mathbb{N}$, $\varepsilon=1,3$ and $\delta=1,2$. Also let $S_8(4,C)=\{2^a+2^b+2^c-1\mid a,b,c\in\mathbb{N}$ with $c>b>a\geq 0\}$. We have*

$$4A_4 := 4\sum_{k\in S_8(4,C)}\binom{n}{2k} \equiv_8 4jy+4j.$$

Proof. Suppose $[n]_2=\langle n_r\ldots n_1n_0\rangle_2$. By the Lucas' Theorem, $\binom{n}{2k}\equiv_2 1$ if and only if $n_l\geq k_{l-1}$ for all $1\leq l\leq r$. Notice that $[k]_2$ consists of $a+2$ digit 1's with $\langle k_ak_{a-1}\ldots k_1k_0\rangle=\langle 01\ldots 11\rangle$. So $\binom{n}{2k}\equiv_2 1$ means $a\leq 2j+1$ and $n_{b+1}=n_{c+1}=1$ with $c>b>a\geq 0$. The following is obtained by counting the possible ordered pair (b,c) with respect to a fixed $a\in[0,2j+1]$, where $y:=d(Y)$.

$$\begin{aligned} A_4(n) &\equiv_2 \sum_{a=0}^{2j}(\binom{y}{2}+y(2j-a)+\binom{2j-a}{2})+\binom{y}{2} \\ &\equiv_2 jy+\frac{1}{3}j(4j^2-1). \end{aligned}$$

We complete this proof after knowing that $\frac{1}{3}j(4j^2-1) \equiv_2 j$ and multiply A_4 by 4 under modulo 8. ■

We abuse the notation W_{st}, once used in the last section, and let $W_{st} := E_8(s, \binom{n}{2k}) + E_8(t, \binom{n}{2k})$. The congruence of $\binom{n}{2k}/2^{\alpha(\binom{n}{2k})}$ is crucial, and we can use the following properties to evaluate it.

$$\begin{array}{lll} \binom{n}{2k}/2^{\alpha(\binom{n}{2k})} \equiv_8 1 & \Leftrightarrow & (W_{35}+1)(W_{37}+1) = W_{35}W_{37} + W_{57} + 1 \equiv_8 1, \\ \binom{n}{2k}/2^{\alpha(\binom{n}{2k})} \equiv_8 3 & \Leftrightarrow & W_{35}(W_{57}+1) \equiv_8 1, \text{ and} \\ \binom{n}{2k}/2^{\alpha(\binom{n}{2k})} \equiv_8 1 \text{ or } 5 & \Leftrightarrow & W_{37} \equiv_8 0. \end{array}$$

The third property has been used in the proof of Theorem 4.2.

In the following we compute $A_1 + 5A_5$ and $2A_2 + 6A_6$ under modulo 8. Indeed, we are able to compute A_1, A_2, A_5, and A_6 independently. But doing them in pairs yields two simple formulas.

Lemma 5.2 *We have*

$$A_1(n) + 5A_5(n) \equiv_8 \begin{cases} 2j+4 & \text{if } \varepsilon = 1 \text{ and } \delta = 1; \\ 6j+2 & \text{if } \varepsilon = 1 \text{ and } \delta = 2; \\ 2j+6 & \text{if } \varepsilon = 3 \text{ and } \delta = 1; \\ 6j & \text{if } \varepsilon = 3 \text{ and } \delta = 2. \end{cases}$$

Proof. We shall see the detail of $\binom{n}{2k}$, for those $n = (4i+\varepsilon)4^{j+1} - \delta$ and $k = 2^a - 1$. Especially, A_1 is obtained when $a = 0$ and $a = 1$. According the value of $\alpha(\binom{n}{2k})$ given in Lemma 3.2, we discuss three cases as follows, while the forth case with $\alpha(\binom{n}{2k}) \geq 3$ is irrelevant.

Case I. We start with the easy case when $\alpha(\binom{n}{2k}) = 2$, *i.e.*, when $a = 2j+2$, $\varepsilon = 1$, and $i \equiv_2 1$, or when $a = 2j+3$ and $i \equiv_2 1$. In both possible conditions $a \geq 2$, so $\binom{n}{2k}$ only contributes to A_5 and the contribution is $4i_0(\chi(\varepsilon = 1) - 1) = 4i_0\chi(\varepsilon = 3) \pmod 8$. For $5 \times 4 \equiv_8 4$, the value $\pmod 8$ contributing to $5A_5$ is the same.

Case II. Suppose $a \leq 2j+1$, *i.e.*, $\alpha(\binom{n}{2k}) = 0$. We will calculate W_{st} in this case, but first deal with r_1, r, zr_1, and zr by plugging in $n - 2k$. (As a special case when $a = 0$, plugging in $n - 2k$ is as same as plugging in n.) To check the following four formula, the reader had better refer to the notations A, B, C, and D which are four consecutive subsequences of $[n]_2 = [(4i+\varepsilon)4^{j+1} - \delta]_2$ in the proof of Lemma 3.2.

$$\begin{aligned}
r_1(n-2k) &= r_1(n) + \chi(\delta=1)\chi(a\geq 1) - \chi(\delta=2)\chi(a=1)\chi(j=0) \\
&\quad +\chi(a=2j)\chi(j\geq 1), \\
r(n-2k) &= r(n) + \chi(\delta=1)\chi(1\leq a\leq 2j)\chi(j\geq 1) - \chi(\delta=2)\chi(a=2j+1), \\
zr_1(n-2k) &= zr_1(n) + \chi(\delta=1)\chi(a=1)\chi(j\geq 1) - \chi(\delta=2)\chi(a\geq 1) \\
&\quad -\chi(\varepsilon=3)\chi(a=2j+1), \\
zr(n-2k) &= zr(n) + \chi(\delta=1)\chi(1\leq a\leq 2j)\chi(j\geq 1) \\
&\quad -\chi(\delta=2)\chi(a=2j+1). \qquad (13)
\end{aligned}$$

Precisely, $\chi(j\geq 1)$ can be removed in the formulas of $r(n-2k)$ and $zr(n-2k)$, but we rather not. In general, we define $(f+g)(n) := f(n)+g(n)$ for briefness. By Lemma 4.1, we have

$$\begin{aligned}
W_{35} &\equiv_2 [(r_1+r+zr_1+zr)(n) - (r_1+r+zr_1+zr)(n-2k)] \\
&\quad -(r_1+r+zr_1+zr)(2k) \\
&\equiv_2 [\chi(\delta=1)\chi(a=1)\chi(j\geq 1) + \chi(\delta=2)\chi(a=1)\chi(j=0) \\
&\quad +\chi(\varepsilon=3)\chi(a=2j+1) + \chi(a=2j)\chi(j\geq 1) + \chi(a\geq 1)] \\
&\quad -\chi(a\geq 2) \\
&\equiv_2 \chi(\delta=1)\chi(a=1)\chi(j=0) + \chi(\delta=2)\chi(a=1)\chi(j\geq 1) \\
&\quad +\chi(\varepsilon=3)\chi(a=2j+1) + \chi(a=2j)\chi(j\geq 1), \\
W_{37} &\equiv_2 [zr(n) - zr(n-2k)] - zr(2k) + [n_1 - (2k)_1 - (n-2k)_1] \\
&\equiv_2 \chi(\delta=1)\chi(1\leq a\leq 2j)\chi(j\geq 1) + \chi(\delta=2)\chi(a=2j+1) + \chi(a\geq 1), \\
W_{57} &\equiv_2 W_{35} + W_{37} \\
&\equiv_2 \chi(\delta=1)\chi(2\leq a\leq 2j)\chi(j\geq 1) + \chi(\delta=2)\chi(a=2j+1)\chi(j\geq 1) \\
&\quad +\chi(\varepsilon=3)\chi(a=2j+1) + \chi(a=2j)\chi(j\geq 1) + \chi(a\geq 2)
\end{aligned}$$

Note that $n_1 - (2k)_1 - (n-2k)_1 = 0$ in the formula of W_{37}. We can also use $W_{57} \equiv_2 [(r_1+r+zr_1)(n) - (r_1+r+zr_1)(n-2k)] - (r_1+r+zr_1)(2k) + [n_1 - (2k)_1 - (n-2k)_1]$ to obtain the last formula.

Respectively, the necessary and sufficient condition for $\binom{n}{2k} \equiv_8 1, 3, 5, 7$ are the oddness of the following four.

$$\begin{aligned}
W_{35}W_{37}+W_{57}+1 &\equiv_2 \chi(\delta=1)[\chi(j=0,a=1)+\chi(j\geq 2,2\leq a\leq 2j-1)]\\
&\quad +\chi(\delta=2)[\chi(j\geq 1,a=1)+\chi(a=2j+1)[\chi(\varepsilon=3)+\chi(j\geq 1)]]\\
&\quad +\chi(a\geq 2)+1,\\
W_{37}(W_{57}+1) &\equiv_2 \chi(\delta=1,a=2j+1)[\chi(j=0)+\chi(\varepsilon=3)]\\
&\quad +\chi(\delta=2,j\geq 1)[\chi(a=2j)+(a=1)],\\
W_{35}(W_{37}+1) &\equiv_2 \chi(\delta=1)\chi(j\geq 1,a=2j)+\chi(\delta=2)\chi(\varepsilon=3,a=2j+1),\\
W_{37}(W_{35}+1) &\equiv_2 \chi(a\geq 1)+\chi(a=2j+1)[1+\chi(\delta=1)[\chi(j=0)+\chi(\varepsilon=1)]]\\
&\quad +\chi(j\geq 1)[\chi(a=2j)+\chi(\delta=1,1\leq a\leq 2j-1)\\
&\quad +\chi(\delta=2,a=1)].
\end{aligned}$$

We complete this case by summarizing the above criteria by a table to evaluate $\binom{n}{2k}$ modulo 8, and also for A_1, A_5, and A_1+5A_5:

	$j=0$, $\varepsilon=1$, $\delta=1$	$j=0$, $\varepsilon=1$, $\delta=2$	$j=0$, $\varepsilon=3$, $\delta=1$	$j=0$, $\varepsilon=3$, $\delta=2$	$j\geq 1$, $\varepsilon=1$, $\delta=1$	$j\geq 1$, $\varepsilon=1$, $\delta=2$	$j\geq 1$, $\varepsilon=3$, $\delta=1$	$j\geq 1$, $\varepsilon=3$, $\delta=2$	$\#\{a\}$
$a=0$ *	1	1	1	1	1	1	1	1	1
$a=1$ *	3	1	7	5	1	3	1	3	1
$2\leq a\leq 2j-1$					1	7	1	7	$2j-2$
$a=2j\geq 2$					5	3	5	3	1
$a=2j+1\geq 3$					7	1	3	5	1
A_1 *	4	2	0	6	2	4	2	4	
A_5	0	0	0	0	$2j+2$	$6j+6$	$2j+6$	$6j+2$	
A_1+5A_5	4	2	0	6	$2j+4$	$6j+2$	$2j$	$6j+6$	

Even though the difference of A_1, the general formulas of A_1+5A_5 when $j\geq 1$ work for the corresponding special cases when $j=0$.

Case III. Suppose $\alpha(\binom{n}{2k})=1$, *i.e.*, $a=2j+2$ and $\varepsilon=3$. The congruences of $\binom{n}{2k}$ in this case are only 2 and 6. The similar situation happens in the proof of Theorem 4.2. We know $\binom{n}{2k}\equiv_8 2$ if and only if $W_{37}\equiv_2 0$. Now $k=a^{2j+2}-1$ and $\varepsilon=3$. With a little adjustment, we derive

$$zr(n-2k) = zr(n) + i_0 - \chi(\delta = 2),$$

and then

$$\begin{aligned} W_{37} &\equiv_2 [zr(n) - zr(n-2k)] - zr(2k) + [n_1 - (2k)_1 - (n-2k)_1] \\ &\equiv_2 i_0 + \chi(\delta = 2) + 1. \end{aligned}$$

Therefore, $\binom{n}{2k} \equiv_8 2$ if $\delta = 1$ and $i_0 = 1$, or $\delta = 2$ and $i_0 = 0$; $\binom{n}{2k} \equiv_8 6$ otherwise. So the value contributed to A_5 as well as to $5A_5$ is $\chi(\delta = 1, \varepsilon = 3)(6+4i_0) + \chi(\delta = 2, \varepsilon = 3)(2+4i_0)$ (mod 8).

Now just sum up the contributions from the above three cases. For example, when $\varepsilon = 3$ and $\delta = 2$ we have

$$A_1 + 5A_5 = 4i_0 + 6j + 6 + 2 + 4i_0 = 6j.$$

The check of the other three is left to the reader. ■

The next lemma will be used to evaluate $2A_2 + 6A_6$. The proof is similar to the one of Lemma 3.2; so we skip it.

Lemma 5.3 *Given $n = (4i + \varepsilon)4^{j+1} - \delta$ and $k = 2^a + 2^b - 1$ for $a, b, i, j \in \mathbb{N}$, $b > a$, $\varepsilon = 1, 3$ and $\delta = 1, 2$, we have*

$$\alpha\left(\binom{n}{2k}\right) = \begin{cases} 0 & \text{if } a \le 2j+1 \text{ and } n_{b+1} = 1; \\ 1 & \begin{array}{l} \text{if } a \le 2j+1,\ n_{b+1} = 0, \text{ and } n_{b+2} = 1, \text{ or} \\ \text{if } a = 2j+2,\ n_{b+1} = 1, \text{ and } \varepsilon = 3. \end{array} \end{cases}$$

otherwise $\alpha(\binom{n}{2k}) \ge 2$.

Lemma 5.4 *We have*

$$2A_2(n) + 6A_6(n) \equiv_8 4y\chi(\varepsilon = 3) + [2j + 4j\chi(\delta = 2)] + [4y(j + \chi(\delta = 2)) + 4\chi(\varepsilon = 3)]$$

or

$$2A_2(n) + 6A_6(n) \equiv_8 \begin{cases} 2j + 4jy & \text{if } \varepsilon = 1 \text{ and } \delta = 1; \\ 6j + 4jy + 4y & \text{if } \varepsilon = 1 \text{ and } \delta = 2; \\ 2j + 4jy + 4y + 4 & \text{if } \varepsilon = 3 \text{ and } \delta = 1; \\ 6j + 4jy + 4 & \text{if } \varepsilon = 3 \text{ and } \delta = 2. \end{cases}$$

Proof. Notice that $S_8(2,C) \cup S_8(6,C) = \{2^a + 2^b - 1 \mid a, b \in \mathbb{N}, b > a\}$. According to the value of $\alpha(\binom{n}{2k})$ given in Lemma 5.3, we discuss two cases as follows, while the third case with $\alpha(\binom{n}{2k}) \geq 2$ is irrelevant.

Case I. Suppose $\alpha(\binom{n}{2k}) = 1$. Because each $\binom{n}{2k}$ is even, so are $A_2(n)$ and $A_6(n)$; and then $2A_2(n) + 6A_6(n) \equiv_8 4(A_2(n)/2) + 4(A_6(n)/2)$. So we need only consider the parity of $A_2(n)/2 + A_6(n)/2$, which is exactly the number of $\binom{n}{2k}$ in this case. The number of those k's satisfying $a \leq 2j+1$, $n_{b+1} = 0$, and $n_{b+2} = 1$ is

$$r(Y) - \chi(\varepsilon = 3) + \sum_{a=0}^{2j} r(Y) \;\equiv_2\; \chi(\varepsilon = 3),$$

because the digit $n_{b+2} = 1$ shall be the last 1 in any run of Y. But there is an unnecessary counting that happens when $\chi(\varepsilon = 3)$ and $a = b = 2j+1$. Given $\varepsilon = 3$, the number of those k's satisfying $a = 2j+2$ and $n_{b+1} = 1$ is $(y-1)\chi(\varepsilon = 3)$. Therefore, the total contribution in this case is

$$4y\chi(\varepsilon = 3). \tag{14}$$

Case II. Suppose $\alpha(\binom{n}{2k}) = 0$, *i.e.*, $k = 2^a + 2^b - 1$ with $a \leq 2j+1$ and $n_{b+1} = 1$. Due to the property of products between $\{2,6\}$ and $\{1,3,5,7\}$ in $\mathbb{Z}_8$, we shall consider $\{1,5\}$ a class and $\{3,7\}$ the other. Also notice that $\binom{n}{2k} \equiv_8 1$ or 5 if $W_{37} \equiv_2 0$; otherwise $\binom{n}{2k} \equiv_8 3$ or 7. Actually, we shall count A_2 and A_6 under modulo 4 not 8, and then multiply them by 2 and 6 respectively under modulo 8. Let us deal with the following two subcases:

(a) Suppose $a+1 \leq b \leq 2j$. This subcase is for $n_{b+1} = 1$ belonging to the subsequence C, so we must have $b \leq 2j$ and then $j \geq 1$ and $a \leq 2j-1$. By referring to (13) and a little adjustment (for example, removing any test for $a \geq 2j$), we find

$$zr(n-2k) \;=\; zr(n) + \chi(\delta = 1, j \geq 1, 1 \leq a \leq 2j-1) + \chi(j \geq 1, b \leq 2j-1).$$

The reader can also derive $zr(n-2k)$ directly. Note that $j \geq 1$ and $a \leq 2j-1$ are always true in this subcase; so they shall be removed from the above formula. Furthermore, we obtain

$$\begin{aligned} W_{37} &\equiv_2 [zr(n) - zr(n-2k)] - zr(2k) + [n_1 - (2k)_1 - (n-2k)_1] \\ &\equiv_2 \chi(\delta = 1, a \geq 1) + \chi(b \leq 2j-1) + \chi(a \geq 1) + 1 \\ &\equiv_2 \chi(\delta = 2, a \geq 1) + \chi(b = 2j). \end{aligned}$$

We complete this subcase by the following table for $\binom{n}{2k}$, A_2 and A_6 (mod 4), and also for $2A_2+6A_6$ (mod 8).

	$j\geq 1$, $\delta=1$	$j\geq 1$, $\delta=2$	$\#\{(a,b)\}$
$b=1,\quad a=0$ *	1	1	1
$2\leq b\leq 2j-1,\quad a=0,$	1	1	$2j-2$
$2\leq b\leq 2j-1,\quad a\geq 1,\quad a+2\leq b$	1	3	$(2j-3)(j-1)$
$2\leq b\leq 2j-1,\quad a\geq 1,\quad a+1=b$ *	1	3	$2j-2$
$b=2j,\quad a=0$	3	3	1
$b=2j,\quad a\geq 1,\quad a+2\leq b$	3	1	$2j-2$
$b=2j,\quad a\geq 1,\quad a+1=b$ *	3	1	1
A_2 * (mod 4)	$2j+2$	$2j$	
A_6 (mod 4)	$j+2$	$3j$	
$2A_2+6A_6$ (mod 8)	$2j$	$6j$	

Notice that $(2j-3)(j-1)=2(j-2)(j-1)+j-1\equiv_4 j+3$. We conclude the contribution of this subcase as a unique formula

$$(2+4\chi(\delta=2))j\chi(j\geq 1)\equiv_8 2j+4j\chi(\delta=2). \tag{15}$$

The value of this formula is 0 when $j=0$; so the formula works without assuming $j\geq 0$.

(b) Suppose $b\geq 2j+1$. This subcase is for $n_{b+1}=1$ belonging to the subsequence Y. Because $n_{2j+2}=0$ and we must have $n_{b+1}=1$, we shall suppose $b\geq 2j+2$. Again, by referring to (13) and a little adjustment, we obtain

$$\begin{aligned} zr(n-2k) &= zr(n)+\chi(\delta=1)\chi(1\leq a\leq 2j)\chi(j\geq 1)-\chi(\delta=2)\chi(a=2j+1) \\ &\quad +\chi(\langle n_{b+2}n_{b+1}n_b\rangle=\langle 111\rangle)-\chi(\langle n_{b+2}n_{b+1}n_b\rangle=\langle 010\rangle). \end{aligned}$$

and then

$$\begin{aligned} W_{37} \equiv_2 &\ \chi(\delta=1,j\geq 1,1\leq a\leq 2j)+\chi(\delta=2,a=2j+1) \\ &+\chi(\langle n_{b+2}n_{b+1}n_b\rangle=\langle 111\rangle \text{ or } \langle 010\rangle)+\chi(a\geq 1)+1. \end{aligned}$$

We shall first deal with $2A_2$. In this subcase, there is only one possible situation to have contribution to $2A_2$, namely the situation with $\varepsilon=3$ and $a+1=b=2j+2$. In addition if $i \equiv_2 0$ then we have $\langle n_{2n+4} n_{2n+3} n_{2n+2} \rangle = \langle 010 \rangle$, and then $w_{37} \equiv_2 \chi(\delta = 2) + 1$ and $\binom{n}{2k} \equiv_4 1 + 2\chi(\delta = 1)$. Similarly, if $i \equiv_2 1$ then $\binom{n}{2k} \equiv_4 1 + 2\chi(\delta = 2)$. Thus, the contribution to $2A_2$ is

$$\begin{aligned} & 2\chi(\varepsilon = 3)[\chi(i \equiv_2 0)(1 + 2\chi(\delta = 1)) + \chi(i \equiv_2 1)(1 + 2\chi(\delta = 2))] \\ \equiv_4 \ & 2\chi(\varepsilon = 3)[1 + 2\chi(i \equiv_2 0)\chi(\delta = 1) + 2\chi(i \equiv_2 1)\chi(\delta = 2)]. \end{aligned} \tag{16}$$

The required condition $\varepsilon = 3$ has already been implanted in this formula.

As for $6A_6$, we analyze W_{37} further. Note that the choices of a and b are independent for $a \leq 2j + 1$. This fact is also revealed in W_{37}. Let us bisect W_{37} into $U(a) := \chi(\delta = 1, j \geq 1, 1 \leq a \leq 2j) + \chi(\delta = 2, a = 2j + 1) + \chi(a \geq 1) + 1$ and $V(b) := \chi(\langle n_{b+2} n_{b+1} n_b \rangle = \langle 111 \rangle$ or $\langle 010 \rangle)$. Because $\binom{n}{2k} \equiv_4 1$ or 3 depends on $W_{37} \equiv_2 0$ or 1. Explicitly, we have

$$\binom{n}{2k} \equiv_4 2 - (-1)^{W_{37}} = 2 - (-1)^{U(a)}(-1)^{V(a)}.$$

Now we assume that every $\binom{n}{2k}$ contributes to $6A_6$ even if the only situation with contribution to $2A_2$ might happen. With this assumption the total contribution to A_6 (not $6A_6$) is

$$\begin{aligned} \sum_{a=0}^{2j+1} \sum_{\substack{b \geq 2j+2 \\ n_{b+1}=1}} (2 - (-1)^{W_{37}}) &= 2(2j+2)y - \sum_{a=0}^{2j+1} (-1)^{U(a)} \sum_{b \geq 2j+2} \chi(n_{b+1} = 1)(-1)^{V(a)} \\ &\equiv_4 - \sum_{a=0}^{2j+1} (-1)^{U(a)} \sum_{b \geq 2j+2} \chi(n_{b+1} = 1)(-1)^{V(a)}. \end{aligned} \tag{17}$$

Notice that n_{2j+2} is always 0; this is why in the above counting Y and y are involved, even though n_{2j+2} is not. Let us deal with the two summations in (17) under modulo 4. The first one can be easily evaluate by a table of $(-1)^{U(a)}$.

	$j = 0$, $\delta = 1$	$j = 0$, $\delta = 2$	$j \geq 1$, $\delta = 1$	$j \geq 1$, $\delta = 2$	$\#\{a\}$
$a = 0$	-1	-1	-1	-1	1
$1 \leq a \leq 2j$			-1	1	$2j$
$a = 2j + 1$	1	-1	1	-1	1
$\sum_{a=0}^{2j+1} (-1)^{U(a)}$	0	2	$2j$	$2j + 2$	

We conclude that

$$\sum_{a=0}^{2j+1} (-1)^{U(a)} \equiv_4 2(j + \chi(\delta = 2)), \tag{18}$$

whether $j \geq 1$ or not. The second summation is evaluated as follows. Notice that the fact $Y_0 = 0$ will be used several times in the following calculations.

$$
\begin{aligned}
& \sum_{b \geq 2j+2} \chi(n_{b+1} = 1)(-1)^{V(a)} \\
= \quad & y - 2\sum_{l \geq 1} \chi(\langle Y_{l+1}Y_lY_{l-1}\rangle = \langle 111\rangle \text{ or } \langle 010\rangle) \\
= \quad & y - 2\sum_{l \geq 1} [\chi(Y_l = 1) - \chi(\langle Y_{l+1}Y_lY_{l-1}\rangle = \langle 011\rangle \text{ or } \langle 110\rangle)] \\
= \quad & y - 2\sum_{l \geq 1} [\chi(Y_l = 1) - \chi(\langle Y_{l+1}Y_l\rangle = \langle 01\rangle) - \chi(\langle Y_lY_{l-1}\rangle = \langle 10\rangle) \\
& \quad + 2\chi(\langle Y_{l+1}Y_lY_{l-1}\rangle = \langle 010\rangle)] \\
= \quad & y - 2[y - 2r(Y) + 2r_1(Y)] \\
\equiv_4 \quad & -y. && (19)
\end{aligned}
$$

By (17), (18) and (19), the contribution to A_6 is $2y(j + \chi(\delta = 2)) \pmod 4$ if we assume that every $\binom{n}{2k}$ contributes to A_6, and then the contribution to $6A_6$ is $4y(j + \chi(\delta = 2)) \pmod 8$.

Now refer to (16) and return the extra value caused by the false assumption in case that $\varepsilon = 3$ and $a + 1 = b = 2j + 2$. Therefore, the real contribution to $2A_2 + 6A_6$ in subcase (b) is

$$
\begin{aligned}
& 4y(j + \chi(\delta = 2)) \\
& \quad -(6-2)\chi(\varepsilon = 3)[1 + 2\chi(i \equiv_2 0)\chi(\delta = 1) + 2\chi(i \equiv_2 1)\chi(\delta = 2)] \\
\equiv_8 \quad & 4y(j + \chi(\delta = 2)) + 4\chi(\varepsilon = 3). && (20)
\end{aligned}
$$

Finally, sum up (14), (15) and (20) to finish the proof. ■

With the values obtained from Lemmas 5.1, 5.2 and 5.4, our second main result is obtained as follows.

Theorem 5.5 *The n-th Motzkin number M_n is even if and only if $n = (4i + \varepsilon)4^{j+1} - \delta$ for $i, j \in \mathbb{N}$, $\varepsilon = 1, 3$ and $\delta = 1, 2$. Moreover, we have*

$$
M_n \equiv_8 \begin{cases} 4 & \text{if } (\varepsilon, \delta) = (1,1) \text{ or } (3,2); \\ 4y + 2 & \text{if } (\varepsilon, \delta) = (1,2) \text{ or } (3,1), \end{cases}
$$

where y is the number of digit 1's in $[4i + \varepsilon - 1]_2$.

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