2) Riemannian geometry. $M$ is an $n$-dimensional manifold. At some point $0$ on $M$ we introduce local coordinates $(x^1, \ldots, x^n)$. $\partial_i = \partial / \partial x_i$, $i = 1, \ldots, n$ is a basis for $TM$ at $0$. $TM$ is an $n$-dimensional vector space with dualspace $T^*M$ at $0$. We denote the basis of $T^*M$ which is dual to $(\partial_1, \ldots, \partial_n)$ by $dx^1, \ldots, dx^n$, i.e. $dx^i$ is the linear functional on $TM$ given by $dx^i(\partial_j) = \delta^i_j$. Given a positive definite symmetric matrix $g_{ij}(x)$, the quadratic form

$$g_{ij} dx^i dx^j = \sum_{ij} g_{ij} dx^i dx^j$$

defines an inner product on $TM$:

$$\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = \sum_{k\ell} g_{ik} \frac{dx^k}{\partial x^i} \frac{dx^\ell}{\partial x^j} = \delta_{ij}$$
The metric $g$ induces an isomorphism $i : \mathcal{T} \to \mathcal{T}^*$

$$i(x)(y) = \langle x, y \rangle.$$ 

If $TM$ has the basis $\partial/\partial x^1, \ldots, \partial/\partial x^n$ and $dx^1, \ldots, dx^n$ is the basis for $T^*M$ then $i : \mathcal{T} \to \mathcal{T}^*$ is represented by the matrix $(g_{ij})$

i.e., $i\left( \sum \xi^i \partial/\partial x^i \right) = \sum \nu^i dx^i$ with $\nu^i = (g_{ij}) \xi^j$.

$g_{ij}$ are the components of the covariant tensor.

In coordinates $x^i$,

$$g_{ij} = g e^{x^k} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}.$$ 

Thus we only need to know the tensor at the point $0$ and the first derivatives of the variable change at $0$. Differentiat
\[ \frac{\partial g^i_{jk}}{\partial x^l} = \frac{\partial g^i_{k}}{\partial x^l} - \frac{\partial g^i_{k}}{\partial x^j} \frac{\partial x^j}{\partial x^l} + \frac{\partial g^i_{k}}{\partial x^j} \frac{\partial x^j}{\partial x^l} \]

and the derivative does not change as tensors do so it is not a tensor. To arrange for a "derivative" which takes tensors to tensors one defines

\[ \Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right). \]
Let \( \gamma \) denote the vector field

\[
\gamma = \sum_{i=1}^{n} \lambda^i \frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \lambda^j \frac{\partial}{\partial x^j},
\]

and set

\[
\lambda^i \frac{\partial}{\partial x^i} = \frac{\partial \lambda^i}{\partial x^i} + \sum_{j=1}^{n} \lambda^j \Gamma^i_{jk} \frac{\partial}{\partial x^j},
\]

\[
\lambda^i \frac{\partial}{\partial x^j} = \frac{\partial \lambda^i}{\partial x^j} + \sum_{k=1}^{n} \lambda^k \Gamma^i_{jk} \frac{\partial}{\partial x^k}.
\]

Then

\[
\lambda^{i^*} = \sum_{i,j=1}^{n} \lambda^i \frac{\partial x^i}{\partial x^{i^*}} \frac{\partial x^{i^*}}{\partial x^j} \frac{\partial x^j}{\partial x^i}.
\]

Thus \( \lambda^i \frac{\partial}{\partial x^i} \) change as components of a (1,1)-tensor. \( \xi^i \) is the covariant derivative and \( \Gamma^i_{jk} \) is an (Levi-Civita) affine connection. Under coordinate changes \( \Gamma^{i^*}_{j^*k^*} \).
The change as follows:

\[ R^{i'}_{\alpha eta} = \sum_{j \neq k} R^{i'}_{ij} \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial x^k} + \sum_{\mu=1}^{n} \frac{\partial^2 x^\mu}{\partial x^i \partial x^j} \frac{\partial x^i}{\partial x^j} \]

To retain the tensor character of a derivativ, we replaced the 2nd derivaties by \( \Gamma^{i'}_{jk} \)'s. \( \Gamma^{i'}_{jk} \)'s are the Christoffel symbols and they are not tensors. As above the derivative \( \partial / \partial x^n \) is replaced by \( \nabla / \partial x^n \), where

\[ \nabla / \partial x^n \sum_{j=1}^{n} \lambda^j \frac{\partial}{\partial x^j} = \sum_{j=1}^{n} \left( \frac{\partial x^j}{\partial x^k} + \sum_{\epsilon=1}^{n} \Gamma^{j}_{k\ell} \lambda^\ell \right) \frac{\partial}{\partial x^j} \]

\[ = \sum_{j=1}^{n} \lambda^j \frac{\partial}{\partial x^j}, \]

and \( \Gamma^{j}_{k\ell} = \Gamma^{j}_{\ell k} \).
\[ \nabla_{\partial_i \partial_j} \frac{\partial}{\partial x^k} = \sum_{k} \Gamma^k_{ij} \frac{\partial}{\partial x^k}. \]

So \( \nabla \) maps \( \Gamma M \to \Gamma M \) linearly; \( \overline{X}, \overline{Y} \in TM \),
\[ \overline{X} = \sum_i x^i \partial_i, \quad \overline{Y} = \sum_j y^j \partial_j, \]
\[ \nabla_{\overline{X}} \overline{Y} = \sum_k \left( \sum_i x^i y^j \Gamma^k_{ij} \right) \partial_k. \]

**Defn.** (Abstract) Given \( \overline{X} \in \Gamma M \), \( \nabla \) : \( \Gamma^r(\Gamma M) \to \Gamma^s(\Gamma M) \)
do that

(i) \( \nabla_{\overline{X}} \overline{Y} \) is linear in \( \overline{X} \) and \( \overline{Y} \),

(ii) \( \nabla_{f \overline{X}} \overline{Y} = f \nabla_{\overline{X}} \overline{Y} \) (tensor in \( \overline{X} \))

(iii) \( \nabla_{\overline{X}} (f \overline{Y}) = f \nabla_{\overline{X}} \overline{Y} + (\overline{X} f) \overline{Y}. \)

**Thm.** \( \) unique cov. deriv. on a Riemannian

\( \nabla \) \( \) manifold s.t.
\( \nabla^2 \mathbf{X} - \nabla_\mathbf{Y} \mathbf{X} = [\mathbf{X}, \mathbf{Y}] \quad \text{(torsion free)} \)

(3) \( \mathbf{X} \left< \mathbf{Y}, \mathbf{Z} \right> = \left< \nabla_\mathbf{X} \mathbf{Y}, \mathbf{Z} \right> + \left< \mathbf{Y}, \nabla_\mathbf{X} \mathbf{Z} \right> \),

(metric compatible).

Note that

\( \left[ \mathbf{X}, \mathbf{Y} \right] = \mathbf{X} \mathbf{Y} - \mathbf{Y} \mathbf{X} \)

is the Lie derivative \( \mathbf{L}_\mathbf{X} : \Gamma(TM) \rightarrow \Gamma(TM) \).

In lieu of proof note that \( \mathbf{d} \Rightarrow \)

\( \left< \nabla_\mathbf{X} \mathbf{Y}, \mathbf{Z} \right> = \frac{1}{2} \left\{ \mathbf{X} \left< \mathbf{Y}, \mathbf{Z} \right> + \mathbf{Y} \left< \mathbf{X}, \mathbf{Z} \right> - \mathbf{Z} \left< \mathbf{X}, \mathbf{Y} \right> \right\} \)

\( -\left< \mathbf{Y}, [\mathbf{X}, \mathbf{Z}] \right> - \left< \mathbf{Z}, [\mathbf{Y}, \mathbf{X}] \right> + \left< [\mathbf{X}, \mathbf{Y}], \mathbf{Z} \right> \).

This implies that \( \nabla_\mathbf{X} \) if it exists is unique.

On the other hand, defining \( \nabla_\mathbf{X} \mathbf{Y} \) by the
RHS yields a connection which satisfies
\ref{eq:connection} and \ref{eq:tensor}. □

In some local coordinate system let $\mathbf{X}_i = g$
$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle$ and $g^{ij} g_{ik} = \delta_{k}^{i}$. Since
$[\mathbf{X}_i, \mathbf{X}_k] = 0$, using $\nabla_i = \nabla_{g_i} x_i$, the above formula yields
$\langle \nabla_i \mathbf{X}_j, \mathbf{X}_k \rangle = \frac{1}{2} \left\{ \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right\}.$

Setting
$\nabla_i \mathbf{X}_j = \sum_e \Gamma^e_{ij} X_e \quad \Rightarrow \langle \nabla_i \mathbf{X}_j, \mathbf{X}_k \rangle = \sum_e \Gamma^e_{ij} g_{ek}$

$\Rightarrow \Gamma^e_{ij} = \sum_k g^{ek} \langle \nabla_i \mathbf{X}_j, \mathbf{X}_k \rangle$,

and we are back where we started. Note that when the covariant derivatives are
described in terms of an S.H. frame of vector fields, and not in terms of local coordinates, than the $\Gamma_{ij}^k$ may not be symmetric in $ij$.

Next one introduces the Riemann curvature tensor which occurs after 2 covariant deviations. In general, $\tilde{T}^{\ j_1...j_q}_{i_1...i_p}$ is a tensor with $j_1...j_q$ contravariant and $i_1...i_p$ covariant, i.e.

$$\tilde{T}^{\ j_1...j_q}_{i_1...i_p} = \frac{\partial x^{a_1}}{\partial x'_{i_1}} ... \frac{\partial x^{a_p}}{\partial x'_{i_p}} \frac{\partial x^{b_1}}{\partial x'_{j_1}} ... \frac{\partial x^{b_q}}{\partial x'_{j_q}}$$

Then

$$T^{\ j_1...j_q}_{i_1...i_p, k} = \frac{\partial x^{a_1}}{\partial x'_{i_1}} ... \frac{\partial x^{a_p}}{\partial x'_{i_p}} \frac{\partial x^{b_1}}{\partial x'_{j_1}} ... \frac{\partial x^{b_q}}{\partial x'_{j_q}}$$
where
\[ T_{i_1 \ldots i_p}^{j_1 \ldots j_q} = \frac{\partial T_{i_1 \ldots i_p}^{j_1 \ldots j_q}}{\partial x^k} + \sum_{n=1}^{q} T_{i_1 \ldots i_p}^{j_1 \ldots j_{n-1} e i_{n+1} \ldots j_q} \Gamma^{j_n}_{k \ell} \]
\[ - \sum_{s=1}^{p} T_{i_1 \ldots i_s-1 e i_{s+1} \ldots i_p}^{j_1 \ldots j_q} \Gamma^e_{i_s k} \]
and \( T' \) is defined similarly. So the derived tensor has one more covariant index, hence the name. One also has
\[ T_{i_1 \ldots i_p}^{j_1 \ldots j_q} - T_{i_1 \ldots i_p, t, s}^{j_1 \ldots j_q} \]
\[ = \sum_{n=1}^{q} T_{i_1 \ldots i_p}^{j_1 \ldots j_{n-1} a j_{n+1} \ldots j_q} \Gamma^a_{n \ell t} \]
\[ - \sum_{s=1}^{p} T_{i_1 \ldots i_{p-1} a i_{p+1} \ldots i_p}^{j_1 \ldots j_q} \Gamma^a_{i_p s} \]
\[ \Gamma^i_{rst} = \frac{\partial \Gamma^i_{rn}}{\partial t} - \frac{\partial \Gamma^i_{rt}}{\partial s} + \Gamma^a_{rn} \Gamma^i_{at} - \Gamma^a_{rt} \Gamma^i_{as} \]
is a tensor. So is the Riemann–Christoffel tensor

\[ R_{ijke} = g^a i R_{a jke} . \]

Covariant derivatives of \( g_{ij} \) and \( g \) vanish. \( R_{ijke} = 0 \iff \) there is a coordinate system \((x_1, \ldots, x^n)\) in which the metric tensor takes the Euclidean form \( dx^1^2 + \ldots + dx^n^2 \). Of course that is not the case in general. Still, one can always construct a coordinate system in a neighbourhood of any point \( O \), call it the origin, such that

\[ g_{ii}(0) = 1, \quad g_{ij}(0) = 0, \quad i \neq j, \quad \frac{\partial g_{ij}}{\partial x^k}(0) = 0 \, . \]

In particular,

\[ \Gamma^l_{ijk}(0) = 0 \, . \]
Such coordinates are referred to as a geodesic coordinate system. Also,

(i) the first covariant derivative of a tensor is equal to the ordinary derivative at the origin of geodesic coordinates,

(ii) at the origin of geodesic coordinates

$$R_{ijke} = \frac{1}{2} \left( \frac{\partial^2 g_{ie}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ie}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{in}}{\partial x^j \partial x^e} + \frac{\partial^2 g_{in}}{\partial x^j \partial x^e} \right).$$

We shall give a somewhat easier defn. of the action of the covariant derivative on differential forms. \( \mathcal{X} \in \Gamma(TM), \omega \in \Gamma^*M, \) the

\[
(\nabla_\mathcal{X} \omega)(Y) = \mathcal{X}(\omega(Y)) - \omega(\nabla_\mathcal{X} Y),
\]

\[
\nabla_\mathcal{X} (\alpha \wedge \beta) = (\nabla_\mathcal{X} \alpha) \wedge \beta + \alpha \wedge (\nabla_\mathcal{X} \beta).
\]
Note that
\[ \nabla_i (dx^j) = - \sum_l \Gamma^j_{ie} dx^e, \]
so if
\[ \alpha = \sum_{i_1 < \ldots < i_p} \alpha_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p}, \]
then
\[ (\nabla_j \alpha)_{i_1 \ldots i_p} = \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial x_j} - \sum_{l, \nu} \Gamma^l_{ji} \alpha_{i_1 \ldots i_{\nu-1}, l, i_{\nu+1} \ldots i_p} \]
with the convention that \( \alpha \) is extended from \( i_1 < \ldots < i_p \) to all indices by requiring anti-symmetry. Define locally the operator \((\alpha^i)^*\) acting on \(p\)-forms by
\[(a^i)^* \alpha = dx^i \wedge \alpha,\]

and let \(a^i\) denote the adjoint of \((a^i)^*\),

\[a^i \, dx^i \wedge \ldots \wedge dx^k = \sum_{k=1}^{p} (-1)^{k-1} g \, \wedge dx^i \wedge \ldots \wedge dx^k\]

\(\wedge\) deleted. Set \(J = (j_1, \ldots, j_p), K = (k_1, \ldots, k_{p-1}), \Theta\)

\[\langle (a^i)^* dx^j, dx^k \rangle = \langle dx^j, a^i dx^k \rangle\]

follows from expanding the determinant on the left along the first row, see next page. It follows that

\[\{a^i, (a^j)^*\}_g = a^i (a^j)^* + (a^j)^* a^i = g^{ij} \cdot\]

Prop. \(d = \sum_{i=1}^{n} (a^i)^* \bar{V}_i\).

Proof. Clearly,
Note that

\[ \langle dx^i \wedge dx^j \wedge \cdots \wedge dx^k, dt^1 \wedge \cdots \wedge dt^{k+1} \rangle \]

\[ = \det \begin{pmatrix} g^{i_1 k_1} & \cdots & g^{i_{k+1} k_{k+1}} \\ g^{i_1 k_1} & \cdots & g^{i_{k+1} k_{k+1}} \\ \vdots & \ddots & \vdots \\ g^{i_1 k_1} & \cdots & g^{i_{k+1} k_{k+1}} \end{pmatrix} \]

\[ = \cdots \]