Nonlinear Asymptotic Stability of the Lane-Emden Solution for the Viscous Gaseous Star Problem

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Consider a volume of compressible gases expanding into vacuum with self-gravitation:

\[ \bar{n} \Omega(t) \]

\[ \rho > 0 \]

\[ \Gamma(t) = \partial \Omega(t) \]

\[ \rho \equiv 0 \]

This models in the evolution dynamics of viscous gaseous stars which are governed by
A Free boundary problem of compressible Navier-stokes-Poisson equations:

\[ \rho_t + \text{div}(\rho u) = 0, \ \text{in} \ \Omega(t), \]
\[ (\rho u)_t + \text{div}(\rho u \otimes u) + \text{div} \mathcal{S} = -\rho \nabla_x \Psi \ \text{in} \ \Omega(t), \]
\[ \rho > 0 \ \text{in} \ \Omega(t), \]
\[ \rho = 0 \ \text{and} \ \mathcal{S}n = 0 \ \text{on} \ \Gamma(t) := \partial \Omega(t), \]
\[ \mathcal{N}(\Gamma(t)) = u \cdot n, \]
\[ (\rho, u) = (\rho_0, u_0) \ \text{on} \ \Omega := \Omega(0). \]
where \((x, t) \in \mathbb{R}^3 \times [0, \infty),\)

\(\rho:\) density,

\(u:\) velocity,

\(S:\) stress tensor,

\(\Psi:\) gravitational potential;

\(\Omega(t) \subset \mathbb{R}^3:\) the changing volume occupied by a fluid at time \(t\),

\(\Gamma(t) :\) moving interface of fluids and vacuum states,

\(n:\) exterior unit normal vector to \(\Gamma(t)\)

\(\mathcal{V}(\Gamma(t)):\) normal velocity of \(\Gamma(t)\).

The gravitational potential is described by

\[
\Psi(x, t) = -G \int_{\Omega(t)} \frac{\rho(y, t)}{|x - y|} dy \text{ satisfying } \Delta \Psi = 4\pi G \rho \text{ in } \Omega(t)
\]

(0.3)
$G$: the gravitational constant, set $G = 1$; The stress tensor is taken as:

$$G = pl_3 - \lambda_1 \left( \nabla u + \nabla u^t - \frac{2}{3} (\text{div} u) I_3 \right) - \lambda_2 (\text{div} u) I_3,$$

where $I_3$: $3 \times 3$ identical matrix,
$p$: the pressure of the gas,
$\lambda_1 > 0$: the shear viscosity,
$\lambda_2 > 0$: the bulk viscosity,
$\nabla u^t$: the transpose of $\nabla u$.

The equation of state is given by:

$$p = p(\rho) = K \rho^\gamma,$$

where $K > 0$ is a constant (set $K=1$), $\gamma > 1$: the adiabatic exponent.

In most important physical situations for gaseous stars: $\gamma < 2$. 
Stationary Solutions (Lane-Emden Solutions):
For a non-rotating gaseous star, for $\gamma > 4/3$, it is proved by Lieb & H. T. Yau that any equilibrium configuration which minimizes the energy among all possible configurations must be spherically symmetric. A spherically symmetric stationary solution to the above vacuum free boundary problem is called a Lane-Emden solution.

Lane-Emden Solutions:
$(\rho, u)(x) = (\tilde{\rho}(|x|), 0)$ satisfies

$$\partial_r (\tilde{\rho}^\gamma) + 4\pi r^{-2} \tilde{\rho} \int_0^r \tilde{\rho}(s) s^2 ds = 0. \quad (0.4)$$
**Facts:**
The solutions to (0.4) can be characterized by the values of $\gamma$ and its total mass. Indeed, for given finite total mass $M > 0$,

(i) if $\gamma \in (6/5, 2)$, there exists at least one compactly supported solution,

(ii) for $\gamma \in (4/3, 2)$, every solution is compactly supported and unique (which minimizes the energy functional),

(iii) if $\gamma = 6/5$, the unique solution admits an explicit expression, and it has infinite support,

(iv) for $\gamma \in (1, 6/5)$, there are no solutions with finite total mass,
(v) for $\gamma > 6/5$, let $\bar{R}$ be the radius of the stationary star giving by the Lane-Emden solution and
\[ M = 4\pi \int_0^{\bar{R}} \bar{\rho}(s) s^2 ds, \] then, on the vacuum surface $r = \bar{R}$,
\[ -\infty < \nabla_n(c^2(\bar{\rho})) = -\frac{\gamma-1}{\gamma} \frac{M}{\bar{R}^2} < 0, \] so
\begin{equation}
\bar{\rho}^{\gamma-1}(r) \sim \bar{R} - r \text{ as } r \text{ close to } \bar{R}, \tag{0.5}
\end{equation}
i.e., the sound speed is $C^{1/2}$ Hölder continuous across the vacuum boundary.
The Free boundary Problem in The Spherically Symmetric Setting:

The evolving domain $\Omega(t)$ is a ball with the changing radius $R(t)$,

$$\rho(x, t) = \rho(r, t) \quad \text{and} \quad u(x, t) = u(r, t) x/r$$

with $r = |x| \in (0, R(t))$;
system (0.1) can then be rewritten as

\[(r^2 \rho)_t + (r^2 \rho u)_r = 0 \quad \text{in} \quad (0, R(t)),\]

\[\rho(u_t + uu_r) + p_r + 4\pi \rho r^{-2} \int_0^r \rho(s, t)s^2 ds = \mu \left(\frac{(r^2 u)_r}{r^2}\right)_r\]

\[\quad \text{in} \quad (0, R(t)),\]

\[\rho > 0 \quad \text{in} \quad [0, R(t)),\]

\[\rho = 0 \quad \text{and} \quad \frac{4}{3} \lambda_1 \left(u_r - \frac{u}{r}\right) + \lambda_2 \left(u_r + 2\frac{u}{r}\right) = 0 \quad \text{for} \quad r = R(t),\]

\[\dot{R}(t) = u(R(t), t) \quad \text{with} \quad R(0) = R_0, \quad u(0, t) = 0,\]

\[(\rho, u) = (\rho_0, u_0) \quad \text{on} \quad (0, R_0),\]

\[
\mu = \frac{4\lambda_1}{3} + \lambda_2 > 0.
\]
Aim:
1) prove the global- in-time regularity of solutions to the free boundary uniformly up to the vacuum boundary of solutions when \( 4/3 < \gamma < 2 \) (the stable index) capturing the behavior that the sound speed \( c = \sqrt{p'(\rho)} \) is \( C^{1/2} \) Hölder continuous near the vacuum boundary, as long as the initial datum is a suitably small perturbation of the Lane-Emden solution with the same total mass.

2) Establish the large time asymptotic convergence of the global strong solution, in particular, the convergence of the vacuum boundary and the density, to the the Lane-Emden solutions with the detailed convergence rate as the time goes to infinity.
Initial Density:
To capture the interesting behavior mentioned behavior near the vacuum boundary, the initial density is supposed to satisfy:

\[
\rho_0(r) > 0 \quad \text{for} \quad 0 \leq r < R_0, \quad \rho_0(R_0) = 0
\]

and \( -\infty < (\rho_0^{\gamma^{-1}})_r < 0 \) at \( r = R_0 \), \hspace{1cm} (0.7)

\[
\rho_0^{\gamma^{-1}}(r) \sim R_0 - r \quad \text{as} \quad r \quad \text{close to} \quad R_0, \hspace{1cm} (0.8)
\]
Issues in the study of the above vacuum free boundary problem:
The behavior (0.7) near the vacuum boundary causes a high degeneracy in system (0.6), and it is difficult to deal with such a degeneracy even for the local-in-time existence. Indeed, the local-in-time wellposedness of smooth solutions to the vacuum free boundary problems with the physical vacuum was only established recently for compressible inviscid flows:

(a) 1-d local-in-time wellposedness theory:
Jang-Masmoudi (CPAM 2009),
Coutand- Shkoller (CPAM 2011).
(b) 3-d theory:
Coutand-Lindblad-Shkoller (CMP 2010): A priori estimates;
Coutand-Shkoller (ARMA 2012): local-in-time wellposedness theory;
Jang-Masmoudi (Preprint): local-in-time existence theory
(another approach).

(c) Luo-Xin-Zeng (ARMA 2014): New Well-posedness theory
for compressible Euler or Euler-Poisson equations.

For the vaccum free boundary problem (0.6) of the
compressible Navier-Stokes-Poisson equations featuring the
behavior (0.7) near the vacuum boundary, a local-in-time
wellposedness theory of strong solutions was established in
2010 by Jang.
Previous Related Works On the Stability of Gaseous Stars

Physical Theory:
The stability problem has been important in the theory of gaseous stars which has been studied extensively by astrophysists (Chandresekhar, Weinberg and Lebovitz & Lifschitz).

Mathematical Theory:
1) Linear stability of Lane-Emden solutions: S. S. Lin (SIMA1997).
2) A conditional nonlinear Lyapnov type stability theory of stationary solutions for $\gamma > 4/3$: G. Rein (ARMA2003) (using a variational approach) (For rotating stars by Smoller-Luo(ARMA2009)): For compressible Euler-Poisson equations, in the framework of Cauchy problems in the entire $\mathbb{R}^3$-space by assuming the existence of global solutions of the Cauchy problem for the 3-d compressible Euler-Poisson equations. The stability results are of Lyapnov type involving a Lyapnov functional which is essentially equivalent to a $L^p$-norm of difference of solutions. The vacuum boundary cannot be traced in this setting.
3) For $\gamma \in (6/5, 4/3)$, the nonlinear dynamical instability of Lane-Emden solutions was proved in the framework of free boundary problems for Euler-Poisson by Jang (2014) and Navier-Stokes-Poisson equations by Jang & Tice (APDE2013), respectively, and a nonliear instability for $\gamma = \frac{6}{5}$ was proved by Jang (2014).

4) For $\gamma = \frac{4}{3}$, an instability was identified by Deng, Liu, Yang and Yao (ARMA2002) that a small perturbation can cause part of the mass to go off to infinity for inviscid flows.
5) For a vacuum free boundary problem of a modified compressible Navier-Stokes-Poisson equations with spherical symmetry, Fang and Zhang (ARMA2009) proved the existence of the global weak solution for a reduced initial boundary value problem after using the Lagrangian mass coordinates, under some constraints on the ratio of the coefficients of the shear viscosity and bulk viscosity. Due to the lack of regularity near the vacuum boundary, for the global weak solutions, only the uniform convergence of the velocity \( u(r, t) \) is proved. The convergence of the vacuum boundary and the uniform convergence of density are missing.
Key idea in our approach:
1) Establish the global-in-time regularity uniformly up to the vacuum boundary, which ensures the large time asymptotic convergence of the evolving vacuum boundary, density and velocity to those of the Lane-Emden solution with detailed convergence rates, and detailed large time behaviors of solutions near the vacuum boundary. In particular, we show that every spherical surface inside the evolving domain converges to the sphere enclosing the same mass inside the domain of the Lane-Emden solution and the large time asymptotic states for the vacuum free boundary problem (0.6) are determined by the initial mass distribution and total mass.

2) Overcome the degeneracy and singular behavior near the vacuum free boundary and coordinates singularity at the symmetry center.
Main ingredients of the analysis:
Combinations of some new weighted nonlinear functionals (involving both lower order and higher order derivatives) and space-time weighted energy estimates. The constructions of these weighted nonlinear functionals and space-time weights depend crucially on the structures of the Lane-Emden solution, the balance of pressure and gravitation, and the dissipation.

- Choose a proper Lagrangian particle trajectory formulation to transform the original free boundary value problem to an initial boundary value problem on a fixed domain so that the domain of the Lane-Emden solution becomes the reference domain and the initial particle position is given by the same mass distribution (0.15). Let \( r = r(x, t) \) be the mass position at time \( t \) with the same mass of the Lane-Emden solution at \( x \).
• The basic uniform estimates are obtained by finding detailed balance between the pressure and the self-gravitation which are detected by the following two multipliers:

1. weighted moment multiplier

\[ w_\alpha^2 \equiv \int_0^x \tilde{\rho}^{-\alpha}(y)(r^3(y, t) - y^3)_y dy, \quad (0 \leq \alpha < \gamma - 1) \]

2. weight energy multiplier

\[ w_\alpha^1 = \int_0^x \tilde{\rho}^{-\alpha}(y)(r^2(y, t)v(y, t))_y dy, \quad (0 \leq \alpha < \gamma - 1) \]

(here the first multiplier is motivated by the virial equation in the study of stellar dynamics.)
To obtain the uniform higher order estimates up to the vacuum boundary, we introduce the entropy relative to the Lane-Emden solution

$$\mathcal{G}(x, t) = \ln\left(\frac{\bar{\rho}(x)}{\rho(r(x, t), t)}\right)$$

In terms of $\mathcal{G}$, the main governing equation becomes a damping transport equation with a degenerate damping rate which can be studied by bootstrap argument based on new energy multiplies of the form

$$x^2 \bar{\rho}^{\gamma-2} \mathcal{B}_t$$

with

$$\mathcal{B} = \gamma \left(\frac{x^2 \bar{\rho}}{r^2 r_x}\right)^\gamma g_x + \left[\left(\frac{x^2}{r^2 r_x}\right)^\gamma - \left(\frac{x}{r}\right)^4\right] x \varphi \bar{\rho}$$

These weights are crucial for high order estimates.
Lagrangian Formulation and Main Results

More on Lane-Emden solutions:
For $\gamma \in (4/3, 2)$ and any given finite positive total mass, there exists a unique solution to equation (0.4) whose support is compact. Without abusing notations, we use $x$ to denote the distance from the origin for the Lane-Emden solution. Therefore, for any $M \in (0, \infty)$, there exists a unique function $\bar{\rho}(x)$ such that

$$\bar{\rho}_0 := \bar{\rho}(0) > 0, \quad \bar{\rho}(x) > 0 \quad \text{for} \quad x \in (0, \bar{R}),$$

$$\bar{\rho}(\bar{R}) = 0, \quad M = \int_0^{\bar{R}} 4\pi \bar{\rho}(s)s^2 ds; \quad (0.9)$$
\[-\infty < \bar{\rho}_x < 0 \text{ for } x \in (0, \bar{R}) \]
and \[\bar{\rho}(x) \leq \bar{\rho}_0 \text{ for } x \in (0, \bar{R}) ;\] (0.10)

\[(\bar{\rho}^\gamma)_x = -x \phi \bar{\rho},\]
where \[\phi := x^{-3} \int_0^x 4\pi \bar{\rho}(s)s^2 ds \in \left[ M/\bar{R}^3, 4\pi \bar{\rho}_0/3 \right];\] (0.11)

for a certain finite positive constant \(\bar{R}\) (indeed, \(\bar{R}\) is determined by \(M\) and \(\gamma\)). Note that

\[(\bar{\rho}^{\gamma-1})_x = \frac{\gamma - 1}{\gamma} \rho^{\gamma-1} (\bar{\rho}^\gamma)_x = -\frac{\gamma - 1}{\gamma} x\phi.\] (0.12)
It then follows from (0.9) and (0.11) that $\bar{\rho}$ satisfies the physical vacuum condition, i.e.,

$$C^{-1} (\bar{R} - x) \leq \bar{\rho}^{\gamma - 1}(x) \leq C (\bar{R} - x), \quad x \in (0, \bar{R}).$$

(0.13)

For the convenience of presentation, we set $\bar{R} = 1$ and

$$I = (0, \bar{R}) = (0, 1).$$
Lagrangian formulation:
Let $x$ be the reference variable and define the Lagrangian
variable $r(x, t)$ by

$$r_t(x, t) = u(r(x, t), t) \text{ for } t > 0 \text{ and } r(x, 0) = r_0(x), \ x \in I. \quad (0.14)$$

($r(x, t)$ is the position of the particle starting at $r_0(x)$ at $t = 0$.)

Choice of the initial position $r_0(x)$:
Let $r_0 : \overline{I} \to [0, R_0]$ ($\overline{I} = [0, 1]$) be the diffeomorphism defined by

$$\int_{r_0(x)}^x \rho_0(s)s^2 \, ds = \int_{r_0(x)}^x \bar{\rho}(s)s^2 \, ds, \ x \in \overline{I}, \quad (0.15)$$
so that

\[ \rho_0(r_0(x))(r_0(x))^2 r'_0(x) = \bar{\rho}(x)x^2, \ x \in \bar{I}. \tag{0.16} \]

((0.15) means that the initial mass in the ball with the radius \( r_0(x) \) is the same as that of the Lane-Emden solution in the ball with the radius \( x \).)

**Conservation of Mass:**

\[ \int_0^{r(x,t)} \rho(s, t)s^2 ds = \int_0^{r_0(x)} \rho_0(s)s^2 ds, \ x \in I. \tag{0.17} \]
Lagrangian density and velocity:

\[ f(x, t) = \rho(r(x, t), t) \quad \text{and} \quad v(x, t) = u(r(x, t), t). \]

\[ f(x, t) = \frac{x^2 \bar{\rho}(x)}{r^2(x, t)r_x(x, t)}, \quad x \in I. \quad (0.18) \]
System (0.6) on the reference domain $I = (0, 1)$:

$$
\bar{\rho} \left( \frac{x}{r} \right)^2 v_t + \left[ \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right)^\gamma \right]_x + \frac{x^2}{r^4} \bar{\rho} \int_0^x 4\pi \bar{\rho} y^2 \, dy = \mu \left( \frac{(r^2 v)_x}{r^2 r_x} \right)_x
$$

in $I \times (0, T]$,

$v(0, t) = 0$, $\mathfrak{B}(1, t) = 0$ on $(0, T]$,

$(r, v)(x, 0) = (r_0(x), u_0(r_0(x)))$ on $I \times \{t = 0\}$.

(0.19)
where $\mathcal{B}$ is the norm stress at the boundary given by

$$
\mathcal{B} = \frac{4}{3} \lambda_1 \left( \frac{v_x}{r_x} - \frac{v}{r} \right) + \lambda_2 \left( \frac{v_x}{r_x} + 2 \frac{v}{r} \right) = \frac{4}{3} \lambda_1 \frac{r}{r_x} \left( \frac{v}{r} \right)_x + \lambda_2 \frac{(r^2 v)_x}{r_x r^2}.
$$

Equation for $G$

Note that the relative entropy $G$ satisfies

$$
G = \ln r_x + 2 \ln \left( \frac{r}{x} \right),
$$

so that

$$
G_t = \frac{v_x}{r_x} + 2\frac{v}{r} \quad \text{and} \quad G_x = \frac{r_{xx}}{r_x} + 2\frac{x}{r} \left( \frac{r}{x} \right)_x.
$$
The transformation between $G$ and $r$ is one-to-one, we can solve for $r$ in terms of $G$ by

$$r(x, t) = \left(3 \int_0^x y^2 \exp(G(y, t)) dy\right)^{1/3}, \ x \in \overline{I}, \ t \geq 0.$$  

(0.22)

Equation (0.19)$_1$ can also be written in the form of

$$\frac{x^2}{r^2} \bar{\rho} v_t - \gamma \left(\frac{x^2 \bar{\rho}}{r^2 r_x}\right) G_x - \left[\left(\frac{x^2}{r^2 r_x}\right) \gamma - \left(\frac{x}{r}\right)^4\right] x \phi \bar{\rho} = \mu G_{xt}.$$  

(0.23)

which can be regarded as a linear equation for $G_x$ with a degenerate damping.
Functional Space:
Let $\alpha \in [0, \gamma - 1)$ be any fixed constant. Denote

$$
\mathcal{E}(t) = \|(r_x - 1, v_x)(\cdot, t)\|^2_{L^{\infty}} + \|\bar{\rho}^{\gamma - \frac{1}{2}} G_x(\cdot, t)\|^2 \\
+ \|\bar{\rho}^{-1/2} G_{xt}(\cdot, t)\|^2,
$$

$$
\mathcal{F}(t) = \mathcal{E}(t) + \|\bar{\rho}^{\gamma - 1 - \alpha/2} G_x(\cdot, t)\|^2.
$$

Notation:

$$
\| \cdot \| = \| \cdot \|_{L^2(I)}, \int = \int_I.
$$

Remark The functional $\mathcal{E}(t)$ contains the $L^2$-norms of all terms in equation (0.23). So the finiteness of $\mathcal{E}(t)$ for all $t \geq 0$ ensures the global existence of a strong solution.
Definition of strong solutions:
\( v \in C([0, T]; H^2_{loc}[0, 1]) \cap C([0, T]; W^{1,\infty}[0, 1]) \) with

\[
 r(x, t) = r_0(x) + \int_0^t v(x, s)ds, \quad x \in [0, 1], \ t \in [0, T] \quad (0.24)
\]

satisfying the initial condition (0.19)\(_3\) is called a strong solution of problem (0.19) in \([0, T]\) if the following hold:

1. \( r_x(x, t) > 0 \) for \((x, t) \in [0, 1] \times [0, T]\);
2. \( \bar{\rho}^{-\frac{1}{2}} (v_{xx}, (\frac{v}{x})_x) \in C([0, T]; L^2([0, 1])), \)
   \( \bar{\rho}^{-\frac{1}{2}} (\frac{r^2 v}{x})_x \in C([0, T]; H^1[0, 1]), \mathcal{B} \in C([0, T]; L^2([\delta, 1])) \) for any \( \delta > 0 \) with \( \mathcal{B} \) defined in (0.20);
3. \( \bar{\rho}^{\frac{1}{2}} v \in C^1([0, T]; L^2[0, 1]) \);
4. \( v(0, t) = 0 \) and \( \mathcal{B}(1, t) = 0 \) hold in the sense of \( W^{1,\infty} \) trace and \( H^1 \) trace, respectively for \( t \in [0, T] \);
5. (0.19)\(_1\) holds for \( t \in [0, T] \) and \( x \in [0, 1], \) a.e..
Theorem of the global existence of the strong solution

**Theorem 0.1**

Let $\gamma \in (4/3, 2)$ and $\bar{\rho}$ be the Lane-Emden solution satisfying (0.9)-(0.11). Assume that the initial density $\bar{\rho}_0$ satisfies (0.7). There exists a constant $\bar{\delta} > 0$ such that if

$$E(0) \leq \bar{\delta},$$

then problem (0.19) admits a unique strong solution in $I \times [0, \infty)$ with

$$E(t) \leq C E(0)$$

for $t \geq 0$ and some constant $C$. 

(0.25)
Nonlinear Asymptotic Stability Theorem
For any \( t \geq 0 \), since \( r_x(x, t) > 0 \) for \( x \in \bar{I} \), \( r(x, t) \) defines a diffeomorphism from the \( \bar{I} \) to the domain \( \{0 \leq r \leq R(t)\} \) with the boundary

\[
R(t) = r(1, t). \tag{0.26}
\]

It also induces a diffeomorphism from the initial domain \( \bar{B}_{R_0}(0) =: \{x \in \mathbb{R}^3 : |x| \leq R_0\} \) to the evolving domain \( \bar{B}_{R(t)}(0) =: \{x \in \mathbb{R}^3 : |x| \leq R(t)\} \) for all \( t \geq 0 \):

\[
x \neq 0 \in \bar{B}_{R_0}(0) \rightarrow r \left( r_0^{-1}(|x|), t \right) \frac{x}{|x|} \in \bar{B}_{R(t)}(0),
\]
where \( r_0^{-1} \) is the inverse map of \( r_0 \) defined in (0.15). Denote the inverse of the map \( r(x, t) \) by \( R_t \) for \( t \geq 0 \) so that

\[
\text{if } r = r(x, t) \text{ for } 0 \leq r \leq R(t), \text{ then } x = R_t(r). \quad (0.27)
\]

Now for the strong solution \((r, \nu)\) given in Theorem 34, one sets for \( 0 \leq r \leq R(t) \) and \( t \geq 0 \),

\[
\left\{ \begin{array}{c}
\rho(r, t) = \frac{x^2 \bar{\rho}(x)}{r^2(x,t) r_x(x,t)} \text{ for } x = R_t(r), \\
u(r, t) = \nu(R_t(r), t).
\end{array} \right. \quad (0.28)
\]

Then the triple \((\rho, u, R(t)) \) \( (t \geq 0) \) defines a global strong solution to the free boundary problem (0.6).
Theorem 0.2
Under the assumptions in Theorem 34. Then the triple 
\((\rho, u, R(t))\) for \(t \geq 0\) defined by (0.26) and (0.28) is the 
unique global strong solution to the free boundary problem 
(0.1) satisfying \(R \in W^{1,\infty}[0, +\infty)\). Moreover, the solution 
satisfies the following decay estimates:

(i) for any \(0 < \theta < \min\{\frac{2(\gamma-1)}{3\gamma}, \frac{4-2\gamma}{\gamma}\}\), there exists a positive 
constant \(C(\theta)\) independent of \(t\) such that, for \(t \in [0, \infty)\),

\[
\sup_{x \in I} |r(x, t) - x| \leq C(\theta)(1 + t)^{-\frac{\gamma-1}{\gamma} + \frac{\theta}{2}} \sqrt{\mathcal{E}(0)}, \quad (0.29)
\]

\[
\sup_{0 \leq r \leq R(t)} |u(r, t)| \leq C(\theta)(1 + t)^{-\frac{1}{4}\left(\frac{2\gamma-1}{\gamma} - \theta\right)} \sqrt{\mathcal{E}(0)}, \quad (0.30)
\]
Theorem 0.2 (continued)

\[
\sup_{0 \leq x \leq \bar{R}} |\rho(r(x, t), t) - \bar{\rho}(x)| \\
\leq C(\theta)(1 + t)^{-\frac{2-\gamma}{4\gamma} + \frac{\theta}{8} \frac{\gamma+1-\gamma\theta}{2\gamma-1-\frac{\gamma\theta}{2}}} \sqrt{\mathcal{E}(0)}; \quad (0.31)
\]

(ii) Suppose that \( \mathcal{F}(0) < \infty \) for some \( \alpha \in [0, \gamma - 1) \). Set \( \kappa = 0 \) when \( \alpha = 0 \) and \( \kappa = \alpha/\gamma - \theta \) when \( \alpha > 0 \). Then for any \( 0 < \theta < \min\{2(\gamma - 1)/(3\gamma), 2(\gamma - 1 - \alpha)/\gamma, (4 - 2\gamma)/\gamma\} \), there exists a positive constant \( C(\theta) \) such that the solution obtained in i) satisfies the following decay estimates:
Theorem 0.2 (continued)

\[
\sup_{0 \leq r \leq R(t)} |u(r, t)| \leq C(\theta)(1 + t)^{-\frac{3}{8}} \left( \frac{2\gamma - 1}{\gamma} - \theta \kappa \right) \sqrt{[\mathcal{F}(0) + \mathcal{F}^2(0)]},
\]

(0.32)

\[
\sup_{0 \leq x \leq \bar{R}} \left| \bar{\rho}^{\frac{\gamma}{2} - 1}(x)(\rho(r(x, t), t) - \bar{\rho}(x)) \right|
\leq C(\theta)(1 + t)^{-\min\left\{ \frac{1}{4}(\frac{2\gamma - 1}{\gamma} - \theta + \kappa), \frac{3}{16}(\frac{2\gamma - 1}{\gamma} - \theta + 5\kappa) \right\}} \sqrt{[\mathcal{F}(0) + \mathcal{F}^2(0)]}.
\]

(0.33)
Theorem 0.2 (continued)

Furthermore, if \( \| x \rho - \frac{1}{2} G_{x t t}(\cdot, 0) \| + |v_t(1, 0)| < \infty \), then \( R \in W^{2, \infty}[0, +\infty) \) and

\[
|\dddot{R}(t)| \leq |v_t(1, 0)| + C(\mathcal{E}(0))^{1/4} \left( (\mathcal{F}(0))^{1/4} + \| x \rho - \frac{1}{2} G_{x t t} \|^{1/2}(\cdot, 0) \right),
\]

\[ t \geq 0. \] (0.34)
Some remarks

Remark 0.3
The estimate in (0.33) yields the uniform convergence with rates of the density to (0.1) to that of the Lane-Emden solution for both large time and near the vacuum boundary since $\gamma < 2$.

Remark 0.4
The initial perturbation here includes three parts: the deviation of the initial domain from that of the Lane-Emden solution, the difference of initial density from that of the Lane-Emden solution, and the velocity. Since the Lane-Emden solution is completely determined by the total mass $M$, our nonlinear asymptotic stability result shows that the time asymptotic state of the free boundary problem is determined by the total mass which is conserved in the time evolution and the initial mass distribution.
Remark 0.5
We make comments on the finiteness of functionals $E$ and $F$ at $t = 0$ in terms of the initial data $\rho_0(r)$ and $u_0(r)$. Note that

$$G(x, 0) = \ln \left( \frac{\bar{\rho}(x)}{\rho_0(r_0(x))} \right), \quad x \in I,$$

(0.35)

and near the vacuum boundary the Lane Emden solution $\bar{\rho}$ behaves as

$$\bar{\rho}(x) \sim (1 - x)^{\frac{1}{\gamma-1}}, \quad \bar{\rho}'(x) \sim (1 - x)^{\frac{2-\gamma}{\gamma-1}}, \quad \text{as } x \to 1^-.$$

If the initial density $\rho_0$ obeys the same behavior near the vacuum boundary as the Lane-Emden solution, then
Remark 0.5 (continued)

\[ |\bar{\rho}^{\gamma - \frac{1}{2}} G_x(x, 0)| \leq C(1 - x)^{\frac{1}{2(\gamma - 1)}} \text{ and} \]
\[ |\bar{\rho}^{\gamma - 1 - \frac{\alpha}{2}} G_x(x, 0)| \leq C(1 - x)^{\frac{-\alpha}{2(\gamma - 1)}} \text{ for } x \in I. \]

So \( \|\bar{\rho}^{\gamma - \frac{1}{2}} G_x(x, 0)\| < \infty \) and \( \|\bar{\rho}^{\gamma - 1 - \frac{\alpha}{2}} G_x(\cdot, 0)\|^2 \) in \( \mathcal{G}(0) \) is finite for \( \alpha \in [0, \gamma - 1) \). The finiteness of \( \|\bar{\rho}^{-1/2} G_{xt}(\cdot, 0)\| \) is a requirement on the initial velocity \( v(x, 0) \) (and thus \( u_0 \)).
Remark 0.6
The condition $\|x\tilde{\rho}^{-\frac{1}{2}}G_{xtt}(\cdot, 0)\| + |v_t(1, 0)| < \infty$ in ii) in the above theorem to ensure $R \in W^{2,\infty}[0, +\infty)$ (uniform boundedness of the acceleration of the vacuum boundary) is a higher order compatibility condition of the initial data with the vacuum boundary. Indeed, one may check from the proof that every particle moving with the fluid has the bounded acceleration for $t \in [0, \infty)$ if it does so initially.

Remark 0.7
The results obtained in this paper are among few results of the global strong solutions to vacuum free boundary problems of compressible fluids capturing the singular behavior of (0.7).
Main Ideas and Steps of the proof
Indeed, we have the following more detailed decay estimates, which are necessary for the nonlinear asymptotic stability:

**Theorem 0.8**

Let \((r, \upsilon)\) be the global strong solution to problem (0.19) as stated in Theorem 34. Then it holds that:

(i) for any \(0 < \theta < \min\{\frac{2(\gamma - 1)}{3\gamma}, \frac{4-2\gamma}{\gamma}\}\) and \(\delta \in (0, 1)\), there exist positive constants \(C(\theta)\) and \(C(\theta, \delta)\) independent of \(t\) such that

\[
\| \tilde{\rho}^{\frac{\gamma \theta}{4} - \frac{\gamma - 1}{2}} \left( r - x, x r_x - x \right)(\cdot, t) \|^2 \leq C(\theta) \mathcal{E}(0),
\]

(0.36)
Theorem 0.8 (continued)

\[
(1 + t)^{\frac{2(\gamma - 1)}{\gamma} - \theta} \| (r - x)(\cdot, t) \|^2_{L_\infty} \\
+ (1 + t)^{\frac{1}{2}} \left( \frac{2\gamma - 1}{\gamma} - \theta \right) \| (v, xv_x)(\cdot, t) \|^2_{L_\infty} \\
+ (1 + t)^{\frac{2 - \gamma}{2\gamma} - \frac{\theta}{4} \left( \frac{\gamma + 1 - \gamma \theta}{2} \right)} \| \bar{\rho} G(\cdot, t) \|^2_{L_\infty} \\
+ (1 + t)^{\frac{\gamma - 1}{\gamma} - \theta} \left\| \left( r_x - 1, \frac{r}{x} - 1, \bar{\rho}^\frac{1}{2} v_t \right)(\cdot, t) \right\|^2 \\
+ (1 + t)^{\frac{2\gamma - 1}{\gamma} - \theta} \left( \left\| \left( x\bar{\rho}^\frac{1}{2} v_t, v, xv_x \right)(\cdot, t) \right\|^2 + \\
\left\| \bar{\rho}^\frac{\gamma}{2} (r - x, xr_x - x)(\cdot, t) \right\|^2 \right) \leq C(\theta) \mathcal{E}(0), \tag{0.37}
\]
Theorem 0.8 (continued)

\[(1 + t)^{\frac{\gamma-1}{\gamma} - \theta} \left( \left\| \left( r_x - 1, \frac{r}{x} - 1, v_x, \frac{v}{x} \right) \right\|_2^{2} \right)_{H^1([0,1-\delta])} \\
+ \left\| (G_x, G_{xt}) (\cdot, t) \right\|^{2}_{L^2([0,1-\delta])} \right) \\
\leq C(\theta, \delta) \mathcal{E}(0), \quad (0.38)
\]

for \(0 \leq t < \infty\).
Theorem 0.8 (continued)

(ii) Under the same assumptions as in (ii) of Theorem 0.2, and \( \delta \in (0, 1) \), there exist positive constants \( C(\theta) \) and \( C(\theta, \delta) \) such that the solution obtained in i) satisfies the following estimates:
\( \mathcal{G}(t) + (1 + t)^\kappa \left\| \bar{\rho}^{\gamma-1} G_x(\cdot, t) \right\|^2 \)
\( + (1 + t)^{\min\left\{ \frac{1}{2} \left( \frac{2\gamma-1}{\gamma} - \theta + \kappa \right), \frac{3}{8} \left( \frac{2\gamma-1}{\gamma} - \theta + 5\kappa \right) \right\}} \left\| \bar{\rho}^{\frac{\gamma}{2}} G(\cdot, t) \right\|^2_{L^\infty} \)
\( + (1 + t)^{\frac{1}{2} \left( \frac{2\gamma-1}{\gamma} - \theta + 3\kappa \right)} \left\| \left( v_x, \frac{\nu}{\chi}, x\bar{\rho}^{\frac{3}{2}\gamma-1} G_x, x\bar{\rho}^{\frac{\gamma}{2}-1} G_{xt}, \bar{\rho}^{\frac{1}{2}} v_t \right) (\cdot, t) \right\|^2 \)
\( + (1 + t)^{\frac{3}{8} \left( \frac{2\gamma-1}{\gamma} - \theta + 5\kappa \right)} \left\| \left( v_x, \frac{\nu}{\chi} \right) (\cdot, t) \right\|^2_{L^\infty} \)
\( + (1 + t)^{\frac{3}{4} \left( \frac{2\gamma-1}{\gamma} - \theta + \kappa \right)} \left\| (xv_x, v)(\cdot, t) \right\|^2_{L^\infty} \)
\leq C(\theta) \left[ \mathcal{G}(0) + \mathcal{G}^2(0) \right], \hspace{1cm} (0.39)
Theorem 0.8 (continued)

\[
(1 + t)^{\frac{1}{2}}\left(\frac{2\gamma - 1}{\gamma} - \theta + 3\kappa\right) \left\| \left( r_{\cdot} - 1, \frac{r}{x} - 1 \right) (\cdot, t) \right\|^2_{L^2([0,1-\delta])} \\
+ (1 + t)^{\frac{1}{4}}\left(\frac{2\gamma - 1}{\gamma} - \theta + 9\kappa\right) \left[ \left\| \left( r_{\cdot}, \frac{r}{x}, \nu_{\cdot}, \frac{\nu}{x} \right) (\cdot, t) \right\|^2_{L^2([0,1-\delta])} \\
+ \left\| (G_{x}, G_{xt}) (\cdot, t) \right\|^2_{L^2([0,1-\delta])} \right] \\
+ (1 + t)^{\frac{3}{8}}\left(\frac{2\gamma - 1}{\gamma} - \theta + 5\kappa\right) \left\| \left( r_{\cdot} - 1, \frac{r}{x} - 1 \right) (\cdot, t) \right\|^2_{L^\infty([0,1-\delta])} \\
\leq C(\theta, \delta) \left[ \mathcal{F}(0) + \mathcal{F}^2(0) \right],
\] 

(0.40)
Theorem 0.8 (continued)

for \(0 \leq t < \infty\). Moreover, if \(\|x\tilde{\rho}^{-\frac{1}{2}}G_{xtt}\|(\cdot, 0) < \infty\), then

\[
\int x^2 \tilde{\rho} v_{tt}^2(x, t)dx + \int_0^\infty \int (v_{tt}^2 + x^2 v_{xtt}^2)dxds \\
\leq C(\mathcal{F}(0) + \|x\tilde{\rho}^{-\frac{1}{2}}G_{xtt}\|^2(\cdot, 0); \ t \geq 0; \quad (0.41)
\]
Theorem 0.8 (continued)

(iii) Furthermore, if we assume $\|G_x(\cdot, 0)\|^2$ is finite, then for any $0 < \theta < \min\{2(\gamma - 1)/(3\gamma), (4 - 2\gamma)/\gamma\}$, there exist positive constants $c$ independent of $\theta$ and $t$ and $C(\theta)$ which is independent of $t$ such that

$$
\|G_x(\cdot, t)\|^2 + \|(r_x, r/x)_x(\cdot, t)\|^2 \\
\leq c \|G_x(\cdot, 0)\|^2 + C(\theta)\mathcal{E}(0)(1 + t)^{1/2 + \theta \frac{\gamma}{2\gamma - 2}},
$$

(0.42)

$$
\|(v_x, v/x)_x(\cdot, t)\|^2 \\
\leq C(\theta) \left( \|G_x(\cdot, 0)\|^2 + \mathcal{E}(0) \right) (1 + t)^{-\frac{7\gamma - 6}{4\gamma} + \theta \left(4 + \frac{\gamma}{2\gamma - 2}\right)},
$$

(0.43)

for $0 \leq t < \infty$. 
Main Ideas in the Proofs:

1. The uniform-in-time *a priori* estimates (based on the local existence theory):

   *a priori* assumptions in the bootstrap arguments:

   \[ |r_x - 1| \leq \epsilon_0 \text{ for } (x, t) \in I \times [0, T], \]  
   \[ |v_x| \leq \epsilon_1 \text{ for } (x, t) \in I \times [0, T], \]

   where \( \epsilon_0 \in (0, 1/2] \) and \( \epsilon_1 \in (0, 1] \) are some small constants.
New functional $\mathcal{E}$

$$
\mathcal{E}(t) = \| (r - x, xr_x - x)(\cdot, t) \|^2 + \| (v, xv_x)(\cdot, t) \|^2 \\
+ \| (r_x - 1)(\cdot, t) \|^2_{L^\infty([1/2, 1])} \\
+ \left\| \bar{\rho}^{-1/2} G_x(\cdot, t) \right\|^2 + \left\| \bar{\rho}^{-1/2} G_{xt}(\cdot, t) \right\|^2.
$$

(0.46)

(we prove it is equivalent to $\mathcal{E}$ under the a priori assumptions (0.44) and (0.45))
The main goal in the proof
To show the higher-order energy functional $\mathcal{E}(t)$ is uniformly bounded by the initial data, that is, $\mathcal{E}(t) \leq C\mathcal{E}(0)$ for all $t \in [0, T]$ and verify the a priori assumptions.

Some key ideas:

1. A crucial point: we can obtain the decay estimate of the unweighted norm of $\|(r_x(x, t) - 1)\|_{L^2_x[0,1]}$ and the uniform boundedness of $|r_x - 1|$ and $|v_x|$, which are consequences of our uniform higher order estimates and are the key to the proof of the convergence of the vacuum boundary and the uniform convergence of the density.
New Multipliers:

In the derivation of the decay estimates of the unweighted norm of $\|(r_x(x, t) - 1)\|_{L^2_x[0, R]}$, the multipliers $w^2_\alpha = \int_0^x \bar{\rho}^{-\alpha}(y)(r(y, t)^3 - y^3)y\,dy$, which is motivated by the virial equations in the study of stellar dynamics and structures and is used to detect the detailed balance between the pressure and self-gravitation, and $w^1_\alpha = \int_0^x \bar{\rho}^{-\alpha}(y, t)(r^2(y, t)v(y, t))_y\,dy$ ($0 \leq \alpha < \gamma - 1$, $v(y, t) = u(r(y, t), t)$) play essential role in the construction of nonlinear functionals. To the best of our knowledge, those multipliers have not been used in the previous literatures.
2. **Uniform estimate of** $\| r_x - 1 \|_{L^\infty[0,1]}$

A combination of the local $L^2$-estimates for $r_{xx}$ away from the vacuum boundary, $\| r_{xx} \|_{L^2[0,1-\delta]}$, and the pointwise estimate away from the origin, $\| r_x(\cdot, t) - 1 \|_{L^\infty[\frac{1}{2},1]}$.

The bound for $\| r_{xx} \|_{L^2[0,1-\delta]}$: obtained by the $L^2$-estimates of $\mathcal{G}_x$.

The bound for $\| r_x(\cdot, t) - 1 \|_{L^\infty[\frac{1}{2},1]}$: by using the fact that the viscosity term can be written as $\mathcal{G}_{xt}$ so that one can integrate the equation with respect to both $x$ and $t$ to get the desired estimates. The monotonicity of the Lane-Emden density plays an important role here.
3. For the higher order estimates, due to the degeneracy of (0.7), the dissipation of the viscosity alone is not enough for the global-in-time estimates, and we have to make full use of the balance between the pressure and the gravitation:

**Decompose the gradient of the pressure**

\[
\left( \frac{x^2}{r^2 r_x} \right)^\gamma x \phi \bar{\rho} - \gamma \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right)^\gamma G_x
\]

\( \phi(x) = x^{-3} \int_0^x 4\pi \bar{\rho}(s)s^2 ds \): the mean density of the Lane-Emden solution inside the ball \( B_x(0) \).

1st part: \( \left( \frac{x^2}{r^2 r_x} \right)^\gamma x \phi \bar{\rho} \), is used to balance the gravitational force.
2nd part: \( \gamma \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right) \gamma G_x \) is the \( t \)-antiderivative of the viscosity multiplied by a weight which is equivalent to \( \bar{\rho}^\gamma(x) \).

(This weight is degenerate on the boundary, but strictly positive in the interior. This degeneracy near the vacuum boundary is one of main obstacles in the higher order estimates.)

In this way, we write the original eq as

\[
\mu G_{xt} + \gamma \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right) \gamma G_x = \frac{x^2}{r^2} \bar{\rho} v_t - \left[ \left( \frac{x^2}{r^2 r_x} \right)^\gamma - \left( \frac{x}{r} \right)^4 \right] x \phi \bar{\rho},
\]

(0.47)

(Note that the relative entropy is a nonlinear transformation to transform the original nonlinear equation of \( r \) to an equation whose principal part is linear in \( G \), motivated by Hopf-Cole transformation for viscous Burgers’ equation.)
The advantage of the approach:

(a) the interplay among the viscosity, pressure and gravitational force can be easily seen as follows:

In Lagrangian coordinates \((x, t)\) for \(x \in [0, \bar{R}]\), the viscosity term becomes \(\mu G_{xt}\), and the gradient of the pressure has the above decomposition.

Regarded as an equation for \(G\), (0.47) has the principal part of the equation, \(\mu G_{xt} + \gamma \left(\frac{x^2 \bar{\rho}}{r^2 r_x}\right)^\gamma G_x\), is linear in \(G_x\) and the term \(\gamma \left(\frac{x^2 \bar{\rho}}{r^2 r_x}\right)^\gamma G_x\) can be regarded as a degenerate damping. This structure leads to desirable estimates on \(G\) and their derivatives.
Another advantage for this formulation is that one can find a multiplier in the form of $x^2 \bar{\rho}^{\gamma - 2}\mathcal{P}_t$ (the $t$-derivative in Lagrangian coordinates) with
\[
\mathcal{P}(x, t) := \gamma \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right)^\gamma G_x + \left[ \left( \frac{x^2}{r^2 r_x} \right)^\gamma - \left( \frac{x}{r} \right)^4 \right] x \phi \bar{\rho},
\]
which is the sum of the gradient of the pressure and gravitational force. This multiplier is important for getting the key decay estimate of $\int_0^\bar{R} \bar{\rho}^{2\gamma - 2} r_{xx}^2(x, t) dx$, which is completely new as far as we know.
Main Strategy:
Lower order estimates:

1. the bound for the basic energy (using the energy weight $w_0^1$)

$$
\left\| x\bar{\rho}^{1/2} v(\cdot, t) \right\|^2 + \left\| x\bar{\rho}^{\gamma/2} \left( \frac{r}{X} - 1, r_x - 1 \right)(\cdot, t) \right\|^2 \\
+ \int_0^t \left\| (v, xv_x)(\cdot, s) \right\|^2 \, ds.
$$
2. the bound for (using the moment weight $w_0^2$)

$$\left\| x \left( \frac{r}{x} - 1, r_x - 1 \right) (\cdot, t) \right\|^2 + \int_0^t \left\| x \bar{\rho}^{\gamma/2} \left( \frac{r}{x} - 1, r_x - 1 \right) (\cdot, s) \right\|^2 ds,$$

which refines the weighted estimate of $\|x \bar{\rho}^{\gamma/2} \left( \frac{r}{x} - 1, r_x - 1 \right) \|$ obtained in the basic energy estimates.
3. The decay estimates for the basic energy: establishing a bound for

\[
(1 + t) \left( \| x \bar{\rho}^{1/2} v(\cdot, t) \|^2 + \| \bar{\rho}^{\gamma/2} (r - x, xr_x - x)(\cdot, t) \|^2 \right) \\
+ \int_0^t (1 + s) \| (v, x v_x)(\cdot, s) \|^2 \, ds,
\]

by using the weight \((1 + t)w_0^1\) and \((1 + t)w_0^2\). With those estimates, we are able to bound \(|r_x - 1|\) away from the origin.
4. the bound

\[
(1 + t) \left( \| x \tilde{\rho}^{1/2} v_t(\cdot, t) \|^2 + \| \tilde{\rho}^{\gamma/2} (v, x v_x)(\cdot, t) \|^2 \right) \\
+ \int_0^t (1 + s) \| (v_t, x v_{tx})(\cdot, s) \|^2 \, ds, \tag{0.48}
\]

was derived by repeating the estimates in step 1-3 for \( v_t \).
5. Further decay estimates for the lower norms, which is important to the derivation of the decay of \( \|r - x\|_{L^\infty(I)} \) in (0.37). This in particular implies the convergence of the evolving boundary \( r = R(t) \) to that of the Lane-Emden stationary solution. Those estimates also show that rates of time decay in various norms depend on the behavior of the initial data near the vacuum boundary. These estimates are derived by using the weights \((1 + t)\nu w_\alpha^1\) and \((1 + t)\nu w_\alpha^2\) \((0 < \alpha < \gamma - 1)\). This is one of most difficult part.
Higher order estimates

1. the uniform bound for
\[
\left\| \left( \bar{\rho}^{\gamma-\frac{1}{2}} G_x, \bar{\rho}^{-\frac{1}{2}} G_{xt} \right) (\cdot, t) \right\|^2.
\]

2. the decay estimates for
\[
\left( \left\| \bar{\rho}^{\frac{1}{2}} v_t (\cdot, t) \right\|^2 + \left\| (G_x, G_{xt}) (\cdot, t) \right\|_{L^2([0,\frac{1}{2}])}^2 \right).
\]

This completes the proof of the uniform in time bounds for the higher-order energy functional \( \mathcal{E}(t) \), which also verifies the a priori assumptions (0.44) and (0.45) due to the equivalence of \( \mathcal{E}(t) \) and \( \mathcal{E}(t) \), and consequently, the global existence of the strong solution is obtained.
Stability Analysis:

1. With the decay estimates for the lower-order norms and the higher order estimates, we prove the decay estimates for

\[ \left\| \left( r_x - 1, \frac{r}{x} - 1 \right)(\cdot, t) \right\|^2 \]

and

\[ \| (v, xv_x) (\cdot, t) \|_{L^\infty}^2 \]

and

\[ \| (r - x)(\cdot, t) \|_{L^\infty}^2 . \]
2. The faster decay estimates under the assumption of the finiteness of $\mathcal{F}(0)$:

(i) The decay estimates for

$$\| \bar{\rho}^{\gamma -1} G_x ( \cdot , t ) \|^2 .$$

(ii) The bound for

$$\left\| \bar{\rho}^{\gamma - 1 - \frac{\alpha}{2}} G_x ( \cdot , t ) \right\|^2$$

$(0 < \alpha < \gamma - 1)$.

(iii) The decay estimates for

$$\left\| x \left( \bar{\rho}^{\frac{3}{2} \gamma - 1} G_x , \bar{\rho}^{\frac{\gamma}{2} - 1} G_{xt} \right) ( \cdot , t ) \right\|^2 .$$
A key ingredient in the derivation of those estimates to use the new multiplier $x^2 \tilde{\rho}^{\gamma-2} \Psi_t$, where

$$\Psi(x, t) := \gamma \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right) \gamma \mathcal{G}_x + \left[ \left( \frac{x^2}{r^2 r_x} \right)^\gamma - \left( \frac{x}{r} \right)^4 \right] x \phi \bar{\rho},$$

the sum of the gradient of the pressure and gravitation force as a multiplier.
Thank You!