Chapter 2

A Glimpse of Measure and Integration

This chapter gives a quick but precise exposition of essentials of measure and integration so that a global view of the subject is provided in good time.

2.1 Measure and Integration

A triple \((\Omega, \Sigma, \mu)\) is called a measure space if \(\Omega\) is a set, \(\Sigma\) a \(\sigma\)-algebra of subsets of \(\Omega\), and \(\mu: \Sigma \mapsto \mathbb{R}^+ \cup \{+\infty\}\) satisfies the following:

i) \(\mu(\emptyset) = 0\);

ii) If \(\{A_i\} \subset \Sigma\) is disjoint, then \(\mu(\bigcup_i A_i) = \sum_i \mu(A_i)\), where \(\{A_i\}\) is at most countable.

We recall that a family \(\Sigma\) of subsets of a given set \(\Omega\) is called a \(\sigma\)-algebra if the following conditions hold:

1) \(\Omega \in \Sigma\);

2) if \(A \in \Sigma\), then \(A^c \equiv \Omega \setminus A\) is also in \(\Sigma\);

3) if \(\{A_i\} \subset \Sigma\) is at most countable, then \(\bigcup_i A_i\) is also in \(\Sigma\).

From 2) and 3) one can verify easily that if \(\{A_i\} \subset \Sigma\) is at most countable, then \(\bigcap_i A_i \in \Sigma\). Given a measure space \((\Omega, \Sigma, \mu)\), elements of \(\Sigma\) are referred to as measurable sets, and \(\mu\) is called a measure.

Example 2.1.1. Let \(\Omega\) be an arbitrary set and \(\Sigma = 2^\Omega\) the family of all subsets of \(\Omega\). For \(A \subset \Omega\), let \(\mu(A)\) be the number of elements in \(A\) if \(A\) is finite, otherwise let \(\mu(A) = +\infty\). \(\mu\) is obviously a measure, and is called the counting on \(\Omega\).

Example 2.1.2. Let \(\Omega\) be a countable set, say \(\Omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}\), and suppose \(p_1, p_2, \ldots, p_n, \ldots\) is a sequence of nonnegative real numbers with
\[ \sum_{n=1}^{\infty} p_n = 1. \] For \( A \subset \Omega \), let \( \mu(A) = \sum_{\omega_n \in A} p_n \), \((\Omega, 2^\Omega, \mu)\) is a measure space called a discrete probability space.

A function \( f : \Omega \mapsto \mathbb{R} \) is said to be measurable if \( \{ f > \alpha \} \equiv \{ x \in \Omega : f(x) > \alpha \} \in \Sigma \) \( \forall \alpha \in \mathbb{R} \). The family of all measurable functions is a real vector space. Also, this family is closed under limit, i.e. if \( \{ f_n \} \) is a sequence of measurable functions, which converges pointwise to a finite-valued function \( f \), then \( f \) is measurable. For these facts, we refer to Exercises 2.1.1 and 2.1.3.

For \( A \in \Sigma \), \( I_A \), the indicator function of \( A \), is measurable. Elements of \( \langle I_A : A \in \Sigma \rangle \) are called simple functions. We recall that \( \langle W \rangle \) denotes the smallest vector subspace containing \( W \) in a vector space. A simple function \( f \) can be expressed as \( f = \sum_{i=1}^{k} \alpha_i I_{A_i} \), where \( \alpha_1, \ldots, \alpha_k \) are the different values assumed by \( f \) and \( A_i = \{ f = \alpha_i \} \), we define then
\[
\int_{\Omega} f \, d\mu = \sum_{i=1}^{k} \alpha_i \mu(A_i), \tag{2.1}
\]
if the right hand side of (2.1) has a meaning. We recall some usual conventions concerning algebraic operations involving \( +\infty \) and \( -\infty \): \( +\infty + \infty = \infty, -\infty + (-\infty) = -\infty, a + \infty = -(a - \infty) = \infty \) if \( a \) is a finite number, \( a \cdot \infty = (-a) \cdot (-\infty) = \infty \), or \( -\infty \) depending on whether \( a > 0 \) or \( a < 0 \), and \( 0 \cdot \infty = 0 \cdot (-\infty) = 0 \) where \( +\infty \) is sometimes written simply as \( \infty \). It is easy to see that whenever \( f \) is expressed as \( f = \sum_{i=1}^{l} \beta_i I_{B_i} \), where \( B_1, \ldots, B_l \) are in \( \Sigma \) and are disjoint, then
\[
\int_{\Omega} f \, d\mu = \sum_{i=1}^{l} \beta_i \mu(B_i).
\]
In particular, \( \int_{\Omega} f \, d\mu \) has a meaning if \( f \) is simple and nonnegative, although it is possible that \( \int_{\Omega} f \, d\mu = +\infty \).

If \( f \) is measurable and nonnegative define
\[
\int_{\Omega} f \, d\mu = \sup \int_{\Omega} g \, d\mu,
\]
where the supremum is taken over all simple functions \( g \) with \( 0 \leq g \leq f \). Obviously if \( f \) is nonnegative and simple, this coincides with the previously defined \( \int_{\Omega} f \, d\mu \) for simple functions.

For any measurable function \( f \), write \( f = f^+ - f^- \), where
\[
\begin{align*}
f^+(x) &= f(x) \text{ if } f(x) \geq 0, \\
&= 0 \text{ otherwise;}
\end{align*}
\[
\begin{align*}
f^-(x) &= -f(x) \text{ if } f(x) \leq 0, \\
&= 0 \text{ otherwise,}
\end{align*}
\]
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and define 
\[
\int_{\Omega} f \, d\mu = \int_{\Omega} f^+ \, d\mu - \int_{\Omega} f^- \, d\mu
\]
if the right hand side has a meaning. In this case, \(\int_{\Omega} f \, d\mu\) is called the integral of \(f\). If \(\int_{\Omega} f \, d\mu\) is finite, then \(f\) is said to be integrable. It will be shown later that a measurable function \(f\) is integrable if and only if \(|f|\) is integrable (See Theorem 2.3.3). We note that if \(f\) is measurable then so are \(f^+,\ f^-,\) and \(|f|\).

Example 2.1.3. Let \(\Omega\) be an arbitrary set and consider the counting measure \(\mu\) on \(\Omega\), then every real function \(f\) on \(\Omega\) is measurable and \(f\) is integrable if and only if \(\{f(x)\}_{x \in \Omega}\) is summable.

Example 2.1.4. Consider the discrete probability space of Example 2.1.2. Let \(f\) be a real function on \(\Omega\). Since every subset of \(\Omega\) is measurable, \(f\) is measurable and is called a random variable. If \(\int_{\Omega} f \, d\mu\) exists, it is called the expectation of \(f\).

Exercise 2.1.1. If \(f,\ g\) are measurable, then \(f + g\) is also measurable. (Hint: Let \(Q\) be the set of all rational numbers, then for \(\alpha \in \mathbb{R}\) 
\[
\{f + g > \alpha\} = \bigcup_{\beta \in Q} \{f > \alpha - \beta\} \cap \{g > \beta\}.
\]

Exercise 2.1.2. If \(f\) is measurable, and \(\alpha, \beta \in \mathbb{R}\) with \(\alpha < \beta\), then the sets 
\[
\{f \geq \alpha\}, \{f < \alpha\}, \{\alpha < f < \beta\}, \{\alpha \leq f < \beta\}, \{\alpha \leq f \leq \beta\}, \{\alpha < f \leq \beta\}
\]
are all in \(\Sigma\). (All these sets are defined similarly as \(\{f > \alpha\}\))

Exercise 2.1.3. Let \(f_1, f_2, \ldots : \Omega \mapsto \mathbb{R}\) be measurable and \(\lim_{n \to \infty} f_n = f\) pointwise and \(f(x)\) is finite for each \(x \in \Omega\). Show that \(f\) is measurable.

\[
\Bigg(hint: \text{Show that } \{f > \alpha\} = \bigcup_{m \geq 1} \bigcup_{k \geq 1} \bigcap_{n \geq k} \left\{f_n > \alpha + \frac{1}{m}\right\}.\Bigg)
\]

Exercise 2.1.4. Let \(f : \Omega \mapsto \mathbb{R}\) be measurable. For each positive integer \(n\) let 
\[
A^{(n)} = \{f < -n\}, \ C^{(n)} = \{f \geq n\}, \ B_i^{(n)} = \{-n + \frac{i}{n} \leq f < -n + \frac{i+1}{n}\}, \ i = 0, 1, 2, \ldots, 2n^2 - 1,
\]
and let 
\[
g_n = -nI_{A^{(n)}} + \sum_{i=0}^{2n^2-1} \left(-n + \frac{i+1}{n}\right)I_{B_i^{(n)}} + nI_{C^{(n)}}.
\]
Show that \(g_n \to f\) pointwise and then show that if \(f,\ g\) are measurable then \(fg\) is measurable, furthermore if \(g \neq 0\) everywhere on \(X\), then \(f/g\) is also measurable.
Exercise 2.1.5. If \( f \) and \( g \) are nonnegative simple functions and \( \alpha, \beta \geq 0 \), then
\[
\int_{\Omega} (\alpha f + \beta g) \, d\mu = \alpha \int_{\Omega} f \, d\mu + \beta \int_{\Omega} g \, d\mu.
\]

Exercise 2.1.6. If \( f \leq g \) are two nonnegative measurable functions, show that
\[
\int_{\Omega} f \, d\mu \leq \int_{\Omega} g \, d\mu.
\]

2.2 Egoroff Theorem and Monotone Convergence Theorem

Suppose \( \Omega \) is a set and \( \{ A_n \}_{n=1}^{\infty} \) is a sequence of subsets of \( \Omega \), define
\[
\limsup_{n \to \infty} A_n = \bigcap_{1 \leq k < \infty} \bigcup_{n \geq k} A_k;
\]
\[
\liminf_{n \to \infty} A_n = \bigcup_{1 \leq k < \infty} \bigcap_{n \geq k} A_k.
\]

If \( \limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n \), then we say that the limit of the sequence \( A_n \) exists and has the common set as its limit which is denoted by \( \lim_{n \to \infty} A_n \).

In particular, if \( A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \) i.e. \( \{ A_n \} \) is monotone increasing, or \( A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots \) i.e. \( \{ A_n \} \) is monotone decreasing, then \( \lim_{n \to \infty} A_n \) exits and equals \( \bigcup_{1 \leq k < \infty} A_k \) and \( \lim_{n \to \infty} A_n = \lim_{n \to \infty} \bigcap_{k \geq n} A_k \) respectively. Hence \( \limsup_{n \to \infty} A_n = \lim_{n \to \infty} \bigcup_{k \geq n} A_k \) and \( \liminf_{n \to \infty} A_n = \lim_{n \to \infty} \bigcap_{k \geq n} A_k \).

In the following a measure space \( (\Omega, \Sigma, \mu) \) is considered and fixed throughout.

Lemma 2.2.1. Let \( \{ A_n \}_{n=1}^{\infty} \subset \Sigma \) be monotone increasing, then
\[
\mu\left( \lim_{n \to \infty} A_n \right) = \mu\left( \bigcup_{1 \leq n < \infty} A_n \right) = \lim_{n \to \infty} \mu(A_n).
\]

Proof. For each positive integer \( n \) let \( B_n = A_n \setminus A_{n-1} \), where we put \( A_0 = \emptyset \), and for convenience let \( A = \bigcup_{1 \leq n < \infty} A_n \). Then \( A_n = \bigcup_{1 \leq k \leq n} B_k \) and \( A = \bigcup_{1 \leq k < \infty} B_k \). Since \( \{ B_k \} \) is disjoint, we have
\[
\mu(A) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n \to \infty} \mu(A_n).
\]

Lemma 2.2.2. Let \( \{ A_n \}_{n=1}^{\infty} \subset \Sigma \) be monotone decreasing and \( \mu(A_1) < \infty \), then
\[
\mu\left( \lim_{n \to \infty} A_n \right) = \mu\left( \bigcap_{1 \leq n < \infty} A_n \right) = \lim_{n \to \infty} \mu(A_n).
\]
Proof. For each positive integer \( n \) let \( B_n = A_1 \setminus A_n \) and for convenience let \( A = \bigcap_{1 \leq n < \infty} A_n \). Then \( \{B_k\} \) is monotone increasing and \( A_1 \setminus \bigcap_{1 \leq n < \infty} A_n = \bigcup_{1 \leq k < \infty} B_k \). From Lemma 2.2.1, we have
\[
\mu \left( A_1 \setminus \bigcap_{1 \leq n < \infty} A_n \right) = \mu \left( \bigcup_{1 \leq k < \infty} B_k \right) = \lim_{n \to \infty} \mu(B_n).
\]
But \( \mu(A_1 \setminus \bigcap_{1 \leq n < \infty} A_n) = \mu(A_1) - \mu(\bigcap_{1 \leq n < \infty} A_n) \) and \( \mu(B_n) = \mu(A_1) - \mu(A_n) \), this completes the proof of the lemma.

**Theorem 2.2.1.** (Egoroff Theorem) If \( \{f_n\} \) is a sequence of measurable functions and \( f_n \to f \) with finite limit on \( A \in \Sigma \), where \( \mu(A) < +\infty \), then for \( \varepsilon > 0 \), there is \( B \subset A \), \( B \in \Sigma \) such that \( \mu(A \setminus B) < \varepsilon \) and \( f_n \to f \) uniformly on \( B \).

Proof.

[Step 1] For \( \varepsilon > 0 \), \( \eta > 0 \), there are integer \( N > 0 \) and \( C \in \Sigma \) such that \( C \subset A \). \( \mu(A \setminus C) < \varepsilon \) and \( \sup_{x \in C} |f(x) - f_n(x)| \leq \eta \) whenever \( n \geq N \).

To show this, for each \( n \) let \( C_n = \bigcap_{m \geq n} \{x \in A : |f(x) - f_m(x)| \leq \eta\} \). Then \( C_n \not\subset A \). Hence there is \( N \) such that \( \mu(A \setminus C_N) < \varepsilon \). Take \( C = C_N \).

[Step 2] Now given \( \varepsilon > 0 \). By [Step 1] for each positive integer \( m \) there are integer \( N_m \) and \( C_m \subset A \) with \( C_m \in \Sigma \) such that
\[
\mu(A \setminus C_m) < \varepsilon/2^m
\]
and
\[
\sup_{x \in C_m} |f(x) - f_n(x)| \leq \frac{1}{m}
\]
whenever \( n \geq C_m \).

Then take \( B = \bigcap_{m=1}^{\infty} C_m \) to complete the proof.

The proof of the following theorem is didactic in nature, it is designed to reveal the nature if Monotone Convergence Theorem.

**Theorem 2.2.2.** (Monotone convergence Theorem) Let \( \{f_n\} \) be a monotone nondecreasing sequence of nonnegative measurable functions. Suppose
\[
\lim_{n \to \infty} f_n = f
\]
is finite valued, then
\[
\int_{\Omega} f \, d\mu = \lim_{n \to \infty} \int_{\Omega} f_n \, d\mu.
\]

Proof. It is obvious that
\[
\lim_{n \to \infty} \int_{\Omega} f_n \, d\mu \leq \int_{\Omega} f \, d\mu.
\]
We assume that \( \int_{\Omega} f \, d\mu < +\infty \). Then given \( \varepsilon > 0 \) there is a simple function \( g \), \( 0 \leq g \leq f \) such that
\[
\int_{\Omega} f \, d\mu < \int_{\Omega} g \, d\mu + \varepsilon.
\]
Let us write \( g = \sum_{i=1}^{k} \alpha_i I_{A_i} \), where \( \{A_i\} \) is disjointed and \( \alpha_i > 0 \). Since
\[
\int_{\Omega} g \, d\mu \leq \int_{\Omega} f \, d\mu < +\infty, \quad \mu(A_i) < +\infty, \quad i = 1, \ldots, k,
\]
hence, by replacing each \( \alpha_i \) by a smaller number if necessary, we may assume that on each \( A_i \)
\[
\inf_{x \in A_i} |f(x) - g(x)| > \eta > 0.
\]
From Egoroff Theorem there are integer \( N \) and \( B_i \subset A_i, \ B_i \in \Sigma, \ i = 1, \ldots, k \) such that \( f_N > g \) on each \( B_i \) and
\[
\int_{\Omega} f \, d\mu < \int_{\Omega} g' \, d\mu + 2\varepsilon,
\]
where \( g' = \sum_{i=1}^{k} \alpha_i I_{B_i} \). Then
\[
\int_{\Omega} f_N \, d\mu \geq \int_{\Omega} g' \, d\mu > \int_{\Omega} f \, d\mu - 2\varepsilon,
\]
and therefore
\[
\lim_{n \to \infty} \int_{\Omega} f_n \, d\mu \geq \int_{\Omega} f_N \, d\mu > \int_{\Omega} f \, d\mu - 2\varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, we have \( \lim_{n \to \infty} \int_{\Omega} f_n \, d\mu \geq \int_{\Omega} f \, d\mu \).
Hence
\[
\int_{\Omega} f \, d\mu = \lim_{n \to \infty} \int_{\Omega} f_n \, d\mu.
\]
\( \square \)

Exercise 2.2.1. Complete the proof of Monotone Convergence Theorem by considering the case \( \int_{\Omega} f \, d\mu = +\infty \).

2.3 Concepts Related to Sets of Measure Zero

We now remark on concepts connected with measure zero sets. Let \( A = \{x \in \Omega : x \text{ does not have a property } P\} \), if \( \mu(A) = 0 \), we say that the property \( P \) holds almost everywhere on \( \Omega \) (or simply \( P \) holds almost everywhere). For example, if outside a \( \mu \)-measure zero set, \( f \) is finite, then we say that \( f \) is finite almost everywhere; also if \( \lim_{n \to \infty} f_n(x) = f(x) \) exists for each \( x \) outside a \( \mu \)-measure zero set, then we say that \( f_n \) converges almost everywhere. If a property \( P \) holds almost everywhere, we simply say that \( P \) holds a.e.

From now on we allow our functions to be extended real-valued, i.e. \( +\infty \) and \( -\infty \) are allowed to be taken as function values. Measurability of extended
### 2.3. CONCEPTS RELATED TO SETS OF MEASURE ZERO

Real valued functions is defined similarly. It is then easy to see that if two measurable functions differ on a set of measure zero, then they have the same integral if their integrals exist.

**Exercise 2.3.1.** If $f$ is measurable, then $\{f = +\infty\}$ and $\{f = -\infty\}$ are in $\Sigma$.

All the results we have established so far remains true if the pointwise conditions are replaced by conditions held almost everywhere. For examples:

**Theorem 2.3.1.** (Egoroff Theorem) If a sequence $\{f_n\}$ of almost everywhere finite measurable functions converges a.e. to a finite function $f$ on $A$, where $A \in \Sigma$, and $\mu(A) < +\infty$, then for every $\epsilon > 0$, there is $B \in \Sigma$, $B \subset A$ such that $\mu(A \setminus B) < \epsilon$ and $f_n \to f$ uniformly on $B$.

**Theorem 2.3.2.** (Monotone convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions which are nonnegative and nondecreasing a.e., then

$$\int_{\Omega} \lim_{n \to \infty} f_n d\mu = \lim_{n \to \infty} \int_{\Omega} f_n d\mu.$$

**Exercise 2.3.2.**

i) If $f \geq 0$ a.e., measurable, then there is a sequence $\{g_n\}$ of simple functions such that $0 \leq g_n \leq f$ and $g_n \not\to f$ a.e.

ii) If $f$, $g \geq 0$ a.e. and measurable, then

$$\int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu.$$

iii) If $f$ is measurable and $f = f_1 - f_2 = g_1 - g_2$, where $f_1$, $f_2$, $g_1$, and $g_2$ are measurable and nonnegative a.e., then

$$\int_{\Omega} f_1 d\mu - \int_{\Omega} f_2 d\mu = \int_{\Omega} g_1 d\mu - \int_{\Omega} g_2 d\mu$$

if they are meaningful, i.e. $\int_{\Omega} f d\mu$ does not depend on how $f$ is written as the difference of two nonnegative measurable functions. Also show that if $f = f_1 - f_2$, where $f_1$ and $f_2$ are nonnegative, then $f^+ \leq f_1$ and $f^- \leq f_2$, and hence, if $\int_{\Omega} f_1 d\mu - \int_{\Omega} f_2 d\mu$ is meaningful, so is $\int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu$ and $\int_{\Omega} f_1 d\mu - \int_{\Omega} f_2 d\mu = \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu$.

iv) If $f, g$ are measurable functions such that $\int_{\Omega} f d\mu$, $\int_{\Omega} g d\mu$, $\int_{\Omega} f d\mu + \int_{\Omega} g d\mu$ are meaningful, then

$$\int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu.$$

In particular, this holds true if $f$ and $g$ are integrable.

v) In Monotone Convergence Theorem, that $\{f_n\}$ is a sequence of nonnegative measurable functions can be replaced by “$\{f_n\}$ is bounded from below by an integrable function a.e.”. Also show that Monotone Convergence Theorem holds also for extended real valued functions.
Exercise 2.3.3. If \( f \geq 0 \) a.e. measurable, then \( \int_{\Omega} f d\mu = 0 \) implies that \( f = 0 \) a.e.

The following theorem is a consequence of Exercise 2.3.2 iv):

**Theorem 2.3.3.** \( f \) is integrable if and only if \( |f| \) is integrable.

### 2.4 Fatou Lemma and Lebesgue Dominated Convergence Theorem

**Theorem 2.4.1.** (Fatou Lemma) Let \( \{f_n\} \) be a sequence of extended real-valued measurable functions which is bounded from below by an integrable function. Then

\[
\int_{\Omega} \liminf_{n \to \infty} f_n d\mu \leq \liminf_{n \to \infty} \int_{\Omega} f_n d\mu.
\]

**Proof.** Let \( g_n = \inf_{k \geq n} f_k \), then \( g_n \) is nondecreasing and is bounded from below by an integrable function. By Monotone Convergence Theorem (see Exercise 2.3.2 v))

\[
\int_{\Omega} \liminf_{n \to \infty} f_n d\mu = \int_{\Omega} \lim_{n \to \infty} g_n d\mu \\
= \lim_{n \to \infty} \int_{\Omega} g_n d\mu \leq \liminf_{n \to \infty} \int_{\Omega} f_n d\mu.
\]

**Exercise 2.4.1.** Show that in Fatou Lemma, if \( \{f_n\} \) is bounded from above by an integrable function, then

\[
\int_{\Omega} \limsup_{n \to \infty} f_n d\mu \geq \limsup_{n \to \infty} \int_{\Omega} f_n d\mu.
\]

**Theorem 2.4.2.** (Lebesgue Dominated Convergence Theorem (LDCT)) If \( f_n, \ n = 1, 2, \cdots \) and \( f \) are measurable functions and \( f_n \to f \) a.e. Suppose further that \( |f_n| \leq g \) a.e. with \( g \) being an integrable function for all \( n \). Then

\[
\int_{\Omega} f d\mu = \lim_{n \to \infty} \int_{\Omega} f_n d\mu.
\]

**Proof.** \( \{f_n\} \) is bounded from below and from above by integrable functions. Hence by Fatou Lemma

\[
\limsup_{n \to \infty} \int_{\Omega} f_n d\mu \leq \int_{\Omega} \limsup_{n \to \infty} f_n d\mu \leq \liminf_{n \to \infty} \int_{\Omega} f_n d\mu,
\]

and consequently \( \int_{\Omega} \lim_{n \to \infty} f_n d\mu = \lim_{n \to \infty} \int_{\Omega} f_n d\mu. \)
Corollary 2.4.1. Under the same conditions as in LDCT we have
\[ \lim_{n \to \infty} \int_\Omega |f_n - f| \, d\mu = 0. \]

2.5 The Space \( \mathbb{L}^p(\Omega, \Sigma, \mu) \)

Unless stated otherwise, hereafter functions are extended real-valued.

For a measurable function \( f \), let
\[
\|f\|_p = \left( \int_\Omega |f|^p \, d\mu \right)^{1/p} \quad \text{if } 1 \leq p < +\infty;
\]
\[
\|f\|_\infty = \inf \{ M \geq 0 : |f| \leq M \text{ a.e.} \}.
\]

\( \|f\|_\infty \) is called the essential supremum of \( f \).

Exercise 2.5.1. Show that \( |f| \leq \|f\|_\infty \text{ a.e.} \)

Recall that if \( p, q \geq 1 \) are such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then they are called conjugate exponents.

Theorem 2.5.1. (Hölder’s Inequality) If \( p, q \geq 1 \) are conjugate exponents, then
\[
\int_\Omega |f| \, d\mu = \|f\|_1 \leq \|f\|_p \|g\|_q.
\]

Proof. We may assume that \( 0 < \|f\|_p, \|g\|_q < +\infty \), hence \( |f|, |g| < \infty \text{ a.e.} \).

We may further assume \( 1 < p, q < \infty \). Now let \( \zeta = \left( \frac{|f|}{\|f\|_p} \right)^\beta, \eta = \left( \frac{|g|}{\|g\|_q} \right)^\alpha \), \( \alpha = \frac{1}{p}, \) and \( \beta = \frac{1}{q} \) in Lemma 1.6.1, we have
\[
\frac{|f| |g|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g|^q}{\|g\|_q^q}
\]
a.e. on \( \Omega \), from which on integrating both sides we complete the proof.

Exercise 2.5.2. From our proof of Hölder’s inequality find a necessary and sufficient condition for the inequality to become an equality.

Theorem 2.5.2. (Minkowski’s Inequality) Let \( f, g \) be measurable, \( 1 \leq p \leq +\infty \), then
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p,
\]
whenever \( f + g \) is meaningful a.e. on \( \Omega \).
Proof. This is obvious when \( p = 1 \) or \( +\infty \). We now consider the case \( 1 < p < +\infty \), then

\[
\|f + g\|_p^p = \int_\Omega |f + g|^p d\mu = \int_\Omega |f + g|^{p-1} |f + g| d\mu
\]

\[
\leq \int_\Omega |f + g|^{p-1} |f| d\mu + \int_\Omega |f + g|^{p-1} |g| d\mu
\]

\[
\leq \left[ \int_\Omega |f + g|^{(p-1)q} d\mu \right]^{1/q} \{\|f\|_p + \|g\|_p\}
\]

\[
= \|f + g\|_p^{p/q} \{\|f\|_p + \|g\|_p\}
\]

by Hölder’s inequality, where \( \frac{1}{p} + \frac{1}{q} = 1 \). The theorem follows by dividing extreme ends of the above sequence of inequalities by \( \|f + g\|_p^{-1} \), because we may assume that \( \|f + g\|_p < \infty \).

Exercise 2.5.3. Verify the last statement. (Hint: Show that if \( \|f\|_p + \|g\|_p < +\infty \), then \( \|f + g\|_p < +\infty \) by using Exercise 1.6.2.)

Let now \( \mathcal{L}^p(\Omega, \Sigma, \mu) \) be the family of all measurable functions \( f \) with \( \|f\|_p < +\infty \). From Minkowski inequality, it is readily seen that \( \mathcal{L}^p(\Omega, \Sigma, \mu) \) is a real vector space. If we let

\[
\mathcal{N} = \{ f \in \mathcal{L}^p(\Omega, \Sigma, \mu) : \|f\|_p = 0 \},
\]

then \( f \in \mathcal{N} \) if and only if \( f = 0 \) a.e. on \( \Omega \). Consider now the space \( \mathcal{L}^p(\Omega, \Sigma, \mu) = \mathcal{L}^p(\Omega, \Sigma, \mu)/\mathcal{N} \), then \( \mathcal{L}^p(\Omega, \Sigma, \mu) \) is a vector space which consists of equivalent classes of \( \mathcal{L}^p(\Omega, \Sigma, \mu) \) w.r.t the equivalent relation \( \sim \) defined by \( f \sim g \) if and only if \( f = g \) a.e. on \( \Omega \).

We shall allow ourselves the liberty of not distinguishing between a class of functions in \( \mathcal{L}^p(\Omega, \Sigma, \mu) \) and a function representing the class; hence by \( f \in \mathcal{L}^p(\Omega, \Sigma, \mu) \) we shall mean that \( f \) is to be considered as a class of equivalent functions in \( \mathcal{L}^p(\Omega, \Sigma, \mu) \) as well as any function from that class.

For \( f \in \mathcal{L}^p(\Omega, \Sigma, \mu) \), let

\[
\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p} \text{ if } 1 \leq p < +\infty ,
\]

and

\[
\|f\|_\infty = \text{ essential supremum of } f.
\]

Remember that in the definition above, \( f \) on the left hand side is a class of function and \( f \) on the right hand side is a function representing that class. We note that the definition above is well-meant.
2.5. **THE SPACE $L^p(\Omega, \Sigma, \mu)$**

**Theorem 2.5.3.** $L^p(\Omega, \Sigma, \mu)$ with norm $\| \cdot \|_p$ is a Banach Space.

**Proof.** This is obvious when $p = +\infty$.

Assume now $1 \leq p < +\infty$ and let $\{f_n\}$ be a Cauchy sequence in $L^p(\Omega, \Sigma, \mu)$. There is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$, $k = 1, 2, \ldots$. Put $g = \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$, Monotone Convergence Theorem and Minkowski inequality imply

$$\|g\|_p \leq \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_p \leq 1,$$

hence $g \in L^p(\Omega, \Sigma, \mu)$. Observe that $f_{n_k} \to f$ a.e. with $f$ finite a.e. But $|f_{n_k}| \leq |f_{n_1}| + g$, $k = 1, 2, \ldots$, which implies $f \in L^p(\Omega, \Sigma, \mu)$ by LDCT. Now $|f_{n_k} - f|^p \leq (|f| + |f_{n_1}| + g)^p$ a.e. thus by LDCT again we know $\|f_{n_k} - f\|_p \to 0$ as $k \to \infty$, this fact together with $\{f_n\}$ being a Cauchy sequence imply $\|f_n - f\|_p \to 0$ as $n \to \infty$. Hence $L^p(\Omega, \Sigma, \mu)$ is complete. \qed