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§ 1 Axiomatization of geometries.

Projective geometry was long known to be a geometry of incidence relations — a point is on a line or a line passes through a point. The routine procedures of this geometry are to construct a line through two distrinct points and to take the intersection of two lines or combinations of these two operations (for example, a projection); i.e., joining and intersecting of elements. For this reason, K. Menger started in 1928 to develop this geometry from algebraic postulates concerning these two operations. These two operations are actually the two operations $\cup$ and $\cap$ in lattice theory, and G. Birkhoff also characterized in 1935 the incidence relations among linear subspace of projective space lattice-theoretically. K. Menger also has done this independently and moreover carried out corresponding study for affine geometry with Alt and Schreiber in 1936. Since then, K. Menger and his school have also set up the foundation of hyperbolic geometry upon algebraic postulates concerning these two operations. Mean-while the corresponding problem for inversise geometry (conformal geometry or circle and sphere geometry) has been studied by S. Inumi (1940) and A. J. Hoffmann (1951).

These geometries were assumed to be of finite dimensional, but O. Frink (1946) has studied the characterjation of projective geometry of infinite dimension. W. Prenorvitz (1948) has studied that of perspective geometry of infinite dimension. Then U. Sasaki

(1952) has studied the corresponding case of affine geometry (infinite dimensional) and also studied the unified treatment of projective and affine geometry of infinite dimension, Finally R. Wille (1967) has given the unified theory of all projective, affine and inversive geometries infinite dimension.

On the other hand F. Maeda gave formulation of an abstract geometry and studied its properties (1951). And Jonnson (1959) gave the definition of geometry which is most closely related to the one given in the book of Crapo and Rota (1970).

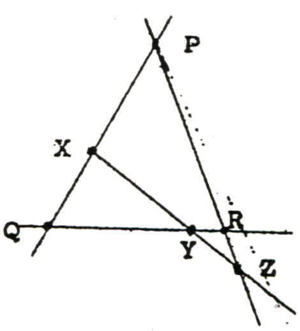
In the discussion of these materials, there are two aspects closely related; one is the geometrical part and the other is the lattice theoretical one.

For the geometrical part. let us examine the axioms of geometries which deal with only incidence relations.

Following Veblen and Young, a projective geometry consists of a set S of elements called points and a family of subsets (called lines) of S which satisfy the following pastulates:

P1. Two distinct points are on one and only one line

P2. If the points P, Q and R are not collinear while



the points P, Q, X are collinear;
points Q, R, Y are collinear;
and $X \neq Y$,
then there is a point Z such that
points P, R, Z are collinear
and points X, Y, Z are collinear.

The following set of postulates given by Sasaki is said to be equivalent to the set of postulates $I_1 - I_8$ of connection given by D. Hilbert:

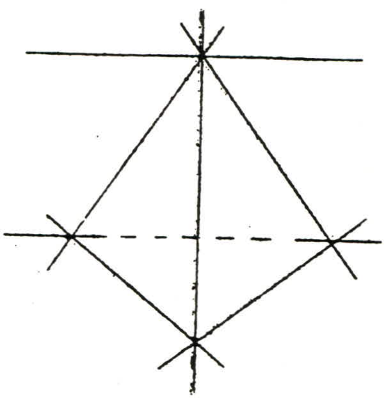
A1. For any pair of distinct points p, q, there exists a line containing p, q and two distinct points on a line determine the line.

A2. For any triple of points p, q, r which are not on a line, there is a plane containing these points, and three non-collinear points on a plane determine the plane.

A2'. The line through two distinct points on a plane is contained in the plane.

A4. If two planes contained in a 3-space have a point in common, then they have at least one more point in common.

But actually this set of pastulates in not quite equivalent to the set of axioms of connection given by Hilbert.



The set of 4 points, 7 lines and 4 planes given in the figure satisfies A1, A2, A2' and A4, but it does not satisfy the following axiom in the set of axioms of connection:

I_3 Every line contains at least two points.

Thus, to get an equivalent system, we have to change A1 to A1' as follows:

A1′. Any two distinct points are contained in exactly a line, and each line contains at least 2 points.

These postulates are common to projective and affine geometry. The postulate which is special to affine geometry is the following one:

A3. If p, q, r are non-collinear points, then there exists one and only one line through r which is parallel to the line determined by p, q.

Here two lines pq, rs are called to be parallel to each other (denoted by pq || rs) if they are contained in the same plane and have no point in common.

In the axiomatic foundation of 2-dimensional inversive geometry given by Van der Waerden and higher dimensional one given by Hoffman, the postulates were chosen by taking into acount the fact (or its two dimensional version) that a three dimensional inversive geometry over an ordered field V in which every non negative number is a square may be defined as a set π of objects called points, circles, sphaes and inversive space with the properties:

(i) if p is any point, then there is an affine geometry whose "points", "lines", "planes" and "3-space" are respective by the points of π other than p, the circle containing p, the sphars containing p, and the inversive space,

(ii) the underlying field of this affine geometry is V,

(iii) this affine geometry can be made a euclidean geometry in such a way that the "circles" and "spheres" of the euclidean geometry, are respectively the circles of π not containing p, and the spheres of π not containing p.

Taking (i) into acount, and consider the corresponding affine geometry, we can get some foundamental propositions for inversive geometry.

A-4 of affine geometry inplies the following for inversive geometry:

If two spheres through the points p are contained in an inversive sphae space and have one more point in common, then they have at least another point in common.

Since p is an arbitrarily selected point, so this statement can also be put as follows:

S4. If two shperes in a 3-dimensional inversive space have two distinct points in common, they have at least one more point in common.

By similar argument, from A1′, A2, A2′ we get respectively the followings:

S1. These distinct points on a circle determine the circle, and each circle contains at least three distinct points.

S2. Four points which are not on a circle determine a sphere.

S3. If three distinct points are contained in a sphere then the circle determined by these three points is contained in the sphere.

To include all the above mentioned geometries, R. Wille (1967) set up the following postulates:

In a set S of points, a family of subsets of S, each of which is called a <u>curve</u>, and another family of subsets of S, each of which is called a <u>surface</u> are specified out such that the following postulates are satisfied:

W1. $n + 1$ distinct points are contained in exactly one curve, and each curve contains at least $n + 1$ distinct points.

W2. $n + 2$ distinct points which are not contained in a curve, are contained in exactly one surface, and in each surface, there are at least $n + 2$ points which are not contained in a curve.

W3. Along with the $(n+1)$ distinct points contained in a surface, the curve which is determined by these points is also contained in the surface.

To formulate the fourth postulate, we give the following definitions:

<u>Def.</u> A point set is called a subspace (or a flat) if it contains the curves and surfaces determined by W1 and W2 with $n + 1$ or $N + 2$ points contained in the set.

<u>Def.</u> The intersection of all subspace which contain a set A is said the subspace generated by A.

W4. The intersection of two surfaces which are contained in the subspace generated by $n + 3$ points, will never consists of

[illegible]

exactly n distinct points.*)

Def. The family of points S and the curves and surfaces which satisfies the postulates W1 — W4 is called a (an incidence) geometry (Wille geometry) of grade n (n = 0,1,···,··) on a set S.

Examples. a) The geometries of grade 0 are projective geometries, provided that curves and surfaces are intupreted as points and lines (Menyer, Birghoff, Tring, Prenowitz, Maeda).

b_1) For n = 1 we get the so-called strongly planar geometries (Sasaki, Jonnson).

A geometry is called planar if a subspace is defind by requiring that the plane is contained in the set whenever the three non-collinear points defining it are contained in the set. A geometry is called strongly planar if it is planar and the postulate A4 is satisfied. Note that in the difinition of subspaces of a projective geometry, we only require that a line is contained in the set if the two points determining the line are contained in the set.

b_2) If we also assume the parallel axiom, then we get affine geometries (Menyer, Alt, Schreiber, Sasaki).

c) The geometries of grade 2 are, for examples, the inversive geometries (Izumi, Hoffman).

d) I guess it might also be considered the set of all ellipses

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*) It can contain less than n points, as two parallel planes in an affine 3-space show.

and ellipsoids (or parabolas and paraboloids) in projective (or affine) space.

It is interesting now to observe that if we consider a set of points (consisting of $n+1$ or more points) and a family of subsets called blocks which satisfies only the first of the above mentioned 4 postulates, that is;

P_a1 Any $n+1$ distinct elements (points) of S are contained in exactly one block and every block contains at least $(n+1)$ distinct elements.

Def. Such a collection of subsets (blocks) of S is called a partition of type $n+1$ $(n \geq 0)$ on the set S.

Thus every Wille geometry of grade n is a partition of type $n+1$. A partition of type 1 is an equivalence relation on S. The lattice of such partitions was extensively studied by O. Ore (1942) and later by D. Sachs (1961). Partitions of type n $(n \geq 2)$ were studied by J. Hartmanis 1951 - 61.

It is intended to give a kind of combined treatment of partitions and geometries in the following.

§ 2 Geometries in terms of closure operations.

In the discussion of the incidence relations of the simplest figures - linear subspaces - of geometries, we have to deal with a function which assign a subspace to a subset of points. For example, A1 assigns a line to a set of points consisting of two distinct points, and A2 assigns a plane to a set of points consisting of three distinct points which are not on a line. Since the set union of two subspaces is not a subspace, so we also have to assign a subspace to it. Thus we assign to a subset A the smallest subspace $\overline{A}$ which contains A. This function satisfies obviously

1) $A \subseteq \overline{A}$

and A2′ jirstifies to introduce the following postulate:

2′) $A \subseteq \overline{B}$ implies $\overline{A} \subseteq \overline{B}$.

These two postulates imply the following:

2) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$

3) $\overline{A} = \overline{\overline{A}}$.

Thus the operation $A \to \overline{A}$ is a closure operation, since 1), 2), 3) are satisfied.

For finite dimensional geometries, we can add as in the book of Crapo and Rota, the following postulate on finite basis:

Any subset $A \subseteq S$ has a finite subset $A_f \subseteq A$ such that $\overline{A_f} = \overline{A}$.

But, since we want to study geometries without the restriction of finite dimension, we have to replace the postulate on finite basis by a weaker one.

To formulate such a postulate for geometries, we need some more difinitions:

Def. A property of subsets of a set S is called a closure property if (i) S has the property

(ii) Any intersection of subsets having the given property itself has this property.

Def. The property of the subset A that $A = \bar{A}$ is a closure property which is called the closure property associated with a closure operation.

Def. A closure property Φ associated with a closure operation $A \to \bar{A}$ on the subsets of a set S is called finitary when the condition $A \in \Phi$ is equivalent to the condition that $K \subseteq A$ and K finite imply $\bar{K} \subset A$.

Prop. The condition that the closure property Φ associated with the closure operation is finitary is equivalent to the condition that $\bar{S} = \cup \bar{K}_y$ is the set union of the closure $\bar{K}_y$ of the finite subsets K_y of S.

This can be proved by using the following lemma 1:

Lemma 1. For any finitary closure operation, if a set D of closed subsets S_δ is directed by set-inclusion, then their set-union $\cup_D S_\delta$ is closed.

Here a directed set is defined by

Def. A directed set is a poset in which any two elements, and hence any finite subset of elements has an upper bound in the set.

The following lemma is also useful:

Lemma 2. If for any directed set D of closed subsets S_δ (directed by set-inclusion) the set union $\bigcup_D S_\delta$ is closed, then Φ is finitary.

Examples (of finitary closure property): (1) Being a subalgebra of a (finitary) algebra A. Here an algebra A is a pair $[S,F]$ where S is a non-empty set of elements and F is a specified set of operations f_α, each mapping a power $S^{n(\alpha)}$ of S into S, for some appropriate non-negative finite integer $n(\alpha)$. That is, f_α assigns to every $n(\alpha)$-ple $(x_1,\cdots mx_{n(\alpha)})$ of elements fo S a value $f\,(x_1,\cdots,x_{n(\alpha)})$ in S.

By a subalgebra of an abstract algebra $A = [S,F]$, it is meant a subset T (passibly void, but if A contains a least non-void subalgebra the void set is not considered to be a subalgebra) of S which is closed under the operations of F, or F-closed. That is if $f_\alpha \in F$ and $x_1,\cdots,x_{n(\alpha)} \in T$, then $f_\alpha(x_1,\cdots,x_{n(\alpha)}) \in T$.

That being a subalgebra of an algebra A is a finitary closure property follows from lemma 2.

(2) Being a subrelative of a relative. A subset R of k-tuples of M is called a relation. For $\langle x_1,\cdots,x_k\rangle \in R$ we write simply $R_{x_1:\cdots x_k}$ ("$x_1,\cdots,x_k$" stand in the relation R).

M' M is said to be closed with respect to a (k+1)-places relation R, if $x_1,\cdots,x_k \in M'$ and $R_{x_1\cdots x_k x}$ in M then $x \in M'$.

A relative is a set m with a well-ordered set of relations R_α which are of $n(\alpha)$ place. A subrelative is defined to be a subset of m which is closed with respect to all relations of m.

To each k-arguments operation ϕ of m we can assign a (k+1)-places relation R in the following way:

$$\phi(x_1,\cdots,x_k) = x \rightarrow R_{x_1\cdots x_k x}.$$

Conversely, to each (k+1)-places relation R we can assign a k-arguments operation ϕ as follows:

$$\phi(\xi_1,\cdots,\ {}_k) = \begin{cases} \eta & \text{if } R_{\xi_1\cdots\xi_k\eta} \\ \xi_1 & \text{otherwise.} \end{cases}$$

Then a subset is closed with respect to ϕ if and only if it is closed with respect to R.

E. Masda had defined (1951) the abstract geometry with finitary operation as follows:

Def. Let G be a set of points. If for any finite points $p_1,\cdots,p_n$ of G, there exists a subset $p_1 +\cdots+ p_n$ of G containing p_i $(i = 1,2,\cdots,n)$ which satisfies

(1°) $p_1 = p_2$ implies $p_1+p_2+\cdots+p_n = p_2+\cdots+p_n$,

(2°) for any permutation $p_{i_1},\cdots,p_{i_n}$ of $p_1,\cdots,p_n$,

$$p+\cdots+p_n = p_{i_1}+\cdots+p_{i_n},$$

(3°) $q_i \in p_1^{(i)}+\cdots+p_{n_i}^{(i)}$ $(i = 1,\cdots,m)$ imply

$$q_1+\cdots+q_m \subseteq p_1^{(1)}+\cdots+p_{n_1}^{(1)}+p_1^{(2)}+\cdots+p_1^{(m)}+\cdots+p_{n_m}^{(m)}.$$

Then G is called an abstract geometry with finitary operation.

Def. A subset H of G is called a subgeometry if $p_1,\cdots,p_n \in H$ imply $p_1+\cdots+p_n \subseteq H$.

It follows that $p_1+\cdots+p_n$ is a subgeometry.

In an abstract geometry G with finite operation, we can define a closure operation as follows:

Let B be any subset of G, then define $\overline{B}$ to be the smallest subgeometry containing B. Then $B \to \overline{B}$ is a closure operation. Its associated closure property is finitary, since $B = \overline{B}$ means that B is a subgeometry, and B is a subgeometry if and only if $K = \{p_1,\cdots,p_n\} \subset B$ implies $\overline{K} = p+\cdots+p_n \subseteq B$.

This definition has an ambiguity that if the number of points $p_1,\cdots,p_n$ of G reduces to only one point (i.e., n = 1) then whether the subset p_1 assigined to $\{p_1\}$ is $\{p_1\}$ or not?

It may be natural to assume so. If we understand so, then $\{\bar{p}\} = \{p\}$ or writing simply $\bar{p} = p$. But we can not prove $\bar{p} = p$ just from only the given conditions for an abstract georetry.

If we consider the set G of all vectors in a vector space (not necessarily be finite dimensional), then for a set of vectors $\{v_1, \cdots, v_k\}$, it is quite natural to define $v_1 + \cdots + v_k$ to be the linear subspace spanned by $\{v_1, \cdots, v_k\}$. Then naturally, for $k = 1$, v_1 should represent the linear subspace spanned by v_1. Then obviously (1°), (2°) and (3°) are satisfied. Thus G is an abstract geometry with finitary operation. This shows that $\{\bar{p}\} = \{p\}$ can bot be proved only from the conditions of an abs abstract geometry. Actually, in this case, a subgeometry i a linear subspace, and $\{\bar{v}\}$ = (the vector space spanned by v) $\neq \{v\}$ if $v \neq 0$ but $\{\bar{0}\} = \{0\}$. And the least linear subspace containing ϕ is $\{0\}$ os $\bar{\phi} = \{0\} \neq \phi$.

Conversely, if $\{\bar{p}\} = \{p\}$ is not assumed for an abstract geometry, a set m with a closure operation whose associated closure property is finitary is an abstract geometry with finitary operation.

Jonnson (1959) gave the definition of geometry in terms of closure operation as follows:

<u>Def.</u> By a geometry we mean an ordered pair $\langle S,C \rangle$ consiting of a set S (whose elements are called points) and a function C (called the closure) which associates with every subset X of S another subset C(X) of S in such a way that the following condition are satisfied:

(i) $X \subset C(X) = C(C(X))$ for every subset X of S,

(ii) $C(p) = p$, i.e., $C(\{p\}) = \{p\}$ for every point of S,

(iii) $C(\phi) = \phi$,

(iv) For every subset X of S, C(X) is the union of all subsets of the form C(Y) with Y a finite subset of X.

From (iv) it follows that

(iv') If $X \subseteq Y \subseteq X$, then $C(X) \subseteq C(Y)$.

The condition (i) and (iv') show that $C : X \to C(X)$ is acthally a closure operation. So we will also use $\overline{X}$ instead of C(X) in the sequel. (iv) means, by above proposition, that the closure property associated with this closure operation is <u>finitary</u>. Obviously (iv) is weaker than the postulate of finite basis. This is a nice difinition of geometry, because it is related to so many topics in other branches mathematics.

Def. Suppose $\langle S,C\rangle$ is a geometry.

(i) An element of S is called a point of the geometry $\langle S,C\rangle$.

(ii) A set of the form $C(X)$ with $X \subseteq S$ is called a subspace of $\langle S,C\rangle$. If $Y = C(X)$ then Y is said to be spanned by X.

(iii) A subspace of $\langle S,C\rangle$ is said to be n-dimensional if it is spanned by a set with $n+1$ elements but is not spanned by any set with fewer than $n+1$ elements.

We are now in the position to show that Wille geometry is a geometry in this sense.

For any subset A in the Wille geometry, take $\bar{A}$ to be the least subspace which contains A. It is obvious that $A \to \bar{A}$ is a closure operation. The closure property associated with this closure operation can be shown to be finitary by lemma 2: Suppose that D is any directed family of subspaces S_δ, then it is easy to see that their ser-union $\bigcup_D S_\delta$ is also a subspace. Thus (iv) is satisfied by the above proposition. Since any set of k $(0 \leqslant k \leqslant n)$ distinct points can be seen as a subspace in Wille geometry, the set of a single point is in particular a subspace so $\bar{p} = p$. And obviously $\bar{\phi} = \phi$.

It can be shown similarly that a partition of type $n + 1$ on a set S is also a geometry in this sense. We define a subspace to be subset with the property that if $n + 1$ distinct points are contained in the set, the block determined by them is contained in the set. Then define closure as in Wille geometry.

§ 3. Tieing up of geometries and lattices.

We are now interested in the set $\mathcal{L}(G(S))$ of all subspaces of a geometry which is obviously a poset with respect to set inclusion. It is a complete lattice since every subset A of $\mathcal{L}(G(S))$ has an inf A (that is the set intersection $\cap A$).

<u>Def.</u> A subset A of a complete lattice V is called a <u>hull-system</u> in V, if for each subset M of A the inf M formed in V is also an element of A.

Hull-system is a complete lattice and the universe element of V also belongsto the hull-system.

Suppose that $G = \langle S,C \rangle$ is a geometry. Since the power set $P(S)$ of S is a complete lattice with respect to set-inclusion and set-union, and for any subset M of $\mathcal{L}(G(s))$ the inf $M = \cap M$ (set-intersection) which is also a subspace, so $\mathcal{L}(G(S))$ is a hull-system in the complete lattice $P(S)$.

For any two posets H_1, H_2, a mapping $\phi : H_1 \rightarrow H_2$ is called an order homomorphism if $a \leq b$ implies $\phi(a) \leq \phi(b)$. An endomorphism is an order homomorphism from a poset into itself.

<u>Def.</u> An endomorphism $\tau : V \rightarrow V$ of a complete lattice V is called a <u>hull-operation</u> if for each element $x \in V$ the following two conditions are satisfied:

$$x \leq \tau(x),$$
$$\tau(\tau(x)) \leq \tau(x).$$

Cor. These two conditions simply that $\tau(x) = \tau(\tau(x))$.

Remark. $x \leq y$ implies $\tau(x) \leq \tau(y)$, since τ is an endomorphism.

Thus a hull-operation of the complete lattice $P(S)$ is actually a closure operation.

Theorem 3.1. The set of all elements of a complete lattice V (with operation $\cdot$ and $+$) which are closed with respect to a hull-operation (i.e., $\tau(x) = x$) is a hull-system which is a complete lattice having $\tau(\Sigma x_\alpha)$ as its lattice union. Conversely, to every hull-system A in a complete lattice V, there corresponds a hull-operation such that the given system is the set of all closed elements with respect to the hull-operation obtained.

Proof of Theorem 3.1.

1st part. Let A be the set of elements x of V (with operations $\cdot$ and $+$) such that $x = \tau(x)$. It is to be shown first that A is a hull system. Let M be any subset of A, and $x_\alpha \in M$. Since $\pi_M x_\alpha \leq x_\alpha$, so $\tau(\pi_M x_\alpha) \leq \tau(x_\alpha) = x_\alpha$ for every x_α of M. Thus $\tau(\pi_M x_\alpha) \leq \pi_M x_\alpha$, from which and $\tau(\pi_M x_\alpha) \geq \pi_M x_\alpha$ it follows that $\tau(\pi_M x_\alpha) = \pi_M x_\alpha$. To show that $\tau(\Sigma_M x_\alpha)$ is the lattice join of the complete lattice A, let y be any upper bound of M in A. Then $x_\alpha \leq y$, so $\Sigma_M x_\alpha \leq y$ and $\tau(\Sigma_M x_\alpha) \leq \tau(y) = y$. On the other hand, $x_\alpha \leq \Sigma_M x_\alpha$, hence

$x_\alpha = \tau(x_\alpha) \leq \tau(\Sigma_M x_\alpha)$ which is in A. So $\tau(\Sigma_M x_\alpha)$ is the least upper bound of M in A.

<u>2nd part</u>. Let A be a hull-system. For any $x \in V$, define $\tau(x) = \pi y$ for all $y \in A$ any $y \geq x$. Then $\tau(x) \in A$, since A is a hull-system. Define the mapping $x \to \tau(x)$ of V into itself. Obviously $x \leq \tau(x)$ and $\tau(\tau(x)) = \tau(x)$ (since $\tau(x) \in A$) so the mapping $x \to \tau(x)$ is a hull-operation. By the definition of $\tau(x)$, $x = \tau(x)$ if and only if $x \in A$. Thus A is the set of all closed element under the hull-operation $x \to \tau(x)$. Q. E. D.

Seeing L(G(S)) as a hull-system of P(S), the corresponding hull-operation asserted in the above theorem is just the closure operation defined in the previous section.

It is now obvious that lemmas 1 and 2 of the previous section can be put together to give the following:

<u>Theorem 3.2</u>. Let A be a hull-system of P(S). Then the following two conditions are equivalent:

a) The closure property associated with the closure-operation (hull-operation) corresponding to A (as asserted in previous theorem) is finitary.

b). For each directed set M contained in A, the set union $\cup M$ is also contained in A.

In this theorem it is characterized when the closure property associated with a closure operation is finitary. To charactrize a hull-system A of P(S) with such "finitary property" lattice theoretically we read some more definitions:

Def. An element "a" of a poset H is said to be accessible if there is a directed subset A of H such that $a \notin A$ and $\Sigma A = a$. Otherwise "a" is called inaccessible.

Theorem 3.3. The following three conditions for a lattice V are necessary and sufficient for the existence of a set S and a hull-operation on P(S) whose associate closure property is finitary so that V is isomorphic to the lattice A of all subsets of S which are closed under the hull-operation:

(1) V is complete.

(2) $x(\Sigma y_\rho) = \Sigma\Sigma x y_\rho$ for each $x \in V$ and every directed set $\{y_\rho\}$ of elements of V.

(3) Every element of V is the (lattice) join of a set of inaccessible elements.

Def. A complete lattice satisfying the condition (2) of the above theorem is said to be ypper-continuous (or meet-continuous).

Proof of Theorem 3.3. Suppose first that V is isomorphic to the lattice A of all subsets of S which are closed under the hull-operation of P(S) whose associate closure property is finitary.

As shown in the first part of Theorem 3.1., A is a hull-system, so it is a complete lattice. Thus (1) is proved.

By Theorem 3.2., for each directed set $M \subset A$, the set union $\cup M$ is also contained in A, that is, $\cup M$ is closed. Thus $\cup M = \Sigma M$ the lattice join obtained in A. Let $M = \{y_\rho\}$, then $\Sigma_M y_\rho = \cup_M y_\rho$. Let $\alpha \in S$, and x be any element of A. Since A is a hull-system of $P(S)$, each element of A is a subset of S and the operation $\cdot$ in A is actually the set-intersection $\cap$. Thus

$$\alpha \in x(\Sigma_M y_\rho) \rightarrow \alpha \in x \text{ and } \alpha \in \Sigma_M y_\rho = \cup_M y_\rho$$

$$\rightarrow \alpha \in x \text{ and } \alpha \in y_\rho \text{ for a certain } y_\rho$$

$$\rightarrow \alpha \in x \cap y_\rho = xy_\rho$$

$$\rightarrow \alpha \in \cup_M(xy_\rho) = \Sigma_M(xy_\rho),$$

since $\{(xy_\rho)\}$ is also a directed set contained in A, so $\cup_M(xy_\rho) = \Sigma_M(xy_\rho)$ by Theorem 3.2 again. Thus $x(\Sigma_M y_\rho) \subseteq \Sigma_M(xy_\rho)$, hence $x(\Sigma_M y_\rho) \leqslant \Sigma_M(xy_\rho)$, since in A order relation $\leqslant$ is actually the set-inclusion. On the other hand, for any complete lattice A, $y_\rho \leqslant \Sigma_M y_\rho$ and hence $xy_\rho \leqslant x(\Sigma_M y_\rho)$, form which it follows $\Sigma_M(xy_\rho) \leqslant x(\Sigma_M y_\rho)$. Thus $x(\Sigma_M y_\rho) = \Sigma_M(xy_\rho)$ and hence (2) is proved.

To show (3), consider the set $\overline{\{\alpha\}}$, where $\bar{y}$ denotes the smallest element of A which contains $y \in P(S)$. It can be shown that for any element x of A, $x = \Sigma\overline{\{\alpha\}}$ for all α contained in x: For each $\alpha \in x$, $\alpha \leq x$ and $\overline{\{\alpha\}} \leq \overline{x} = x$, hence $\Sigma\overline{\{\alpha\}} \leq x$. On the other hand, $\Sigma\overline{\alpha}$, as a set, contains every point of x, thus $\Sigma\overline{\alpha} \supseteq x$, that is, $\Sigma\overline{\{\alpha\}} \geq x$. Thus $x = \Sigma\overline{\{\alpha\}}$. It remains

now to be shown
now to be shown that $\overline{\{\alpha\}}$ is inaccessible. Assume that $\overline{\{\alpha\}} = \Sigma y_\rho$ for a directed set of elements $y_\rho \in A$. As shown above $\Sigma y_\rho = \cup y_\rho$. Now $\alpha \in \overline{\{\alpha\}}$ implies that $\alpha \in \cup y_\rho$, so there is a y_ρ such that $\alpha \in y_\rho$, i.e., $\alpha \leq y_\rho$. Then $\overline{\{\alpha\}} \leq \overline{y}_\rho = y_\rho$. On the other hand $\overline{\{\alpha\}} = \cup y_\rho \supseteq y_\rho$ i.e., $\overline{\{\alpha\}} \geq y_\rho$. Hence $y_\rho = \overline{\{\alpha\}}$ and $\overline{\{\alpha\}}$ is inaccessible.

Conversely, suppose that a lattice V satisfies the conditions (1), (2), (3). Define S to be the set of all inaccessible elements u in V. For any subset $M \subseteq S$, define $\overline{M} = \{u : \text{inaccessible} \mid u \leq \Sigma M\}$. Then $M \to \overline{M}$ is a hull-operation:

(i) If $u \in M$, then $u \leqslant \Sigma M$. Hence $u \in \overline{M}$ and $M \subseteq \overline{M}$.

(ii) If $M_1 \subseteq M_2$ then $\Sigma M_1 \leqslant \Sigma M_2$. Hence $\overline{M}_1 \subseteq \overline{M}_2$.

(iii) Be definition $\overline{\overline{M}} = \left\{ \nu\text{:inaccessible } v \leqslant \Sigma\overline{M} \right\}$. But, if $u \in \overline{M}$, then $u \leqslant \Sigma M$, hence $\Sigma\overline{M} \leqslant \Sigma M$. On the other hand $\overline{M} \supseteq M$, so $\Sigma\overline{M} \geqslant \Sigma M$. Thus $\overline{\Sigma M} = \Sigma M$ and $\overline{\overline{M}} = \overline{M}$.

It can be shown further that the closure property associated with the closure operation $M \to \overline{M}$ is finitary.

For this, it suffices to show that if $K = \left\{ u_1, \cdots, u_n \right\} \subset M$ implies $\overline{K} \subseteq M$, then $M = \overline{M}$. That is, to show that if $u \leqslant u_1 + \cdots + u_n$ implies $u \in M$ for $u_1, \cdots, u_n \in M$, then $M = \overline{M}$. To show $M = \overline{M}$ we need only to show $M \supseteq \overline{M}$. Let $u \notin \overline{M}$ i.e., $u \leqslant \Sigma M$. It suffices now to show that there exist $u_1, \cdots, u_n$ in M such that $u \leqslant u_1 + \cdots + u_n$ (then $u \in M$ by the assumption that $\overline{K} \subseteq M$). Consider the set $\left\{ y_\rho \right\}$ of all lattice joins y_ρ of finite elements of M. Then the set $\left\{ y_\rho \right\}$ is a directed set, and since $y_\rho = \Sigma u_i \leqslant \Sigma M$, so $\Sigma y_\rho \leqslant \Sigma M$. On the other hand, each element u of M is contained in a y_ρ (i.e., $u \leqslant y_\rho$), so $\Sigma M \leqslant \Sigma y_\rho$, Thus $\Sigma M = \Sigma y_\rho$. Since $u \leqslant \Sigma M$, we have $u = u \cdot (\Sigma M) = u \cdot (\Sigma y_\rho) = \Sigma (u y_\rho)$ by (2). Since $\left\{ u y_\rho \right\}$ is also a directed set and u is inaccessible, there is a (uy_ρ) such that $u = uy_\rho$ that is, $u \leq y_\rho$. This means that there exist $u_1, \cdots, u_n$ such that $u \leq u_1 + \cdots + u_n = y_\rho$.

Let A be the set of all subsets of S which are closed under the above hull-operation $M \to \overline{M}$. Now, to each element $a \in V$ we assign a set $\phi(a)$ of inaccessible elements by $\phi(a) = \left\{u\text{:inaccessible} \mid u \leq a\right\}$. We claim that this correspondence ϕ is an isomorphism between V and A. For this it suffices to show:

(a) ϕ is one-to-one,

(b) $\phi(ab) = \phi(a) \quad \phi(b)$,

(c) for each a, $\phi(a)$ is a closed subset of S with respect to the hull-operation $M \to \overline{M}$,

(d) to each closed subset M' of S, there is an element $a \in V$ such that $\phi(a) = M'$.

(e) If $M'_1 \quad M'_2$ are closed, then $\phi^{-1}(M'_1) \leq \phi^{-1}(M'_2)$.

(c) Since $\phi(a) = \left\{u \mid u \leq a\right\}$, $\Sigma\phi(a) \leq a$ which implies that

$\overline{\phi(a)} = \left\{u \mid u \leq \Sigma\phi(a) \leq a\right\} \quad \phi(a)$. Thus $\phi(a) = \overline{\phi(a)}$.

(a) Since a is a lattice join of some inaccessible elements under a which are actually contained in $\phi(a)$, we have $a \leq \Sigma\phi(a)$. Now, if $\phi(a) = \phi(b)$, then $a = \phi(a) = \phi(b) = b$.

(b) In any lattice, $u \leq ab \leftrightarrow u \leq a$ and $u \leq b$

$\leftrightarrow u \in \phi(a)$ and $u \in \phi(b)$

$\leftrightarrow u \in \phi(a) \cap \phi(b)$.

That is, $u \in \phi(ab)$ if and only if $u \in \phi(a) \cap \phi(b)$. Hence $\phi(ab) = \phi(a) \cap \phi(b)$.

(d) Let M' be any closed subset of S, and let $a = \Sigma M'$. Then $M' = \overline{M'} = \{u \mid u \leq \Sigma M' = a\} = \phi(a)$ by definition of ϕ.

(e) Let M'_1, M'_2 by closed subsets of S with $M'_1 \subseteq M'_2$. Let $a_1 = \Sigma M'_1$ and $a_2 = \Sigma M'_2$, then $M'_1 = \phi(a_1)$ and $M'_2 = \phi(a_2)$ by d). Then $\phi^{-1}(M'_1) = a_1 = \Sigma M'_1 \leq \Sigma M'_2 = a_2 = \phi^{-1}(M'_2)$, i.e., $\phi^{-1}(M'_1) \leq \phi^{-1}(M'_2)$. Q.E.D.

This theorem can be seen as a version in abstract geometry of the following Birkhoff-Frink's theorem on lattice of subalgebras.

<u>Theorem</u>. A lattice L is isomorphic with a subalgebra-lattice if and only if L is complete, meet continuous, and every element of L is a join of join-inaccessible elements.

Apply the above version (Theorem 3.3.) of Birkhooff-Frink's theorem to a geometry in the sense of Jonnson's definition (also called merely finitary geometry) by taking $\overline{p} = p$ into acount, we get the following:

<u>Corollary</u>. Tha lattice of flats $(A = \overline{A})$ of a merely finitary geometry <mC> is complete, upper-continuous and atomistic.

<u>Def.</u> A lattice L is said to be atomistic (or point, or relatively atomic) if L has 0 and every element a $(\neq 0)$ of L is the join of its contained points.

Conversely, if a lattice $\mathcal{L}$ is complete, upper-continuous and atomistic, then since an atom in such a lattice is inaccessible, it is isomorphic to the lattice $\mathcal{L}'$ of subsets of m' consisting of all inaccessible elements of $\mathcal{L}$ which are closed under the closure operation $C' : B \to \overline{B}' = \{u \mid u\text{:inaccessible and } u \leq \Sigma B\}$ with finitary closure property.

If u is inaccessible and $B = \{u\}$, then $\overline{B}' = \{v \mid v\text{:inaccessible and } v \leq \Sigma B = u\}$. Thus if u is not an atom, then it is possible that $\overline{B}' \neq B$ (i.e., $\overline{u}' \neq u$). For example, if $\mathcal{L}$ is the lattice of subspaces (flats) of a projetive geometry and u is a line (it is obviously inaccessible), then for $B = \{u\}$, $\overline{B}'$ = the set of all points on the line plus the line itself, so $B \subsetneq \overline{B}'$. But if $B = \{v \mid v\text{:inaccessible and } v \leq u\}$ then $\overline{B}' = \{v \mid v\text{:accessible and } v \leq \Sigma B = u\}$, so $B = \overline{B}'$. Thus $\langle m'C' \rangle$ is not a merely finitary geometry.

If m is the set of all atoms in $\mathcal{L}$ and $C : A \to \overline{A} = \{p \mid p\text{:atom and } p \leq \Sigma A\}$ for $A \subseteq m$, then $\langle mC \rangle$ is a merely finitary geometry.

This can be easily seen if in the corresponding part of the proof of Theorem 3.3 we make the following changes:

m = the set of all atoms p in

$\overline{S} = \left\{ p \mid p\text{:atom and } p \leq \Sigma s \right\}$ for s m

and replace inaccessible elements $u, u_1, \cdots, u_n$ everywhere by p points $p, p_1, \cdots, p_n$ respectively. It is obvious from this definition of $\overline{S}$ that

$$\overline{\{p\}} = \{ q \mid q\text{:atom and } q \leq \Sigma p = p \} = \{p\}$$

and
$$\overline{\phi} = \{ p \mid p\text{:atom and } p \leq \Sigma \phi = \phi \} = \phi.$$

Continuing the same changes and also changeing $\phi(a)$ as $\phi(a) = \{ p \mid p\text{:atom and } p \leq a \}$ in the part of proof of isomorphism in Theorem 3., we can obtain the following:

<u>Theorem 3.4</u>. A lattice is geometric if and only if it is complete, upper-continuous and atomistic.

In the statement of this theorem, a geometric lattice means the lattice which is isomorphic to the lattice of all subspaces of some merely finitary geometry.

<u>Remark</u>: F. Maeda has proved the following:

Theorem: The set of all subgeometries of an abstract geometry G with finitary operation, is a "relatively atomic", upper-continuous lattice.

In this proof of the fact that the lattice is "relatively atomic", a point p is considered to be a subgeometry, but this is not always the case as the above example (p. 14) shows. Actually this theorem must be modified as the Theorem 3.3.

<u>Remark</u>. Before giving an example, we note the following fact which will be used in the example and also frequently in the sequel:

If $\langle mC' \rangle$ is another merely finitary geometry, then $\langle mC' \rangle = \langle mC \rangle$ if and only if $C(K) = C'(K)$ for all finite subsets K of m. Since $\langle mC \rangle$ is a merely finitary geometry, so $C(S) = \cup C(K_\gamma)$ for all finite subset K_γ of S, but since $C(K_\gamma) = C'(K_\gamma)$ so $C(S) = \cup C(K_\gamma) = \cup C'(K_\gamma) = C'(S)$. This fact will be used often in the sequel.

<u>Examples</u>. Modifying the linear geometry of Prenowitz, Maeda defined the generalized linear geometry as follows:

Let G be a set of points, in which a 3-term relation (pqr) called <u>order</u> is defined such that

1. If (p,q,r) then p, q, r are distinct.
2. If (pqr) then (rqp).
3. If $x \in y+r$ and $y \in p+q$, then there exists a point z such that $x \in p+z$ and $z \in q+r$.

Then G is called a <u>generalized linear geometry</u>.

In the postulate 3, we denote

$$p+q = \begin{cases} \{x \in G \mid (pxq)\} & \text{if } p \neq q \\ p & \text{if } p = q. \end{cases}$$

Cor. Postulate (3) implies that if (yxq) and (pyq) then (pxq).

From the above definition we have

1° $p+p = p$ and

2° $p+q = q+p$.

Def. Let X, Y be any two subsets of G, then we define

$$X+Y = \begin{cases} \cup\{x+y \text{ point } x \in X \text{ and point } y \in Y\} & \text{if } X \neq \phi, y \neq \phi \\ X & \text{if } Y = \phi, \text{ and} \\ Y & \text{if } X = \phi. \end{cases}$$

3° $X+Y = Y+X$.

4° $p+(q+r) = (p+q)+r$.

Proof. If $x \in (p+q)+r$, then there exists $y \in p+q$ such that $x \in y+r$. Then, by postulate 3, there exists a point z such that $x \in p+z$ and $z \in q+r$. Thus $x \in p+(q+r)$, hence $(p+q)+r \subseteq p+(q+r)$. Similarly, $(r+q)+p \subseteq r+(q+p)$ and hence $p+(q+r) = (p+q)+r$.

5° For any subsets X, Y, Z of G

$$X+(Y+Z) = (X+Y)+Z.$$

Def. A subset A of G is called a convex set or an additively closed set of G if $p, q \in A$ implies $p+q \subset A$.

6° X is a convex set if and only if $X+X = X$.

7° If both X and Y are convex, then $X + Y$ is also convex.

Since $p+p = p$, so p is convex, thus $p + q$ is convex by 7°.

If for any subset B of G, we define $\overline{B}$ to be the least convex set which contains B, then $B \to \overline{B}$ is a closure operation and the closure property associated with this closure operation is finitary, since the set union of any directed family of convex sets is also a convex set. Since $\{p\}$ is a convex set so $\overline{p} = p$ and obviously $\overline{\phi} = \phi$.

Thus, by the theorem 3.4, the set of all convex subsets of a generalized linear geometry is a complete, upper-continuous and atomistic lattice which is also linear in the following sense:

<u>Def.</u> A lattice L with 0 is called a <u>linear lattice</u> when a point $p \leqslant a+b$, then there exist points $q \leqslant a$ and $r \leqslant b$ such that $p \leqslant q+r$.

By cor. 7°, for any two convex sets X, Y, $X+Y$ is convex so $X + Y$ is the least convex set containing X, Y. Thus $X + Y$ is the "lattice join" of X and Y, from which it follows that the lattice of convex sets is linear.

Conversely, let the lattice L be a complete, relatively atomic, upper-continuous and linear. Denote by G_L the set of all atoms of L.

Then by Theorem 3.4., L is isomorphic to the lattice of flats $B = \overline{B}$ of the merely finitary geometry $\langle G_L C\rangle$ where

$$C : B \to \overline{B} = \{p \mid p\text{:atom and } p \leqslant \Sigma B\}.$$

On the other hand, if we define in G_L the order relation (p,q,r) to mean that $q \leqslant \overline{p+q}$ (the lattice join) and p, q, r

are distinct, then G_L is a generalized linear geometry.

Postulate 1 follows from the definition of (p,q,r). Postulate 2 follows from the definition of (p,q,r) and the fact that $p+q = q+p$ (the lattice join). To show that postulate 3 holds, we note first that the lattice join

$$p+r = \begin{cases} \{q \mid (p,q,r)\} & \text{if } p \neq r, \text{ by the definition of } (p,q,r); \\ p & \text{if } p = r. \end{cases}$$

Thus the lattice join $p + r$ coincides with the definition of $p + r$ used in the statement of postulate 3.

For the lattice join, from $x \leqslant y+r$ and $y \leqslant p+q$, it follows that $x \leqslant (p+q)+r = p+(q+r)$. Since L is linear, $x \leqslant p+(q+r)$ implies that there is a point $z \leqslant q+r$ such that $x \leqslant p+z$. Thus postulate (3) holds.

Since G_L is a generalized linear geometry with respect to the order relation defined above, if the closure operation C' is defined by

$C' : B \rightarrow \overline{B}'$ = the least convex set of the generalized linear geometry containing the set B,

then $\langle G_L C' \rangle$ is a merely finitary geometry as shown above.

It can be shown now that $\langle G_L C' \rangle = \langle G_L C \rangle$ by using the remark of page 28. For this aim, we need only to show that $C'(\{p_1, \cdots, p_n\}) = C(\{p_1, \cdots, p_n\})$, that is, the least convex set containing $p_1, \cdots, p_n$

is the set of atom p such that $p \leq p_1+\cdots+p_n$.

$C(\{p_1,\cdots,p_n\}) \subseteq C'(\{p_1,\cdots,p_n\})$ is shown by induction:

The least convex set $C'(\{p_1,p_2\})$ contains p_1+p_2 which is itself a convex set, since $p_1', p_2' \in p_1+p_2$ (i.e. $p_i' \leq p_1+p_2$) imply $p_1'+p_2' \leq p_1+p_2$ (i.e. $p_1'+p_2' \subset p_1+p_2$). Thus $C(\{p_1,p_2\}) = C'(\{p_1 p_2\})$ and the inclusion relation holds for $n = 2$.

Assuming that the inclusion relation holds for $n-1$. Let $p \leq p_1+\cdots+p_n$. If $p \leq p_1+\cdots+p_{n-1}$, then by induction assumption, $p \in C'(\{p_1,\cdots,p_{n-1}\}) \subset C'(\{p_1,\cdots,p_{n-1},p_n\})$. If $p \not\leq p_1+\cdots+p_{n-1}$, then since L is linear, $p \leq q+p_n$ with a point $q \leq p_1+\cdots+p_{n-1}$ and hence $q \in C'(\{p_1,\cdots,p_{n-1}\}) \subset C'(\{p_1,\cdots,p_{n-1},p_n\})$ by induction assumption. Thus $p \in C'(\{p_1,\cdots,p_n\})$ and $C(\{p_1,\cdots,p_n\}) \subseteq C'(\{p_1,\cdots,p_n\})$. On the other hand, since $p_1 + p_2 +\cdots+ p_n$ is a convex set containing $\{p_1,\cdots,p_n\}$ (since $q,r \in p_1+\cdots+p_n$ implies $q+r \in p_1+\cdots+p_n$ which means $q+r \subseteq p_1+\cdots+p_m$), $C(\{p_1,\cdots,p_n\}) \supseteq C'(\{p_1,\cdots,p_n\})$ by the definition of C'.

Thus the lattice of flats of the merely finitary geometry $\langle G_L C\rangle$ consists of all the convex sets of the generalized linear geometry defined above, so by Theorem 3.4, we obtain the following:

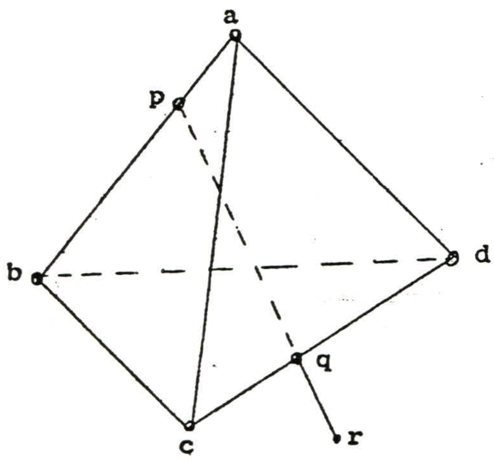
Theorem 3.5. For a lattice L to be isomorphic to the lattice of all convex subsets of a generalized linear geometry, it is necessary and sufficient that the lattice L is complete, upper-continuous, atomistic and linear.

Remark. In a vector space V over the field of real numbers (not necessarily be finite dimensional), for any two elements $p, q \in V$, let us define $p+q = \left\{x\in V \mid x=\alpha p+\beta q \text{ with } \alpha+\beta=1; 0\leqslant\alpha,\beta\leqslant 1\right\}$ and define (pxq) to mean that $x \leqslant p+q$ and p, q, x are distinct. Then postulate 1 and 2 are easily seen to hold. For postulate 3, let $x = \alpha y+\beta r$ with $\alpha+\beta = 1$, $y = \alpha' p+\beta' q$ with $\alpha'+\beta' = 1$. Then

$$x = \alpha y+(1-\alpha)r = \alpha\left\{\alpha' p+(1-\alpha')q\right\}+(1-\alpha)r = \alpha\alpha' p+(1-\alpha\alpha')\left\{\frac{\alpha(1-\alpha')}{1-\alpha\alpha'}q+\frac{1-\alpha}{1-\alpha\alpha'}r\right\}$$

if $\alpha\alpha'-1 \neq 0$. Thus $z = \frac{\alpha(1-\alpha')}{1-\alpha\alpha'}q+\frac{1-\alpha}{1-\alpha\alpha'}r$ makes postulate 3 (if $\alpha\alpha' = 1$, any point $z = \alpha'' q+\beta'' r$, with $\alpha''+\beta'' = 1$ will do) hold. In such a generalized linear geometry, an additively closed set is actually a convex set in the usual sense. Thus for distinct points a, b, c, d, $a+b$ is a segment, $a+b+c$ is a triangular region and $a+b+c+d$ is a solid tetrahedron.

As shown in the figure, let a, b, c, d be vertices of a tetrahedron, p be a point on the edge ab, q be on the edge cd



and p, q, r are collinear.

Then

$$a \cdot b = 0$$

$$(a+b) \cdot c = 0$$

$$(a+b+c) \cdot d = 0$$

$$(a+b+c+d) \cdot r = 0$$

but

$$(r+a+b) \cdot (c+d) = q \neq 0$$

where + and • are lattice operations. This shows that the lattice of convex sets of such a generalized linear geometry is <u>not</u> <u>semi-modular</u>. (By a theorem stated later).

Prenositz has put some additional postulates to characterize the lattice of convex sets of a descriptive geometry of arbitrary dimension, finite or infinte, and later, Bennett has characterized the lattice of convex subsets of a real vector space. Both the works of Maeda and Bennett depend quite heavily on that of Prenowitz.

§ 4. Partition lattices

As define before, a partition of type n $(n \geq 1)$ is a set S of points (consisting of at least $n + 1$ points) with a family of subsets called blocks such that any n distinct points of S are contained in one and only one block, and every block contains at least n distinct points.

It is also shown that a partition of type n on a set S can also be seen as a geometry in the sense of Honnson's definition, so it is sometimes also called a partition geometry of type n.

Def. A block of a partition of type n is said to be non-trivial if it consists of at least $n + 1$ distinct elements. Otherwise it is called trivial.

We shall represent a partition P by the set of its non-trivial blocks, $P = \{S_\alpha\}$.

Let us now consider the set $LP_n(S)$ of all partitions of type n on the same set S.

If $P_1, P_2 \in LP_n(S)$ are two partitions of type n on S, then we define $P_1 \leq P_2$ if and only if every block of P_1 is contained (set inclusion) in a blocks of P_2.

Then $LP_n(S)$ is a partially ordered set with respect to "$\leq$", but we can prove furthermore:

Theorem. $LP_n(S)$ is a complete, atomistic lattice.

<u>Proof.</u> Let $\{P_\alpha | \alpha \in A\}$ be any set of partitions of type n on S. Then, for any n distinct points $x_1, \cdots, x_n$, let $B(x_1, \cdots, x_n) = \bigcap_{\alpha \in A} B_\alpha(x_1, \cdots, x_n)$ (set intersection), where $B_\alpha(x_1, \cdots, x_n)$ is the block of P_α determined by $x_1, \cdots, x_n$. Obviously $\{x_1, \cdots, x_n\} \subset B(x_1, \cdots, x_n)$, and it is uniquely determined by $\{x_1, \cdots, x_n\}$. Thus, the family of $B(x_1, \cdots, x_n)$ is a partition P of type n on S, and $P \leq P_\alpha$ for all $\alpha \in A$. If $Q \leq P_\alpha$ and $B_0(x_1, \cdots, x_n)$ is the block of Q determined by $\{x_1, \cdots, x_n\}$, then $B_0(x_1, \cdots, x_n) \subseteq B_\alpha(x_1, \cdots, x_n)$ for all α, hence $B_0(x_1, \cdots, x_n) \subseteq \cap B_\alpha(x_1, \cdots, x_n)$. Hence $B_0(x_1, \cdots, x_n) \subseteq B(x_1, \cdots, x_n)$, that is, $Q_0 \leq P$. Thus $P = \pi\{P_\alpha | \alpha \in A\}$ and $LP_n(S)$ is a complete lattice.

$P = \{S\}$, the partition with only one non-trivial block S itself, is the universal element of $LP_n(S)$, and the partition with no non-trivial block (i.e., every block is trivial) is the 0 of the lattice $LP_n(S)$.

An atom in this lattice $LP_n(S)$ is a partition with only one non-trivial block which contains $n+1$ distinct points.

Now if $P \neq 0$, then P has has at least one non-trivial block which contains at least $n+1$ distinct points $x_1, \cdots, x_{n+1}$. So, if P_a is the atom with the only one non-trivial block $\{x_1, \cdots, x_{n+1}\}$, then $P_a \leq P$. Suppose next that $P_1 \nleq P_2$. Then there is a block of $P_2 : B_2(x_1, \cdots, x_n) \supsetneq B_1(x_1, \cdots, x_n)$, a block of P_1. Thus there is an element $y \in B_2(x_1, \cdots, x_n)$ but $y \notin B_1(x_1, \cdots, x_n)$. If $P_a = \{(x_1, \cdots, x_n y)\}$ is an atom, then $P_a < P_2$, but $P_a \nleq P_1$. Then $P_1 + P_a$ has the family of non-trivial blocks of P_1 except $B_1(x_1, \cdots, x_n)$ is respect by $B_1(x_1, \cdots, x_n) \cup \{y\}$. Thus $P_1 < P_1 + P_a \leq P_2$.

In the above proof we used the fact that a lattice with 0 is atomistic if and only if $a < b$ implies $a < a+p \leq b$ for some point p.

<u>Theorem</u>. The lattice $LP_n(S)$ is complemented.

<u>Theorem</u>. $LP_n(S)$ is isomorphic to the lattice of subspaces of some partition geometry of type 2.

<u>Proof</u>. Let us call "blocks" of a partition geometry of type 2 "lines". Let S' be the set of all atoms of the lattice $LP_n(S)$.

Define a partition geometry $P_2G(S')$ of type 2 over S' as follows:

Let $X = \left\{(x_1,\cdots,x_n,x_{n+1})\right\}$, $Y = \left\{(y_1,\cdots,y_n,y_{n+1})\right\}$ be two atoms in $LP_n(S)$. Define the line $\ell(X,Y)$ in $P_2G(S')$ determined by X, Y to be the set of all atoms of $LP_n(s)$ where are $\leq X+Y$.

Since

$$X+Y = \begin{cases} \left\{(x_1,\cdots,x_n,x_{n+1},y_{n+1})\right\} & \text{if } \left\{x_1,\cdots,x_n,x_{n+1}\right\}\cap\left\{y_1,\cdots,y_n,y_{n+1}\right\} \\ & = \left\{x_1,\cdots,x_n\right\} = \left\{y_1,\cdots,y_n\right\} \\ \left\{(x_1,\cdots,x_n,x_{n+1}),(y_1,\cdots,y_n,y_{n+1})\right\} & \text{if} \\ \quad \left\{x_1,\cdots,x_n,x_{n+1}\right\}\cap\left\{y_1,\cdots,y_n,y_{n+1}\right\} & \\ \quad \text{contains less than } n \text{ points.} & \end{cases}$$

So, in the former case $\ell(X,Y)$ is non-trivial, and in the latter case it is trivial.

If $\ell(X,Y)$ is non-trivial and X', $Y' \in \ell(X,Y)$, then

$X' = \left\{(x_1,\cdots,\hat{x}_i\cdots x_n x_{n+1} y_{n+1})\right\}$ ($\hat{x}_i$ means that the element x_i is deleted), and $Y' = \left\{(x_1,\cdots\hat{x}_j\cdots x_{n+1} y_{n+1})\right\}$. Thus

$\left\{x_1,\cdots,\hat{x}_i\cdots x_{n+1} y_{n+1}\right\}\cap\left\{x_1,\cdots,\hat{x}_j\cdots x_{n+1} y_{n+1}\right\}$ contains n distinct points. Hence $X'+Y' = \left\{(x_1,\cdots,x_n,x_{n+1},y_{n+1})\right\} = X+Y$.

Thus $P_2G(S')$ is actually a partition geometry of type 2.

We want to show that $LP_n(S)$ is isomorphic to the lattice $\mathcal{L}(P_2G(S'))$ of subspaces of $P_2G(S')$. Consider the correspondence:

$$\begin{array}{ccc} LP_n(S) & \longrightarrow & \mathcal{L}(P_2G(S')) \\ \text{ᴡ} & & \\ P & \longmapsto & \text{set of all atoms under } P. \end{array}$$

The set of all atoms under P is a subspace. Let the atoms $X, Y \leq P$, then $X+Y \leq P$, so every atom under $X + Y$ is under P. Since $LP_n(S)$ is atomistic, this mapping is one-one and order preserving.

It remains now only to show that it is onto. Let $T \in \mathcal{L}(P_2G(S'))$ that is, T is any subspace of $P_2G(S')$. We want to show that there is a partition $P \in LP_n(S)$ such that $P \mapsto T$ under the above correspondence.

Let U be any subset of S such that $\{x_1,\cdots,x_{n+1}\} \subset U$ implies that $\{(x_1,\cdots,x_{n+1})\} \in T$. Let F be the family of all such subsets U.

We can now define a partition of type n on S as follows: Given any n distinct elements $x_1,\cdots,x_n$ of S, let

$$B(x_1,\cdots,x_n) = \begin{cases} \{x_1,\cdots,x_n\} & \text{if there is no } U \in F \text{ such that } \{x_1,\cdots,x_n\} \subset U, \\ \bigcup \text{ set union } \{U \mid \{x_1,\cdots,x_n\} \subset U\} & \text{otherwise.} \end{cases}$$

Then $B(x_1,\cdots,x_n)$ is a maximal set in the family F, if $B(x\ ,\cdots,x_n) \in F$. To show that $B(x_1,\cdots,x_n)$ is an element of the family F, let $y_1,\cdots,y_n,y_{n+1} \in B(x_1,\cdots,x_n)$. We want to show that $\{(y_1,\cdots,y_n,y_{n+1})\} \in T$. $y_i \in B(x_1,\cdots,x_n)$ imply that there exist U_i such that $\{x_1,\cdots,x_n,y_i\} \subset U_i$, hence $\{(x_1,\cdots,x_n,y_i)\}$ $\in T$. From this, it follows that $\{(x_1,\cdots,x_{n-1},y_1,y_j)\} \in T$ for $(j = 2,\cdots,n+1)$, since T is a subspace, so every atom under $\{(x_1,\cdots,x_n,y_1)\}+\{(x_1,\cdots,x_n,y_j)\} = \{(x_1,\cdots,x_n y_1 y_j)\}$ is contained in T. By continuing similar arguments, we can conclude that $\{(x_1,\cdots,x_{n-2},y_1,y_2,y_k)\} \in T$, $(k = 3,\cdots,n+1)$, $\cdots,\{x_1,y_1,\cdots,y_{n-1},y_\ell)\}$ $\in T$ $(\ell = n,n+1)$ and finally, $\{(y_1\cdots y_n y_{n+1})\} \in T$. Thus $B(x_1,\cdots,x_n)$ $\in F$. It is obvious that $B(x_1,\cdots,x_n)$ is a maximal element in F. With all such blocks $B(x_1,\cdots,x_n)$, P is a partition of type n. Let $z_1,\cdots,z_n \in B(x_1,\cdots,x_n)$ be distinct n elements. Since $B(x_1,\cdots,x_n)$ is an element of F, by definition, $B(z_1,\cdots,z_n) \supseteq$ $B(x_1,\cdots,x_n) \supset \{x_1,\cdots,x_n\}$, so conversely we have

$B(z_1,\cdots,z_n) \subseteq B(x_1,\cdots,x_n)$, thus $B(z_1,\cdots,z_n) = B(x_1,\cdots,x_n)$.

Now we can prove that $\{(x_1,\cdots,x_{n+1})\} \in T$ if and only if $\{(x_1,\cdots,x_{n+1})\} \leq P$. Since $\{(x_1,\cdots,x_{n+1})\} \in T$, $U = \{x_1,\cdots,x_{n+1}\} \in F$. Thus $x_1,\cdots,x_n,x_{n+1} \in B(x_1,\cdots,x_n)$, hence $\{(x_1,\cdots,x_n,x_{n+1})\} \leq P$. Conversely, if $\{(x_1,\cdots,x_n,x_{n+1})\} \leq P$, then there is a block $B(x_1,\cdots,x_n) \supseteq \{x_1,\cdots,x_n,x_{n+1}\}$. Since $B(x_1,\cdots,x_n) \in F$, it follows from the difinition of F that $\{(x_1,\cdots,x_n,x_{n+1})\} \in T$. Since $LP_n(S)$ is atomistic, ΣT(lattice union) $= P$.

As $\mathcal{L}(P_2G(S'))$ is complete, upper-continuous and atomistic, we will get the following result whose direct proof will be given below:

<u>Theorem</u>. $LP_n(S)$ is upper-continuous.

<u>Proof</u>. Let $\{P_\alpha\}$ be a direct family in $LP_n(S)$, then $P = \bigcup_\alpha P_\alpha$ is constructed as follows: Let $\{U_{\alpha_i}\}$ be the family of all non-trivial blocks of P_α, and let F be the family of all U_{α_i} for all α. For any n distinct points $x_1,\cdots,x_n$, define

$$B(x_1,\cdots,x_n)=\begin{cases}\{x_1,\cdots,x_n\} \text{ if there is no } U \text{ in } F \text{ which contains } \{x_1,\cdots,x_n\}, \text{ otherwise.} \\ \underset{\text{set union}}{\bigcup}\{U\in F \mid \text{either } \{x_1,\cdots,x_n\}\subset U \text{ or there is a } V \text{ such that } \{x_1,\cdots,x_n\}\subset V \text{ and } V\cap U \text{ contains at least } n \text{ distinct elements}\}\end{cases}$$

Then the collection of all $B(x_1,\cdots,x_n)$ is a partition of type n.

If $\{y_1,\cdots,y_n\}\subset B(x_1,\cdots,x_n)$, then either there is a U such that $\{x_1,\cdots,x_n y_1\}\subset U$ or there exist V and U such that $\{x_1,\cdots,x_n\}\subset V$, $U\cap V=\{z_1,\cdots,z_n\}$ and $y_1\in U$. In the latter case, let U, V be respectively a non-trivial block of P_α and P_β, then since $\{P_\alpha\}$ is a directed set, there is a partition P_γ such that $P_\gamma \geq P_\alpha$, P_β. Thus, among the non-trivial blocks of P_γ, there is one $U_{\gamma_1}\supseteq V$, and another one $U_{\gamma_2}\supseteq U$. Since U_{γ_1}, U_{γ_2} $\supseteq\{z_1,\cdots,z_n\}$, $U_{\gamma_1}=U_{\gamma_2}\equiv U_\gamma$ which contains $x_1,\cdots,x_n z_1,\cdots,z_n y_1$.

Thus in both cases, there is a partition P_{γ_1}, one of its non-trivial block U_{y_1} contains $\{x_1,\cdots,x_n y_1\}$. Similarly there exist $P_{y_2},\cdots,P_{y_n}$ such that P_{y_i} has a block U_{y_i} which contains $\{x_1,\cdots,x_n,y_i\}$. By same argument we get that there is a partition P_δ such that one of its block $U_\delta \supseteq \{x_1,\cdots,x_n,y_1,\cdots,y_n\}$.

Now if $\{y_1,\cdots,y_n\} \subset U'$, then since $U' \cap U_\delta \supseteq \{y_1,\cdots,y_n\}$ so $U' \subseteq B(x_1,\cdots,x_n)$. Suppose next that there is a non-trivial V' belonging to $P_{\delta'}$ and a U' such that $U' \cap V' \supseteq \{w_1,\cdots,w_n\}$, then we can conclude similarly that there is a partition P_ϵ, one of its non-trivial block $U_\epsilon \supseteq \{x_1,\cdots,x_n,y_1,\cdots,y_n,w_1,\cdots,w_n\}$. Thus $U_\epsilon \cap U' = \{w_1,\cdots,w_n\}$ so $U' \subseteq B(x_1,\cdots,x_n)$. Thus it is shown that if $\{y_1,\cdots,y_n\} \subseteq B(x_1,\cdots,x_n)$ then $B(y_1,\cdots,y_n) \subseteq B(x_1,\cdots,x_n)$. Conversely, since $\{x_1,\cdots,x_n,y_1,\cdots,y_n\} \subset U_\delta$, so $U_\delta \subseteq B(y_1,\cdots,y_n)$ and $\{x_1,\cdots,x_n\} \subseteq B(y_1,\cdots,y_n)$. Hence $B(x_1,\cdots,x_n) \subseteq B(y_1,\cdots,y_n)$. Thus $B(x_1,\cdots,x_n) = B(y_1,\cdots,y_n)$, and P is a partition of type n. For any n distinct points $\{x_1,\cdots,x_n\}$ the bolock of $\sum_\alpha P_\alpha$

Thus it is proved that if $\{y_1,\cdots,y_n\} \subset B(x_1,\cdots,x_n)$ then $B(y_1,\cdots,y_n) = B(x_1,\cdots,x_n)$ and it is proved that the collection of $B(x_1,\cdots,x_n)$'s form a partition P of type n.

From the definition of $B(x_1,\cdots,x_n)$ it is obvious that $B(x_1,\cdots,x_n) \supset B_\alpha(x_1,\cdots,x_n)$ that is $P \geqslant P_\alpha$. Thus $P \geqslant \Sigma P_\alpha$. Let $C(x_1,\cdots,x_n)$ be the block of ΣP_α determined by n distinct points $\{x_1,\cdots,x_n\}$. Since $\Sigma P_\alpha \geqslant P_\alpha$, so $C(x_1,\cdots,x_n) \supset B_\alpha(x_1,\cdots,x_n)$ for all α. Hence $C(x_1,\cdots,x_n) \supset \cup B_\alpha(x_1,\cdots,x_n) = B(x_1,\cdots,x_n)$. Thus $\Sigma P_\alpha \geqslant P$ and consequently $P = \Sigma P_\alpha$.

Let $A(x_1,\cdots,x_n)$, $D(x_1,\cdots,x_n)$ be respectively the blocks of the partition Q and $Q\cdot(\Sigma P_\alpha)$ determined by $\{x_1,\cdots,x_n\}$. Then

$$\begin{aligned} D(x_1,\cdots,x_n) &= A(x_1,\cdots,x_n) \cap B(x_1,\cdots,x_n) \\ &= A(x_1,\cdots,x_n) \cap (\bigcup_\alpha B_\alpha(x_1,\cdots,x_n)) \\ &= \bigcup_\alpha (A(x_1,\cdots,x_n) \cap B_\alpha(x_1,\cdots,x_n)). \end{aligned}$$

It is obvious that $A(x_1,\cdots,x_n) \cap B_\alpha(x_1,\cdots,x_n)$ is the block of $Q\cdot P_\alpha$ determined by $\{x_1,\cdots,x_n\}$. Since the collection of $Q\cdot P_\alpha$ is also a directed set, the block of $\sum_\alpha Q\cdot P_\alpha$ determined by $\{x_1,\cdots,x_n\}$ is $\bigcup_\alpha (A(x_1,\cdots,x_n) \cap B_\alpha(x_1,\cdots,x_n))$. Thus the blocks of $Q\cdot(\sum_\alpha P_\alpha)$ and $\sum_\alpha Q\cdot P_\alpha$ determined by $\{x_1,\cdots,x_n\}$ coincide. Thus $Q\cdot(\sum_\alpha P_\alpha) = \sum_\alpha Q\cdot P_\alpha$ is proved, and $LP_n(S)$ is upper-continuous.

determined by $\{x_1, \cdots, x_n\}$ must satisfy the same condition for the definition of $D(x_1, \cdots, x_n)$, so $P = \sum_\alpha P_\alpha$.

Now suppose that $\{(x_1, \cdots, x_n, x_{n+1})\} \leq \sum_\alpha P_\alpha$. Let $D(x_1, \cdots, x_n)$ be the block of $\sum_\alpha P_\alpha = P$ which contains $\{x_1, \cdots, x_n, x_{n+1}\}$. By the difinition of $D(x_1, \cdots, x_n)$, either there is a P_α such that one of its non-trivial block $U_\alpha \supseteq \{x_1, \cdots, x_n, x_{n+1}\}$, that is $P_\alpha \geq \{x_1, \cdots, x_n, x_{n+1}\}$, or there is a non-trivial block V_β of a partition P_β such that $V_\beta \supseteq \{x_1, \cdots, x_n\}$ and a non-tivial block U_γ of P_γ such that $V_\beta \cap U_\gamma \supseteq$ at least n distinct points and $U_\gamma \ni x_{n+1}$. Since $\{P_\alpha\}$ is a directed set, there is a partition P_α with $P_\alpha \geq P_\beta$, P_γ and one of its non-trivial block $U_\alpha \supseteq \{x_1, \cdots, x_n, x_{n+1}\}$, thus $P_\alpha \geq \{(x_1, \cdots, x_{n+1})\}$. Thus we have show above that if $\{(x_1, \cdots, x_n, x_{n+1})\} \leq \sum_\alpha P_\alpha$ where $\{P_\alpha\}$ is a directed family, then ther is P_α in the family with $P_\alpha \geq \{(x_1, \cdots, x_{n+1})\}$.

By using this fact, the upper-continuity can be proved easily:

Suppose that $\{(x_1,\cdots,x_{n+1})\} \leqslant Q\{\Sigma P_\alpha\}$, then $\{(x_1,\cdots,x_{x+1})\} \leqslant Q$ and $\{(x_1,\cdots,x_{n+1})\} \leqslant \underset{\alpha}{\Sigma} P_\alpha$ which implies in trun that there is a P_α with $\{(x_1,\cdots,x_{n+1})\} \leqslant P_\alpha$. Thus $\{(x_1,\cdots,x_{n+1})\} \leqslant QP_\alpha$ and hence $\{(x_1,\cdots,x_{n+1})\} \leqslant \underset{\alpha}{\Sigma} QP_\alpha$. That is, $Q\left(\underset{\alpha}{\Sigma} P_\alpha\right) \leqslant \underset{\alpha}{\Sigma} QP_\alpha$. Since $Q\left(\underset{\alpha}{\Sigma} P_\alpha\right) \geqq \underset{\alpha}{\Sigma} QP_\alpha$ is obvious, we have $Q\left(\underset{\alpha}{\Sigma}\ _\alpha\right) = \underset{\alpha}{\Sigma} QP_\alpha$ and upper-continuity is proved.

As $LP_n(S)$ is a complete, upper-continuous and atomistic lattice, the theorem in p.   also tells us that it is isomorphic to the lattice of all subspaces of some merely finitary geometry. Actually, it can also be shown that this merely finitary geometry is a partition geometry of type 2. For the purpose, we show first that if $P \in LP_n(S)$ is a lattice join of finite number of atoms P_α $(\alpha = 1,\cdots,k) \in LP_n(S)$, then the subspace $\overline{K}$ generated by $K = \{P_1,\cdots,P_k\}$ in $P_2G(S')$ is the set of all atoms under P. Obviously $\{P_a\text{:atom}\,|\,P_a \leqslant P = \underset{\alpha}{\Sigma} P_\alpha\} \supseteqq \overline{K}$. We want to show that for any $P_a \leqslant P$, $P_a \in \overline{K}$ by induction on k. This proposition holds for $k = 2$ by the definition of line in $P_2G(S')$. Assume now that it holds for $k-1$. Let $P_\alpha = \left\{(x_1^\alpha, x_2^\alpha, \cdots, x_{n+1}^\alpha)\right\}$, $(\alpha = 1, \cdots, k)$

and $P_a = \left\{(y_1, \cdots, y_{n+1})\right\}$.

If P contains more than one non-trivial blocks, then $\left\{y_1, \cdots, y_{n+1}\right\}$ is contained in one of them, so $P_a \leq$ lattice join of less than k atoms in $\left\{P_\alpha\right\}$, and the proposition holds by induction assumption. Thus we can assume that P has only one non-trivial block which contains all elements x_i^α, $\alpha = 1, \cdots, k$ and $i = 1, \cdots, n+1$. Suppose that the only one non-trivial block of $\sum_{\beta=1}^{k-1} P_\beta$ does not contain x_{n+1}^k (x_{n+1}^k is the only one element in $\left\{x_1^k, \cdots, x_n^k, x_{n+1}^k\right\}$ which is not contained in the non-trivial block of $\sum_{\beta=1}^{k-1} P_\beta$). If $x_{n+1}^k \notin \left\{y_1, \cdots, y_{n+1}\right\}$, then $P_a \leq \sum_{\beta=1}^{k-1} P_\alpha$ and the proposition holds by induction assumption. Suppose now that $x_{n+1}^k \in \left\{y_1, \cdots, y_{n+1}\right\}$, say $y_{n+1} = x_{n+1}^k$. Suppose moreover that $x_{\ell+1}^k = y_{\ell+1}^k, \cdots, x_n^k = y_n$. Then, $P_k = \left\{(x_1^k, \cdots, x_\ell^k, y_{\ell+1}, \cdots, y_n, y_{n+1}\right\}$ $\in K$. Since $\left\{(x_1^k, \cdots, x_\ell^k y_{\ell+1}, \cdots, y_n, y_\ell\right\} \leq \sum_{\beta=1}^{k-1} P_\beta$ by induction a-assumption $\left\{(x^k_1,\cdots,x^k_\ell,y_{\ell+1},\cdots,y_n y_\ell)\right\} \in \bar{H}$, the subspace generated by $H = \left\{P_1,\cdots,P_{k-1}\right\}$ in $P_2G(S')$. Hence $\left\{(x^k_1,\cdots,x^k_\ell y y_{\ell+1}\cdots y_n\ y_\ell)\right\}$ $\in \bar{K}$, since $\bar{H} \subset \bar{K}$. As $P_k \in K$, and $\bar{K}$ is a subspace, so every atom under $P_k+\left\{(x^k_1,\cdots,x^k_\ell\ y_{\ell+1}\cdots y_n\ y_\ell)\right\} = \left\{(x^k_1,\cdots,x^k_\ell,y_\ell y_{\ell+1},\cdots y_n,y_{n+1})\right\}$ is contained in $\bar{K}$, especially $\left\{(x^k_1,\cdots,x^k_{\ell-1}y_\ell y_{\ell+1}\cdots y_n y_{n+1})\right\} \in \bar{K}$. By continuing similar argument, we get $\left\{(x^k_1,\cdots,x^k_{\ell-2}y_{\ell-1}y_\ell y_{\ell+1}\cdots y_{n+1})\right\} \in \bar{K}$, $\cdots$, and finitally $\left\{(y_1,\cdots,y_n y_{n+1})\right\} \in \bar{K}$.

Now let T be any subspace in $P_2G(S')$, then it can be shown that $T = \left\{P_a:\text{atom} \middle| P_a \leq \Sigma T\right\}$. Suppose that $P_a \leq \Sigma T$, then by upper-continuity there exist atoms $P_1,\cdots,P_k \in T$ such that $P_a \leq \sum_{\alpha=1}^{k} P_\alpha$, then $P_a \in \bar{K} \subseteq T$. Thus the merely finitary geometry is a partition geometry of type 2 (see also remark given in p.).

Remark. In $LP_1(S)$, if $\left\{(ab)\right\} \not\leq P$ and $B(a)$, $B(b)$ are respectively the block of the partition $P \in LP_1(S)$ determined by a and b, then $B(a) \cap B(b) = \phi$. In the partition $P + \left\{(ab)\right\}$ the block containing $\left\{ab\right\}$ is $B'(a,b) = B(a) \cup B(b)$. Obviously, there is no block $B''(a)$ (or $B''(b)$) such that $B(a) \subsetneq B''(a) \subsetneq B'(a,b)$ (or $B(b) \subsetneq B''(b) \subsetneq B'(a,b)$). Thus $P \prec P + \{(ab)\}$ provided $\{(a,b)\} \not\leq P$. This shows that $LP_1(S)$ is semi-modular (see § 5). But $LP_n(S)$ is not semi-modular in general as the following examples shows: (See § 5)

If S contains elements a, b, c, d, f then $LP_2(S)$ contains the following chain:

$$\{(abcdef)\}$$

$$|$$

$$\{(abc)(adc)(bdf)(cef)\}$$

$$|$$

$$\{(abc)(ade)(bdf)\}$$

$$|$$

$$\{(abc)(ade)\}$$

$$|$$

$$\{(abc)\}$$

$$|$$

$$0$$

It is easily seen that

$$\{(cef)\} \leq \{(abc),(ade),(bdf)\} + \{(abd)\}$$

$$= \{(abcdef)\}$$

and

$$\{(cef)\} \nleq \{(abc),(ade),(bdf)\},$$

but

$$\{(abd)\} \nleq \{(abc),(ade),(bdf)\} + \{(cef)\}$$

$$= \{(abc)(ade)(bdf)(cef)\}.$$

Similarly for $LP_n(S)$ (n : general), where S $\{x_1, \cdots, x_{n+4}\}$

$$\{(x_3x_5x_6x_7\cdots x_{n+4})\} \leq \{(x_1x_2x_3x_7\cdots x_{n+4}),(x_2x_4x_6x_7\cdots x_{n+4}),$$

$$(x_1x_4x_5x_7\cdots x_{n+4})\} + \{(x_1x_2x_4x_7\cdots x_{n+4})\}$$

$$= \{(x_1x_2\cdots x_6x_7\cdots x_{n+4})\}$$

and

$$\{(x_3x_5x_6x_7\cdots x_{n+4})\} \nleq \{(x_1x_2x_3x_7\cdots x_{n+4}),(x_1x_4x_5x_7\cdots x_{n+4}),(x_2x_4x_6x_7\cdots x_{n+4})\}$$

but

$$\{(x_1x_2x_4x_7\cdots x_{n+4}\} \nleq \{(x_1x_2x_3x_7\cdots x_{n+4}),(x_1x_4x_5x_7\cdots x_{n+4}),$$

$$(x_2x_4x_6x_7\cdots x_{n+4})\} + \{(x_3x_5x_6x_7\cdots x_{n+4})\}$$

$$= \{(x_1x_2x_3x_7\cdots x_{n+4}),(x_1x_4x_5x_7\cdots x_{n+4}),$$

$$(x_2x_4x_6x_7\cdots x_{n+4}),(x_3x_5x_6x_7\cdots x_{n+4})\}.$$

§ 5. Matroid lattices

It is obvious from the above discussion that complete upper-continuous, atomistic lattices play an important role in hte foundation of geometries. But in many cases of geometries we are interested in, hte corresponding lattices have another distinguished property (the exchange property) appearing in the following definition:

Definition 5.1. By a matroid lattice we mean a geometric lattice (i.e. a complete, upper-continuous, atomistic lattice) L with the following "exchange property":

(O) for any atoms p and q and any element x in L, the conditions $p \leq q+x$ and $p \not\leq x$ jointly imply that $q \leq p+x$.

Matroid lattices have been extensively investigated, and there are numerous equivalent characterijations of this class of lattices:

Theorem 5.1. In any matroid lattice the following conditions hold for all element a, b, c, d and all atoms $p, q, p_0, \cdots, p_n$:

(i) If $a < a+p \leq a+q$, then $a+p = a+q$.

(ii) If $ap = 0$, then $a+p$ covers a.

(iii) If $(a+b)p = 0$, then $(a+p)b = ab$.

(iv) If $(p_0+\cdots+p_{k-1})p_k = 0$ for $k = 1,\cdots,n$, then the system p_i, $(i = 0,\cdots,n)$ is independent.

(v) If a and b covers ab, then a+b covers a and b.

(vi) If a covers ab, then a+b covers b.

(vii) If b covers bc, then $M(b,c)$.

(viii) If $bc < a < c < b+c$, then there exists an element x such that $bc < x \leqslant b$ and $a = (a+x)c$.

(ix) If $bc < a < c < b+c$, then there exists an element x such that $bc < x \leqslant b$ and $(a+x)c < c$.

(x) If $bc < a < c < b+a$, then there exists an element x such that $bc < x \leqslant b$ and $a = (a+x)c$.

Conversely, any geometric lattice which satisfies one of the conditions (i) — (x) is a matroid lattice.

In the statement of the theorem we used the following definitions:

Def. 5.2. A system of elements x_i, $i \in I$ in a complete lattice is said to be independent if

$$\left(\sum_{j \in J} x_j\right)\left(\sum_{k \in K} x_k\right) = 0$$

whenever J and K are disjoint subsets of I.

Def. 5.3. Two elements b and c are said to form a modular pair - in symbol $M(b,c)$ - if

$$(x+b)c = x+bc \quad \text{whenever} \quad x \leqslant c.$$

Def. 5.4. If $M(b,c)$ holds for any two elements b and c, then the lattice is said to be modular.

Def. 5.5. If the relation M is symmetric, that is, if

(xi) for any two elements b and c

$$M(b,c) \rightleftarrows M(c,b)$$

(i.e. M(b,c) and M(c,b) are equivalent)

then the lattice is said to be <u>semi-modular</u>.

<u>Theorem 5.2.</u> In order for a lattice to be a matroid lattice, it is necessary and sufficient that the lattice is complete, atomistic, upper-continuous and semi-modular.

Proof of Theorem 5.1. is carried out by showing each of the implications:

$$\begin{aligned}&(\text{viii}) \to (\text{x}) \to (\text{ix}) \to (\text{vii}) \to (\text{vi})\\ &\to (\text{v}) \to (\text{ii})_f \to (\text{iii})_f \to (\text{iv})\\ &\to (0)_f \xrightarrow[(**)]{} (0) \to (\text{i}) \xrightarrow[(*)]{} (\text{ii})\\ &\to (\text{iii}) \xrightarrow[(*)]{} (\text{viii}).\end{aligned}$$

In this diagram, $(\text{ii})_f$, $(\text{iii})_f$ and $(0)_f$ mean that these conditions are stated only for elements generated by finite number of atoms. And (*) means that in the proof of the implication we need to use the atomisticity, and (**) means that we need to use theatomisticity and upper-continuity.

We brake the proof into several lemmas:

<u>Lemma 5.0.</u> M(b,c) holds if and only if $bc \leqslant a \leqslant c$ implies $(a+b)c = a$.

<u>Proof.</u> M(b,c) implies $(a+b)c = a+bc = a$ for $bc \leqslant a \leqslant c$. Conversely, assume that $bc \leqslant a \leqslant c$ implies $(a+b)c = a$, and let $a' \leqslant c$, and put $a = a'+bc$. Then $bc \leqslant a = a'+bc \leqslant c$. Thus

$(a+b)c = a$, that is, $(a'+bc+b)c = a$ which means that $(a'+b)c = a'+bc$, that is $M(b,c)$.

<u>Lemma 5.1</u>. In a lattice L, we have the following implications:

$$(viii) \rightarrow (x) \rightarrow (ix) \rightarrow (vii).$$

<u>Proof</u>. (viii) → (x). Let $bc < a < c < a+b$, then $a+b \leq b+c$ and $bc < a < c < b+c$. Thus by (viii) there is an element x such that $bc < x \leq b$ and $a = (a+x)c$.

(x) → (ix). Suppose that $bc < a < c < b+c$. Then $b \not\leq c$ and $b \neq bc$, that is, $b > bc$. Put $c' = (a+b)c$, then $c \geq c'$. If $c > c' = (a+b)c$ then (ix) holds for $x = b$. If $c = c'$ then $c = (a+b)c$, hence $c \leq a+b$. Since $b \not\leq c$, we have $c \neq a+b$, hence $c < a+b$. Then $bc < a < c < a+b$, so by (x) there is an x such that $bc < x \leq b$ and $(a+x)c = a < c$. That is (ix) holds.

(ix) → (vii). Suppose that $b \succ bc$ (that is, b covers bc), then $b \not\leq c$. Suppose $bc < a \leq c$. Obviously $(a+b)c \geq a$. But it is shown as follows that $(a+b)c > a$ is impassible, so $a = (a+b)c$. Thus by Lemma 5.0., $M(b,c)$, since if $bc = a$, $(a+b)c = (bc+b)c = bc = a$.

Assume that $c' = (a+b)c > a$. Then since $c' \leq c$ and $b \not\leq c$, it follows that $b \not\leq c'$ and $b+c' > c'$. Furthermore $bc' = b(a+b)c = bc$. Thus $bc = bc' < a < c' < b+c'$. Thus by (ix) there is an x such that $bc' = bc < x \leq b$ and $(a+x)c' < c'$. Since $b \succ bc$, so $x = b$ and $(a+b)c' < c'$. On the other hand $(a+b)c' = (a+b)(a+b)c = (a+b)c = c'$ contradicting the last inequality.

<u>Lemma 5.2</u>. In a lattice L, we have the following implications:

$$(vii) \rightarrow (vi) \rightarrow (v).$$

<u>Proof</u>. (vii) → (vi). Suppose that c a $b+c$, then $bc \leq ab \leq (b+c)b = b$. Since $b \succ bc$ this inequality implies that either $b = ab$ or $bc = ab$. If $b = ab$, then $b \leq a$. But $c \leq a$, so $b+c \leq a$, hence <u>$a = b+c$</u> since $a \leq b+c$ as assumed. If $bc = ab$, then $b \succ ab$, thus $M(b,a)$ by (vii). Then by Lemma 5.0., $(b+c)a = c$ since $ab = bc \leq c \leq a$. But since $a \leq b+c$ it follows $(b+c)a = a$. Consequently <u>$a = c$</u>. Thus we have shown that $c \leq a \leq b+c$ implies either $a = b+c$ or $a = c$. That is $c \prec b+c$.

(vi) → (v). From the assumptions $a \succ ab$ and $b \succ ab$, we have respectively $a+b \succ b$ and $a+b \succ a$ by (vi).

Lemma 5.3. In a lattice with 0, conditions (ii) and (iii) are equivalent.

Proof. (iii) $\rightarrow$ (ii). Suppose $ap = 0$ and $a+p \geq b \geq a$. Then $a+p > a$, $a+b = b$ and $(a+b)p = bp$. Since $p \succ 0$, either $bp = 0$ or $bp = p$. In the first case $(a+p)p = 0$, so by (iii) $(a+p)b = ab = a$. But since $(a+p)b = b$, hence $a = b$. In the second case $b \geq p$ and $(a+p) \geq b \geq a+p$. Hence $a+p = b$.

(ii) $\rightarrow$ (iii). Suppose that $(a+b)p = 0$. Then $ap = 0$, hence $a+p \succ a$ by (ii). Let $b' = (a+p)b$, then $b' \geq ab$. Suppose that $b' > ab$. If $b' > a$ then $a+p \geq b' > a$. Thus $a+p = b = (a+p)b$ since $a+p \succ a$. Then $b \geq a+p \geq p$ contradicting $(a+b)p = 0$. If $b' \leq a$ then $a \geq b' = (a+p)b > ab$. This leads to a contradiction $ab \geq (a+p)b > ab$. Thus $b' > ab$ is impassible and $b' = (a+p)b = ab$.

$(v) \to (ii)_f \to (iii)_f \to (iv) \to (0)_f$.

<u>Proof</u>. $(ii)_f \to (iii)_f$ by <u>Lemma 5.3</u>.

$(iv) \to (0)_f$. Suppose that $p \leq (a_1+\cdots+a_r)+q$ but $p \nleq a_1+\cdots+a_r$. We are showing that $q \leq (a_1+\cdots+a_r)+p$. By deleting redundant elements we can assume that $(a_1+\cdots+a_{s-1})\cdot a_s = 0$ for $s = 2,\cdots,r$. Assume that $q \nleq (a_1+\cdots+a_r)+p$, then $(a_1+\cdots+a_r+p)q = 0$. Since $(a_1+\cdots+a_r)p = 0$, by (iv) we have that $a_1,\cdots,a_r,p,q$ are independent, so $(a_1+\cdots+a_r+q)p = 0$ contradicting the hypothesis $p \leq (a_1+\cdots+a_r)+q$.

$(v) \to (ii)_f$ where $(ii)_f$ asserts that if $p \nleq \sum_{i=1}^{n} p_i$ then $p+\sum_{i=1}^{n} p_i$ covers $\sum_{i=1}^{n} p_i$. We prove this implication by induction on n. For $n = 1$ if $p \nleq p_1$ then $p\cdot p_1 = 0 \prec p,p_1$. Then by (v) $p+p_0 \succ p$ and $p+p_1 \succ p_1$. Thus $(ii)_f$ holds for $n = 1$.

Assume now that $(ii)_f$ holds for n. We are showing that if $p \nleq \sum_{i=1}^{n} p_i+p_{n+1}$ then $\sum_{i=1}^{n} p_i+p_{n+1}+p$ covers $\sum_{i=1}^{n} p_i + p_{n+1}$. If $p_{n+1} \leq \sum_{i=1}^{n} p_i$ then $\sum_{i=1}^{n} p_i+p_{n+1} = \sum_{i=1}^{n} p_i$. Thus $\sum_{i=1}^{n} p_i+p_{n+1}+p = \sum_{i=1}^{n} p_i+p$ covers $\sum_{i=1}^{n} p_i+p_{n+1} = \sum_{i=1}^{n} p_i$. If $p_{n+1} \nleq \sum_{i=1}^{n} p_i$ then $\left(\sum_{i=1}^{n} p_i\right)p_{n+1} = 0$

and $\sum_{i=1}^{n} p_i + p_{n+1}$ covers $\sum_{i=1}^{n} p_i$ by induction assumption. If $p \not\leq \sum_{i=1}^{n} p_i + p_{n+1}$ then $p \not\leq \sum_{i=1}^{n} p_i$, so by induction assumption $\sum_{i=1}^{n} p_i + p$ covers $\sum_{i=1}^{n} p_i$. $\left(\sum_{i=1}^{n} p_i + p\right) \geq \left(\sum_{i=1}^{n} p_i + p_{n+1}\right)\left(\sum_{i=1}^{n} p_i + p\right) \geq \sum_{i=1}^{n} p_i$, so either $\sum_{i=1}^{n} p_i + p = \left(\sum_{i=1}^{n} p_i + p_{n+1}\right)\left(\sum_{i=1}^{n} p_i + p\right)$ or $\left(\sum_{i=1}^{n} p_i + p_{n+1}\right)\left(\sum_{i=1}^{n} p_i + p\right) = \sum_{i=1}^{n} p_i$. Since the former identity implies $\sum_{n=1}^{n} p_i + p_{n+1} \geq \sum_{i=1}^{n} p_i + p \geq p$ contradicting the hypothesis, so $\left(\sum_{i=1}^{n} p_i + p_{n+1}\right)\left(\sum_{i=1}^{n} p_i + p\right) = {}_{i=1}^{n} p_i$ holds. Then by (v) $\left(\sum_{i=1}^{n} p_i + p_{n+1}\right) + \left(\sum_{i=1}^{n} p_i + p\right) = \sum_{i=1}^{n} p_i + p_{n+1} + p = \sum_{j=1}^{n+1} p_i + p$ covers $\sum_{i=1}^{n} p_i + p_{n+1} = \sum_{j=1}^{n+1} p_i$ and $\sum_{i=1}^{n} p_i + p$.

$(iii)_f \rightarrow (iv)$. Let h be the number of terms in the sum of the first factor, and k the number of the terms in the sum of the second factor. Let $k, h \geq 1$ and let $h+k = \ell$.

(1) If $h = k = 1$ then $p_{i_1} p_{j_1} = 0$, since if $j_1 > i_1$ (say) then $p_{i_1} p_{j_1} \leq (p_1 + \cdots + p_{j_1 - 1}) p_{j_1} = 0$.

and $\sum_{i=1}^{n} p_i + p_{n+1}$ covers $\sum_{i=1}^{n} p_i$ by induction assumption. If $p \not\leq \sum_{i=1}^{n} p_i + p_{n+1}$ then $p \not\leq \sum_{i=1}^{n} p_i$, so by induction assumption $\sum_{i=1}^{n} p_i + p$ covers $\sum_{i=1}^{n} p_i$. $\left(\sum_{i=1}^{n} p_i + p\right) \geq \left(\sum_{i=1}^{n} p_i + p_{n+1}\right)\left(\sum_{i=1}^{n} p_i + p\right) \geq \sum_{i=1}^{n} p_i$, so either $\sum_{i=1}^{n} p_i + p = \left(\sum_{i=1}^{n} p_i + p_{n+1}\right)\left(\sum_{i=1}^{n} p_i + p\right)$ or $\left(\sum_{i=1}^{n} p_i + p_{n+1}\right)\left(\sum_{i=1}^{n} p_i + p\right) = \sum_{i=1}^{n} p_i$. Since the former identity implies $\sum_{n=1}^{n} p_i + p_{n+1} \geq \sum_{i=1}^{n} p_i + p \geq p$ contradicting the hypothesis, so $\left(\sum_{i=1}^{n} p_i + p_{n+1}\right)\left(\sum_{i=1}^{n} p_i + p\right) = \sum_{i=1}^{n} p_i$ holds. Then by (v) $\left(\sum_{i=1}^{n} p_i + p_{n+1}\right) + \left(\sum_{i=1}^{n} p_i + p\right) = \sum_{i=1}^{n} p_i + p_{n+1} + p = \sum_{j=1}^{n+1} p_j + p$ covers $\sum_{i=1}^{n} p_i + p_{n+1} = \sum_{j=1}^{n+1} p_j$ and $\sum_{i=1}^{n} p_i + p$.

$(iii)_f \rightarrow (iv)$. Let h be the number of terms in the sum of the first factor, and k the number of the terms in the sum of the second factor. Let $k, h \geq 1$ and let $h+k = \ell$.

(1) If $h = k = 1$ then $p_{i_1}p_{j_1} = 0$, since if $j_1 > i_1$ (say) then $p_{i_1}p_{j_1} \leq (p_1 + \cdots + p_{j_1-1})p_{j_1} = 0$.

(2) Suppose that $(p_{i_1}+\cdots+p_{i_h})(p_{j_1}+\cdots+p_{j_k}) = 0$ for $h+k = \ell-1 \geq 2$. We shall show that it holds for $h+k = \ell$.

Without loss of generality we can assume that there is an m such that $1 \leq m \leq h$ and $i_m > i_1, \ldots, \hat{i}_m, \ldots i_h, j_1, \ldots, j_k$. Then $(p_{i_1}+\cdots+\hat{p}_{i_m}+\cdots+p_{i_h}+p_{j_1}+\cdots+p_{j_k})p_{i_m} \leq (p_1+\cdots+p_{i_m-1})p_{i_m} = 0$, Thus by $(iii)_f$

$$(p_{i_1}+\cdots+p_{i_h})(p_{j_1}+\cdots+p_{j_k})$$

$$= (p_{i_1}+\cdots+p_{i_m}+\cdots+p_{i_h}+p_{i_m})(p_{j_1}+\cdots+p_{j_k})$$

$$= (p_{i_1}+\cdots+p_{i_m}+\cdots+p_{i_h})(p_{j_1}+\cdots+p_{j_k})$$

$$= 0$$

by induction assumption.

Lemma 5.5. In an atomistic and upper-continuous lattice the conditions (0) and $(0)_f$ are equivalent.

Proof. $(0) \to (0)_f$ is obvious.

$(0)_f \to (0)$. Since $p \leq q+a$, $p \leq \ell \cdot u \cdot b S_f$ where S_f is finite and $S_f \subseteq A \cup \{q\}$ in which A is the set of atoms contained in a. If S_f does not contain q then $\ell \cdot u \cdot b S_f \leq a$ contradicting the fact that $p \not\leq a$. Thus $S_f = \{q\} \cup A_f$ with finite A_f Λ.

Thus $p \leq q+\tilde{a}$ where $\tilde{a} \leq a$ is the union of finite points. And $p \not\leq \tilde{a}$ since $p \not\leq a$. Thus by $(0)_f$ $q \leq \tilde{a}+p$ and $q \leq a+p$.

Lemma 5.6. In a lattice with 0, $(0) \rightarrow (i)$.

Proof. Suppose that $a < a+p \leq a+q$. Since $a < a+p$, $p \not\leq a$ holds. And $a+p \leq a+q$ implies $p \leq a+q$. Thus $p \leq a+q$ and $p \not\leq a$, so by (0) $q \leq a+p$ and so $a+q \leq a+p$. Thus with $a+p \leq a+q$ give $a+p = a+q$.

Lemma 5.7. In an atomistic lattice $(i) \rightarrow (ii)$ and $(iii) \rightarrow (viii)$.

Proof. $(i) \rightarrow (ii)$. Suppose that $ap = 0$ and $a+p \geq z \geq a$. If $z \neq a$ (i.e. $z > a$), then there exists q such that $q \leq z$ but $q \not\leq a$. Then $z \geq a+q > a$ and $a+p \geq a+q > a$. Thus by (i) $a+p = a+q$ and hence $a+p = z$.

$(iii) \rightarrow (viii)$. Suppose that $bc < a < c < b+c$. Since $c < b+c$, we have $b \not\leq c$ hence $bc \not\leq b$. Then there exists $p \leq b$, $p \not\leq bc$ hence $p \not\leq c$ that is $pc = 0$. Let $x = bc+p$. Since $((a+bc)+c)\cdot p = cp = 0$, by (iii) we have

$$(a+x)c = (a+bc+p)c = (a+bc)c$$
$$= ac = a.$$

And (viii) is proved.

For the proof of Theorem 5.2., we can show (xi) → (vii) easily and (vii) → (0) by theorem 5.1.. Hence (xi) → (0), To show (0) — (xi) we need the following lemmas and definition.

Lemma 5.8. In an upper-continuous lattice L, a set S of elements of L is an independent system, if and only if every finite subset of S is an independent system.

Lemma 5.9. If P is an independent system of points (atoms) in a motroid lattice and if q is a point with $(\Sigma P)q = 0$, then $P \cup \{q\}$ (set union) is an independent system.

Def. 5.6. An independent system P of points with $\Sigma P = a$ is called a basis of the element a of L.

Lemma 5.10. If L is anmotroid lattice, and if P is any independent system of points with $\Sigma P \leq a$, then there is a set $Q \supset P$ which is a basis of a.

Corollary 5.1. Every element of L (of Lemma 5.10) has a basis.

(xi) → (vii). If $b \succ bc$ then $M(cb)$. Since if $x \leq b$ then either $x \leq bc$ or $bc < x \leq b$, and the latter case implies $x = b$ by $b \succ bc$. If $x \leq bc$ then $(c+x)b \leq (c+bc) \leq bc$. But $(c+x)b \geq bc$, hence $(c+x)b = bc = x+bc$. If $x = b$ then $(c+x)b = (b+c)b = b = b+bc = x+bc$. Thus $M(cb)$. Then by (xi) we have $M(b,c)$.

Proof of Lemma 5.8. We need merely show that if every finite subset of S is an independent system then S is an independent system. Let S_1 and S_2 be any two disjoint subsets of S, ν_1 any finite subset of S_1, ν_2 any finite subset of S_2.

Put $a_{\nu_1} = \Sigma(\nu_1)$, $a_{\nu_2} = \Sigma(\nu_2)$, then $\Sigma(S_1) = \Sigma(a_{\nu_1})$ and and $\Sigma(S_2) = \Sigma(a_{\nu_2})$. Since, by assumption the set sum $\nu_1 \cup \nu_2$ is an independent system, we have $a_{\nu_1} \cdot a_{\nu_2} = 0$ for all ν_2. L being upper continuous, we have $a_{\nu_1}(\Sigma(S_2)) = \Sigma(a_{\nu_1} a_{\nu_2}) = 0$. Similarly $\Sigma(S_1) \cdot \Sigma(S_2) = 0$ and S is an independent system.

Proof of Lemma 5.9. Let ν be any finite subset of $P \cup \{q\}$. If ν does not contain q then ν is independent. If ν conting q, then $\nu = \nu_1 \cup \{q\}$ (set union) with $\nu_1 \subset P$, so ν_1 is independent. Now $a_{\nu_1} = \Sigma(\nu_1) \leq \Sigma(P)$, hence $q \cdot a_{\nu_1} \leq q \cdot \Sigma(P) = 0$. Thus by (iv) ν is independent. Thus by the above Lemma 5.8, $P \cup \{q\}$ is independent.

Proof of Lemma 5.10. Let S be the set of all points contained in a, then $a = \Sigma(S)$ since L is atomistic. Since $\Sigma(P) \leq a$, if $p \in P$ then $p \leq \Sigma(P) \leq a$. Hence $p \in S$ and $P \subseteq S$. Let B be the family of independent subsets of S which contain P, then B is non-empty since $P \in B$. Let K be a chain in B under set inclusion, then $\cup(K)$, the set union of all sets in K, is an upper bound of K which can be shown to be independent. Let ν be any finite subset of $\cup(K)$. Since $\cup(K)$ is the set union there is a subset K_σ in the chain such that $\nu \subseteq K_\sigma$. Then ν is independent since K_σ is independent. Then by the above Lemma 5.8. $\bigcup(K)$ is independent and $\bigcup(K)$ belongs to the family B. By Zorn's lemma there exists a maximal independent subset Q of S with $Q \supseteq P$. If there exists a point $p \in S$ such that $p \cdot \Sigma(Q) = 0$ then by Lemma 5.9. $Q \cup \{p\}$ (set union) is an independent subset of S such that $Q \cup \{p\} \supset P$ which contradicts the property of Q. Hence $p \leq \Sigma(Q)$ for all $p \in S$ and $a = \Sigma(S) \leq \Sigma(Q)$. But $\Sigma(Q) \leq \Sigma(S)$ since $Q \leq S$. Therefore $\Sigma(Q) = a$.

Proof of corollary. If $0 < a$ there is a point $p \leq a$. Then $\{p\} = P$ is an independent subset such that $\Sigma(P) = p \leq a$. Then Lemma 5.10 leads to the corollary.

To complete the proof of the Theorem 5.2., we are showing that in a matroid lattice $M(bc)$ implies $M(cb)$, that is $(0) \rightarrow (xi)$.

Let d be any element with $d \leq b$. If $d \leq bc$ then $(c+d)b \leq b(c+bc) \leq bc = d+bc$. But obviously $(c+d)b \geq d+bc$, hence $(c+d)b = d+bc$, that is $M(c,b)$. Now let d be any element with $bc < d \leq b$. Let p be any point such that $p \leq (d+c)b$. Then by Lemma 5.10 and Corollary 5.1 there exist point sets P, Q and R such that Q, $P \cup Q$ and $Q \cup R$ are respectively bases of bc, d and c. Since $\Sigma(P \cup Q) = \Sigma P + \Sigma Q$, $p \leq d+c = \Sigma(P)+\Sigma(Q)+\Sigma(R)$.

Therefore, by upper-continuity $p \leq p_1+\cdots+p_\ell+q_1+\cdots+q_m+r_1+\cdots+r_n$. Deleting the redundant elements from the above expression we can assume without loss of generality that $(p_1,\cdots,p_\ell,q_1,\cdots,q_m,r_1,\cdots,r_n)$ is an independent set. If $n = 0$ for every $p \leq (d+c)b$ then $(d+c)b \leq d$ and $M(cb)$ holds. If $n > 1$ then r_n, $p \not\leq p_1+\cdots+p_\ell+q_1+\cdots+q_m+r_1+\cdots+r_{n-1}$. Thus by (0) $r_n \leq p_1+\cdots+p_\ell+q_1+\cdots+q_m+r_1+\cdots+r_{n-1}+p \leq b+a$ where $b = p_1+\cdots+p_\ell+q_1+\cdots+q_m$ and $a = r_1+\cdots+r_{n-1}$. Since $r_n \leq c$, $r_n \leq (b+a)c = a+bc$ by $M(b,c)$. Thus $r_n \leq q_1+\cdots+q_m+r_1+\cdots+r_{n-1}$ contradicting that the set $Q \cup R$ is independent. If $n = 1$ then r_1, $p \not\leq p_1+\cdots+p_\ell+q_1+\cdots+q_m$. By (0) again, we have $r_1 \leq p+p_1+\cdots+p_\ell+q_1+\cdots+q_m$ hence $r_1 \leq b$. Since $r_1 \leq c$ it follows that $r_1 \leq bc$ contradicting the fact that $Q \cup R$ is independent.

<u>Examples</u>. <u>1</u>. A geometric linear lattice is not necessarily semi-modular. This is shown by the example given in p. 34 as a condequence of Theorem 5.1, (iv).

2. By Theorem 5.1, (ii) we set that $LP_1(S)$ is semi-modular, but in general $LP_n(S)$ $(n \geq 2)$ is not. See the examples given in p. 48.

Let us now list some properties of matroid lattices.

<u>Theorem 5.3</u>. Every matroid lattice is relatively complemented.

Proof. Let x be any element such that $s \leq x \leq t$, we want to show that there is an element y such that $x+y = t$ and $xy = s$.

Let Y be the set of all element y such that $xy = s$ and $y \leq t$. Y is a non void poset, since $s \in Y$.

It is easy to predict that a maximal element y_0 of Y is a relative complement of x in $[s\ t]$. To show this we need to show $x+y_0 = t$.

Suppose that $x+y_0 \not\leq t$, then since the lattice is atomistic there is a point $p \leq t$ but $p \not\leq x+y_0$ (so of course $p \not\leq x$, $p \not\leq y_0$). Then $y_0 \not\geq y_0+p$ and $y_0+p \leq t$. By the maximality of y_0, $y_0+p \notin Y$, therefore $x(y_0+p) \neq s = xy_0$. Hence there is a point $q \leq x(y_0+p)$ and $q \not\leq s$. Then $q \leq x$ and $q \leq y_0+p$. It is obvious that $q \not\leq y_0$ (otherwise, $q \leq x$ and $q \leq y_0$ imply $q \leq xy_0 = s$, a contradiction). Since the lattice is semi-modular, from $q \leq y_0+p$, $q \not\leq y$ it follows that $p \leq y_0+q$. Then $p \leq y_0+q \leq y_0+x$ ($\because$ $q \leq x$) contradicting $p \not\leq x+y_0$. Thus $x+y_0 = t$.

Now we have still to show that Y has at least a maximal element. For this we need to show that the hypothesis of Zorn's lemma is satisfied for Y. Thus we want to show that every chain K of Y has an upper bound in Y.

As one of such upper bound, we can take the element ℓ.u.bK, the existence of this follows from the fact that the lattice is complete.

We must now show that $k = \ell$.u.bK is an element of Y.

Since each element of $K \leqslant t$, so $k = \cdot u \cdot bK \leqslant t$. To show that $sk = s$, let k_i take all the elements of K $(i \in I)$. Then since the lattice is upper-continuous

$$xk = x\left(\sum_{i \in I} k_i\right) = \sum_{i=I} xk_i.$$

Since $k_i \in K \quad Y$, so $xk_i = s$ for all $i \in I$. Thus $\sum_{i \in I} xk_i = s$. Therefore $xk = s$ and $k \in Y$.

The above proof is a modification of that of "complementedness" for the lattice of subspaces of a projective geometry.

For the lattice of subspaces of a projective geometry, it can be proved to be a direct union of irreducible sublattices. This can be generalized to any matroid lattice.

<u>Def. 5.7.</u> In a lattice L with 0, if p, q are points such that

$$q \leqslant p+x, \quad qx = 0$$

for some element $x \in L$, we say that <u>p is perspective to q</u>, and use the symbol $p \sim q$ to denote it.

<u>Def. 5.8.</u> In a lattice L with 0, by $a \nabla b$ it is meant that

$$ab = 0$$

and

$$(a+x)b = xb \text{ for every } x \in L.$$

<u>Lemma 5.11.</u> In a geometric lattice L the following two properties are equivalent:

(α) $a \nabla b$,

(β) $ab = 0$ and there exist no points p, q such that $p \leq a$, $q \leq b$ and p is perspective to q.

Corollary 5.2. In an atomistic, upper-continuous lattice L, $a \nabla b$, $a_1 \leq a$, $b_1 \leq b$ together imply that $a_1 \nabla b_1$.

Proof of Lemma 5.11. $(\alpha) \rightarrow (\beta)$. $ab = 0$ is evident. Next, assume that there exist points p, q such that $p \leq a$, $q \leq b$ and p is perspective to q. Then there exists an element $x \in L$ such that $q \leq p+x$, $qx = 0$. Hence $(p+x)q = q > 0 = x \cdot q$ which means that $p \nabla q$ is false. On the other hand $a \nabla b$ implies that for every x of L and $a_1 \leq a$, $b_1 \leq b$ we have $(a_1+x)b_1 = (a_1+x) \cdot \underline{(a+x) \cdot b} \cdot b_1 = (a_1+x) \cdot \underline{(x \cdot b)} \cdot b_1 = (((a_1+x) \cdot x) \cdot b) \cdot b_1 = (x \cdot b) \cdot b_1 = x \cdot (b \cdot b_1) = x \cdot b_1$ which is contradictory to $(p+x) \cdot q > x \cdot q$ for $a_1 = p$, $b_1 = q$.

$(\beta) \rightarrow (\alpha)$. It is evident when $a = 0$ or $b = 0$, hence we can assume that $a, b \neq 0$.

If (α) is false, there exists $x = 0$ such that $(a+x) \cdot b > x.b$. Since L is atomistic, there exists a point q such that $(a+x) \cdot b \geq (a \cdot b)+q > x.b$. Then $q \leq b$ since $(a+x) \cdot b \geq q$. And also $(x.b) \cdot q = 0$ since otherwise $(x.b) \cdot q = q$ and $(x.b)+q = x \cdot b$. Furthermore $q \cdot x = 0$ since otherwise $x \cdot q = q$ and $x \geq q$, hence $b \cdot x \geq q$ and $(x.b) \cdot q = q$. Since $q \leq a+x$, we have $q \leq \Sigma(p;p \leq a) + (q;q \leq x)$ by atomisticity. Hence, by upper-continuity, there exist points $p_1, \cdots, p_n$ contained in a and points $q_1, \cdots, q_m$ contained in x such that $q \leq p_1+\cdots+p_n+q_1+\cdots+q_m \leq p_1+\cdots+p_n+x$. Suppose that all superfluous elements were already deleted from $p_1+\cdots+p_m+x$. Then $q \not\leq (p_2+\cdots+p_n)+b$, that is $q(p_2+\cdots+p_n+x) = 0$, but $q \leq p_1+(p_2+\cdots+p_n+b)$. This means that p_1 is perspective to q contradictry (β).

Proof of Corollary 5.2. The corollary is obvious from the last part of the proof of $(\alpha) \to (\beta)$ is Lemma 5.1.1.

Def. 5.9. If S is any subset of L, we denote by S^∇ the set of a such that $a \nabla b$ for all $b \in S$.

Def. 5.10. Let $\{S_\alpha; \alpha\in I\}$ be a family of subsets with 0 of an upper-continuous lattice L. We say that L is a direct sum of S_α $(\alpha \in I)$ and write $L = \Sigma(\oplus S_\alpha; \alpha\in I)$ if

(1°) every $a \in L$ is expressible in the form
$a = \Sigma(a_\alpha; \alpha\in I)$ where $a_\alpha \in S_\alpha$ $(\alpha \in I)$,

(2°) $\alpha \neq \beta$ implies $S_\beta \leq S_\alpha^\nabla$.

S_α is called a component of the direct sum.

Def. 5.11. If p, q are points such that there exists a sequence $p = p_1, p_2, \cdots, p_{n-1}$, $p_n = q$ of points where $p_i \sim p_{i+1}$ or $p_{i+1} \sim p_i$ for $i = 1,\cdots,n-1$, then we say that p and q are connected.

Theorem 5.4. A geometric lattice L is a direct sum of sub-lattices S_α of L, that is $L = \Sigma(\oplus S_\alpha, \alpha\epsilon I)$, and any two points in the same S_α are connected, and two points which are contained in different S_α and S_β are not connected.

Proof of Theorem 5.4. Since the connectedness is obviously an equivalence relation, we can form equivalence classes $\mathfrak{p}_\alpha$ $(\alpha \epsilon I)$ with respect to connectedness. Thus any two points in the same $\mathfrak{p}_\alpha$ are connected and two points which are contained in different $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$ respectively are not connected.

Let S_α be the set which consists of 0 and all elements of L which are expressible as the join of points in $\mathfrak{p}_\alpha$. Obviously, S_α is a sublattice of L.

To show that $L = \Sigma(\oplus S_\alpha; \alpha\epsilon I)$ we show first that (2°) $\alpha \neq \beta$ implies $S_\alpha \leq S_\beta^\nabla$. Let $a_\alpha \epsilon S_\alpha$, $b_\beta \epsilon S_\beta$ $(\alpha \neq \beta)$, and let $p \leq a_\alpha$ and $q \leq b_\beta$. Then by upper-continuity

$$p \leq p_1+\cdots+p_n, \text{ where } p_i \epsilon \mathfrak{p}_\alpha \quad (i = 1,\cdots,n),$$

$$q \leq q_1+\cdots+q_m, \text{ where } q_j \epsilon \mathfrak{p}_\beta \quad (j = 1,\cdots,m).$$

It is supposed here that all the superfluous points were alrealy deleted. Then $p \not\leq (p_2+\cdots+p_n)$ that is ~~$p(p_2+\cdots+p_n) = 0$ but~~ $p \leq p_1+(p_2+\cdots+p_n)$. Thus $p_1 \sim p$. Since $p_1 \epsilon \mathfrak{p}_\alpha$. Similarly $q \epsilon \mathfrak{p}_\beta$.

Therefore p is not perspective to q ($p \not\sim q$). Now $a_\alpha \cdot b_\beta = 0$ since otherwise $a_\alpha \cdot b_\beta \neq 0$, so there exists $r \leqslant a_\alpha$. $r \leqslant b_\beta$ and $r \in p_\alpha \cap p_\beta$. contradicting $p_\alpha \cap p_\beta = \phi$. Thus by Definition $a_\alpha \nabla b_\beta$, that is $S_\alpha \quad S_\beta^\nabla$. To show that (1°) every $a \in L$ is expressible in the form: $a = \Sigma(a_\alpha : \alpha\in I)$ with $a_\alpha \in S_\alpha$ $(\alpha \in I)$, let a_α be the join of all points p such that $p \leqslant a$ and $p \in p_\alpha$. Then by atomisticity $a = \Sigma(a_\alpha, \alpha\in I)$ where $a_\alpha \in S_\alpha$ $(\alpha \in I)$ and consequently $L = \Sigma(\oplus S_\alpha ; \alpha\in I)$.

<u>Theorem 5.5</u>. Let p, q and r be points of a matroid lattice L, then the following statements hold:

(I) $p \sim q$ implies $q \sim p$.

(II) $p \sim q$ and $q \sim r$ imply $p \sim r$.

That is, in a matroid lattice the concepts of perspectvity and connectedness are equivalent.

In the proof of the theorem, we need the following:

<u>Lemma 5.11</u>. Let $\{p_1, \cdots, p_n\}$ be independent, and let $P_i = p + \cdots + p_{i-1} + p_{i+1} + \cdots + p_n$ $(i = 1, \cdots, n)$, then we have

$$\prod_1^r P_i = p_{r+1} + \cdots + p_n \quad \text{for } r = 1, \cdots, n.$$

<u>Proof of the Lemma</u>. $P_1 = p_2 + \cdots + p_n$, so lemma is true for $r = 1$. Suppose that it is true for $r = m$, then we have

$$\prod_1^{m+1} p_i = \left(\prod_1^m p_i\right)\cdot p_{m+1} = (p_{m+1}+\cdots+p_n)\cdot p_{n+1}$$

by induction assumption. By putting $a = p_{m+1}+\cdots+p_n$ and $b = p_{m+1}$ we have

$$a+b = (p_{m+2}+\cdots+p_n)+(p_1+\cdots+p_m+p_{m+2}+\cdots+p_n)$$
$$= (p_1+\cdots+p_n+p_{m+2}+\cdots+p_n) \quad \text{since} \quad a \leqslant b$$
$$\not\geqslant p_{m+1} \quad \text{since} \quad \{p_1,\cdots,p_n\} \quad \text{is independent.}$$

Thus $(a+b)\cdot p_{m+1} = 0$, hence by (iii) of Theorem 5.1., $(a+p_{m+1})\cdot b = a\cdot b$. Then

$$\prod_1^{m+1} p_i = (p_{m+1}+a)\cdot b = a\cdot b = a = p_{m+2}+\cdots+p_n,$$

and the lemma is proved.

<u>Proof of Theorem 5.5.</u> (I) $p \sim q$ means that there exists x such that $q \leqslant p+x$, $q\cdot x = 0$ and $p\cdot x = 0$. Then by semi-modularity $p \leqslant q+x$ which together with $p\cdot x = 0$ gives $q \sim p$.

(II) If any two of p, q and r are coincident, then this is obvious. Hence we may suppose that p, q and r are distinct. Since $p \sim q$ there exists an element $a \in L$ such that $q \leqslant p+a$ and $p\cdot a = q\cdot a = 0$. By upper-continuity there exist x_i $(i = 1,\cdots,m)$ such that

$$(1) \quad \begin{array}{ll} q \leqslant p+x_1+\cdots+x_m & \text{where} \quad x_i \leqslant a \quad (i = 1,\cdots,m) \\ q \not\leqslant px_1+\cdots+x_m, & \text{since} \quad q\cdot a = 0 \end{array}$$

and we may assume that the redundant elements are already dedated in (1). Hence $\{p, x_1, \cdots, x_m\}$ is independent by (iv) of Theorem 5.1. Now put $X = x_1 + \cdots + x_m$, then since $q \cdot X = 0$ by (iv) of Theorem 5.1. again $\{q, x_1, \cdots x_m\}$ is also independent.

Put $X_i = x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_m$, $i = 1, \cdots, m$, then $p \cdot X_i = 0$ since $p \cdot a = 0$. Since $p \cdot X_i \not\geq q$ by the irredundancy of x_i $(i = 1, \cdots, m)$, $\{p, q, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m\}$ is independent for $i = 1, \cdots, m$ by (iv) of theorem 5.1. again.

Now $q \leq p+X$ and $q \cdot X = 0$ [by (1)] imply that $X < q+X \leq p+X$ which is turn implies $q+X = p+X$ by (i) of Theorem 5.5. Moreover $q \leq p+X = (p+X_i)+x_i$ and $q \not\leq p+X_i$, hence $p+X_i < q+(p+X_i) \leq x_i+(p+X_i) = p+X$ which implies $q+(p+X_i) = p+X$ by (i) of Theorem 5.5 again. Thus we have proved

(2) $p+X = q+X = p+q+X_i$ for $i = 1, \cdots, m$.

Now since $q \sim r$, there exist y_i $(i = 1, \cdots, n)$ such that

(3) $r \leq q+Y$, $q \cdot Y = r \cdot Y = 0$ and $Y = y_1 + \cdots + y_n$.

By (1) and (3) we have $r \leq p+X+Y$. Hence if $r(X+Y) = 0$, then $p \sim r$ and (II) is proved.

Thus we assume now that $r \leq X+Y$. Since $r \leq q+Y$, $r \not\leq Y$ it follows that $q \leq r+Y \leq X+Y$ by semi-modularity. Similarly

$q \leq p+X$ and $q \not\leq X$ imply $p \leq q+X \leq X+Y$. Hence

$$X+Y = p+X+Y$$

$$= p+x_1+\cdots+x_m+y_1+\cdots+y_m.$$

Since the set $\{p,x_1,\cdots,x_m\}$ is independent, there exist $y_{i_j} \in \{y_1,\cdots,y_n\}$ for $j = 1,\cdots,k$ such that

$$X+Y = p+x_1+\cdots+x_m+y_{i_1}+\cdots+y_{i_k}$$

by deleting all redundant elements in $\{y_1,\cdots,y_n\}$ and hence the set $\{p,x_1,\cdots,x_m,y_{i_1},\cdots,y_{i_k}\}$ is independent. Now put $Y' = y_{i_1}+\cdots+y_{i_k}$ then by (2)

(4) $X+Y = p+X+Y' = p+q+X_i+Y'$ for $i = 1,\cdots,m$.

Since $\{p,x_1,\cdots,x_m,y_{i_1},\cdots,y_{i_k}\}$ and $\{q,x_1,\cdots,x_m\}$ are both independent and $p+X = q+X$, we have by (iv) of Theorem 5.5 that $\{q,x_1,\cdots,x_m,y_{i_1},\cdots,y_{i_k}\}$ is independent. [Since $(q+x_1+\cdots+x_m)\cdot y_{i_1} = (p+x_1+\cdots+x_m)\cdot y_{i_1} = 0,\cdots,(q+x_1+\cdots+x_m+y_{i_1}+\cdots+y_{i_\ell})\cdot y_{i_{\ell+1}} = (p+x_1+\cdots+x_m+y_{i_1}+\cdots+y_{i_\ell})\cdot y_{i_{\ell+1}} = 0$ as $\{p,x_1,\cdots,x_m,y_{i_1},\cdots,y_{i_k}\}$ is independent]. Hence by Lemma 5.11 we have

$$(X+Y')\cdot\prod_{1}^{m}(q+X_i+Y') = Y',$$

since $\{q,x_1,\cdots,x_m,y_{i_1},\cdots,y_{i_k}\}$ is independent. Now since $r\cdot Y' \leqslant r\cdot Y = 0$ and $Y' = (X+Y)\cdot\prod_{1}^{m}(q+X_i+Y')$ at least one of $X+Y'$ and $q+X_i+Y'$ $(i = 1,\cdots,m)$ does not contain r.

We have by (4) that $r \leqslant X+Y = p+(X+Y') = p+(q+X_i+Y')$. If $r \not\leqslant (X+Y')$ then $r \leqslant p+(X+Y')$ but $r\cdot(X+Y') = 0$, that is $p \sim r$. If $r \not\leqslant q+X_i+Y'$ for an i, then $r \leqslant p+(q+X_i+Y')$ but $r\cdot(q+X_i+Y') = 0$ that is $p \sim r$. Thus the theorem is proved.

Combine Theorem 5.4 and 5.5., we have the following:

Thm. 5.6. A matroid lattice L is a direct union of sub-lattice L_α $(\alpha \in I)$ of L, that is $L = \Sigma(\oplus L_\alpha, \alpha\in I)$, and any two points in the same L_α are perspective, two points which are contained in different L_α and L_β are not perspective.

Moreover, we have

Theorem 5.7. In a matriod lattice L, the following two statements are equivalent:

(α) L is irreducible,

(β) any two points of L are perspective to each other.

Proof of Theorem 5.6. By Theorem 5.5, in a matroid lattice the perspectivity and the connectedness are equivalent, and therefore this theorem follows immediately from Theorem 5.4.

<u>Proof of Theorem 5.7</u>. $(\alpha) \to (\beta)$. Suppose that the statement (β) is false, then there exist at least two points of L which are not perspective. Then by Theorem 5.4, L is a direct union of at least two sublattice and so L is reducible since L has 0. This contradicts (α).

$(\beta) \to (\alpha)$. Suppose that L is reducible. Then since L has 0, L is a direct union of sublattice L_α of L, that is $L = \Sigma(\oplus L_\alpha, \alpha \epsilon I)$, where $L_\alpha \leq L_\beta^\nabla$ provided $\alpha \neq \beta$. Thus $a \in L_\alpha$ satisfies $a \nabla b$ for all $b \in L_\beta$. Let pp $\in L_\alpha$ and $q \in L_\beta$ $(\alpha \neq \beta)$ be two points there $p \nabla q$ and by Lemma 5.11 $p \cdot q = 0$ and $p \not\sim q$ which contradicts (β) saying that any two points of L are perspective to each other. Q. E. D.

Theorem 5.6 and 5.7 lead immediately the following:

Theorem 5.8. Any matroid lattice is a directed union of irreducible matroid lattices.

Following theorems for matroid matroid lattices with finit basis are often very useful:

<u>Theorem 5.9</u>. Let L be a lattice satisfying (v) of Theorem 5.1 and having no infinite chains, and let x, y be any two elements in L such that $x \leq y$, Then all maximal chains from x to y have the same length.

Definition 5.12. A finite chain $x = a_1 > a_2 > \cdots > a_n = y$ is a maximal chain (or a compasition chain) for $[y,x]$ if a_i covers its success or a_{i+1}.

Proof of Theorem 5.9. Let $x = s_0 < s_1 < \cdots < s_n = y$ and $x = t_0 < t_1 < \cdots < t_m = y$ be two maximal chains from x to y. If $n = 0$ or $n = 1$ then $x = y$ or y covers x respectively, hence the chains coincide and the theorem holds. Now assume the truth of the theorem for all pairs x', y' between which there is a maximal chain of length less than n. From $x = s_0 < s_1$ and $x = t_0 < t_1$ it follows that $x \leqslant s_1 \cdot t_1 \leqslant s_1$. Since s_1 covers either $x = s_1 \cdot t_1$ or $s_1 \cdot t_1 = s_1$.

(i) Suppose first that $s_1 \cdot t_1 = s_1$. Then $t_1 \geqslant s_1 > x$, and it follows that $t_1 = s_1$ since t_1 covers x. Then $s_1 < s_2 < \cdots < s_n = y$ and $s_1 = t_1 < t_2 < \cdots < t_m = y$ are two maximal chains, of which the first one is of length $n - 1$. Hence by induction hypothisis $n-1 = m-1$, that is $n = m$.

(ii) If $s_1 \cdot t_1 = x$ then since s_1, t_1 covers $s_1 \cdot t_1 = x$, $s_1 + t_1$ covers both s_1 and t_1 by covering property. Select a maximal chain $s_1 + t_1 < u_2 < \cdots < u_p = y$. Then $s_1 < s_1 + t_1 < u_2 < \cdots < u_p = y$ is also a maximal chain. Compare this with the maximal chain $s_1 < s_2 < \cdots < s_n = y$ and apply the induction assumption we have $p = n$. Comparing the maximal chain with length $p-1 = n-1$:

$t_1 < s_1+t_1 < u_2 < \cdots < u_p = y$ and $t_1 < t_2 < \cdots < t_m = y$ we get similarly that $p-1 = n-1 = m-1$. Thus we have $n = m$.

Definition 5.13. In a latticesatisfying (v) of Thm. 5.1 and havinf no infinite chains, the rank $\lambda(x)$ of an element x is the common length of the maximal chains from 0 to x.

Theorem 5.10. Let L be any atomistic lattice satisfying (v) of Thm. 2 and having no infinite chains, then the rank function λ in L satisfies the inequality:

$$\lambda(x+y)+\lambda(xy) \leqslant \lambda(x)+\lambda(y).$$

Theorem 5.11. Consider the following statements concerning a pair x, y of elements of a lattice L:

(a) $M(y,x)$, that is, for all element $z \leqslant x$

$$(z+y)x = z+xy,$$

(b) x is a minimal relative complement of y in the interval $[xy\ x+y]$,

(c) $\lambda(xy)+\lambda(x+y) = \lambda(x)+\lambda(y)$.

(a) and (b) are equivalent in any relatively complemented lattice.

(a) and (c) are equivalent in any matroid lattice with no infinite chains.

And all three statements are equivalent in a matroid lattice.

Proof of Theorem 5.10. Let x, y be any two element in L. Choose a maximal chain from $x \cdot y$ to x;

$$x\cdot y = x_0 < x_1 < \cdots < x_n = x.$$

Let $y_i = x_i+y$. Since x_i covers x_{i-1} it can be shown that

$y_i = x_i+y$ covers $y_{i-1} = x_{i-1}+y$ or $y_i = y_{i-1}$. Since $x_i > x_{i-1}$ there exists a point p such that $x_i \geq x_{i-1}+p > x_{i-1}$ which implies $x_i = x_{i-1}+p$ by the fact that $x_i \succ x_{i-1}$. Then $y+x_i = (y+x_{i-1})+p$. If $p \leq y+x_i$ then $y+x_i = y+x_{i-1}$. If $p \not\leq y+x_{i-1}$ then $p(y+x_{i-1}) = 0$, they by Thm. 5.1 (ii) (See Lemma 5.4.) $y+x_i = (y+x_{i-1})+p$ covers $y + x_{i-1}$. Thus except for possible repetition of some elements

$$y = y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_n = x+y$$

is a maximal chain from y to $x + y$. Thus

$$\lambda(x+y)-\lambda(y) \leq n = \lambda(x)-\lambda(x\cdot y),$$

that is

$$\lambda(x+y)+\lambda(x\cdot y) \leq \lambda(x)+\lambda(y).$$

Proof of Theorem 5.11. Assume first that L is relatively complemented. We shall prove that (a) $\rightleftarrows$ (b).

(a) $\rightarrow$ (b). Let z be a relative complement of y in the interval $[x\cdot y, x+y]$ with $z < x$, that is we assume that (b) does not hold. Then $z\cdot y = x\cdot y$ and $z+y = z+y$. Thus $(x\cdot y)+z = (z\cdot y)+z = z$ and $x\cdot(y+z) = x\cdot(x+y) = x$. Hence $(x.y)+z = z < x = x\cdot(y+z)$ which is the negation of (a).

(b) $\to$ (a). Assume that (a) does not hold, that is

$$(x.y)+z < x\cdot(y+z) \quad \text{strictly.}$$

Let the element $t < x$ be a relative complement of the element $x(y+z)$ in the interval $[(x\cdot y)+z,x]$. Now since $t \geq x\cdot y+z$, $y+t \geq y+(x\cdot y+z) = (y+x\cdot y)+z = y+z$. Thus $t+y \geq y+z \geq x\cdot(y+z)$ and $t+y \geq t$ imply $t+y \geq t+(x\cdot(y+z)) = x$ since t is a complement of $x(y+z)$ in $[x\cdot y+z,x]$. Thus $t+y \geq x+y$ (since $t+y \geq x$) which together with $t+y \leq x+y$ imply $t+y = x+Y$.

On the other hand, $t \geq x\cdot y+z \geq x\cdot y$ hence $x\cdot y \leq t < x$. Thus $x\cdot y = (x\cdot y)\cdot y \leq t\cdot y. \leq$ Hence Hence $= t\cdot y. =$ Thus Thus t is x is a relative complement of y in the interval $[x\cdot y,x+y]$, and (b) does not hold. Thus (b) $\to$ (a) is proved.

Now assume that L is atomistic, satisfies covering property and has no infinite chains. We shall prove that (a) $\rightleftarrows$ (c).

(a) $\to$ (c). To prove this choose a maximal chain from $x\cdot y$ to x: $x\cdot y = x_0 < x_1 <\cdots< x_n = x$. As in the proof of Theorem 5.10 $\lambda(x+y)-\lambda(y) < \lambda(x)-\lambda(x\cdot y)$ if and only if there is an index i such that $x_{i-1}+y = x_i+y$. If such an index i exists, then

$$x\cdot y+x_{i-1} = x_0+x_{i-1} = x_{i-1} < x_i \leq x\cdot(y+x_i) = x\cdot(y+x_{i-1}).$$

That is $x\cdot y+x_{i-1} < x\cdot(y+x_{i-1})$ with $z = x_{i-1} \leq x$. This shows that (a) is not satisfied and we have proved that (a) $\to$ (c).

Conversely, if the statement (a) is not satisfied for some element $z \leqslant x$, then $s \equiv x\cdot y+z < x\cdot(y+z) \equiv t$. Obviously $xy \leqslant s$ and $t \leqslant x$. Choose a maximal chain:

$$x\cdot y = x_0 <\cdots< x_i = s <\cdots< x_j = t <\cdots< x_n = x.$$

Then $x_i+y = s+y = (x\cdot y+z)+y = z+y$ and $x_j+y = t+y = (x\cdot(y+z))+y \leqslant (y+z)+y = z+y$. Thus $z+y = x_i+y \leqslant x_j+y \leqslant z+y$, hence $x_i+y = x_j+y$. Thus $y_0 = y \leqslant y_1 \leqslant\cdots\leqslant y_n = x+y$ (with $y_i = x_i+y$) which is a maximal chain except for possible repetition of elements actually has repetition $y_i = y_j$. Hence

$$\lambda(x+y)-\lambda(y) < \lambda(x)-\lambda(x\cdot y)$$

and the statement (c) is not satisfied. That is, (c) $\rightarrow$ (a).

§ 6. Matroids

Consider the set M whose elements are all the columns of a matrix. On the linear dependence and linear independence of any subsets (of columns) of M the following two propositions hold:

(I_1) Any subset of an independent set is independent,

(I_2') if N_p and N_{p+1} are independent sets of p and p + 1 columns respectively, then N_p together with some element of N_{p+1} forms on independent set of P + 1 elements.

There are many interesting system obeying these two conditions. Such systems are called matroids according to H. Whitney in his investigations related to matrices and graphs.

Several equivalent definitions of matroids are known, some of which are suggested by graphs which offen remarkable examples of matroids.

Investigation of matroid from a lattice-theoretic point of view supplies an important example of matroid lattices as we shall see soon.

Definition 6.1. A matroid $M = (E,B)$ consists of a non-empty finite set E, together with a non-empty collection B of subsets of E called bases satisfying the following properties:

(B_1) No base properly contains another base,

(B_2) If B_1 and B_2 are bases, and x is an element of B_1, then there exists an element y of B_2 with the property that $\left(B_1-\{x\}\right) \cup \{y\}$ is also a basis.

Corollary 6.1. Any two bases of a matroid contain the same number of elements.

Definition 6.2. A matroid $M = (E,I)$ consists of a non-empty finite set E, together with a non-empty collection I of subsets of E called independent sets satisfying the following properties:

(I_1) Any subset of an independent set is an independent set.

(I_2) If I and J are independent sets, and $|J| > |I|$, then there exists an element x belonging to J but not to I with the property that $I \cup \{x\}$ is an independent set. Here $|I|$ denotes the number of elements in the set I.

Remark. Condition (I_2) can be replaced by the following condition:

$(I\ )$ If $N = \{e_1,\cdots,e_p\}$ and $N' = \{e'_1,\cdots,e'_{p+1}\}$ are two independent sets, then for some i such that e'_i is not in N, $N \cup \{e'_i\}$ is independent.

Definition 6.3. A matroid $M = (E,r)$ consists of a non-empty finite set E, together with an integer-valued function r called its rank function which is defined on the set of subsets of E and which satisfies the following:

(r_1) For each subset A of E, $0 \leq r(A) \leq |A|$.

(r_2) If $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$.

(r_3) For any $A, B \subseteq E$, $r(A \cup B)+r(A \cap B) \leq r(A)+r(B)$.

It is not difficult to show that these three condition put together are equivalent to the followings:

(R_1) $r(\phi) = 0$

(R_2) For any subset N and any element e not in N, $r\left(N \cup \{e\}\right) = r(N)+k$, ($k = 0$ or 1).

(R_3) For any subset N and elements e_1, e_2 not in N, if $r\left(N \cup \{e_1\}\right) = r\left(N \cup \{e_2\}\right) = r(N)$, then $r\left(N \cup \{e_1\} \cup \{e_2\}\right) = r(N)$.

All the above definitions (i.e. Def. 6.1 - Def. 6.3.) are equivalent. For example we shall show that Def. 6.2 and Def. 6.3 are equivalent in the following:

<u>Proof.</u> Def. 6.2. $\Rightarrow$ Def. 6.3. Assume first that $M = (E,I)$ is a matroid defined in terms of its independent sets. We define the rank of A to be the size of the largest independent set contained in A and denote it by $r(A)$. Then a subset of E is independent if and only if $r(A) = |A|$. We shall prove properties (r_1), (r_2), (r_3). (r_1) and (r_2) are obvious from the definition of the rank function r. To prove (r_3), let X be a bass (that is, the maximal independent subset) of $A \cap B$. Since X is an independent subset of A, by applying (I_2) repeatedly, X can

be extended to a base Y of A and then Y extended to a base Z of $A \cup B$. Then $Z - Y$ has no point in common with A, since otherwise Y would not be a maximal independent in A. Since $X \cup (Z-Y)$ is an independent subset of B, it follows that

$$r(B) \geq r(X \cup (Z-Y)) = |X|+(|Z|-|Y|)$$
$$= r(A \cap B)+r(A \cup B)-r(A).$$

Def. 6.3. ⇒ Def. 6.2. Conversely, let $M = (E,r)$ be a matroid defined in terms of a rank function r. Define a subset A of E to be independent if and only if $r(A) = |A|$. To show that (I_1) holds, let $|J| < |I|$. Since I is an independent set $|I| = r(I)$. Let $|I|-|J| = k$. If $k = 1$ then $I = J \cup \{e\}$, $e \notin J$ and by (r_3) we have

$$r(I) = r(J \cup \{e\}) \leq r(J)+r(\{e\})-r(J \cap \{e\})$$
$$= r(J)+r(\{e\})-r(\phi) = r(J)+r(\{e\}).$$

Since $r(\{e\}) \leq |e| = 1$, $r(\{e\}) = 0$ or 1. Thus $r(J) \leq r(I)$ $\leq r(J)+\ell$ with $\ell = 0$ or 1. Suppose now that $r(J) < |J|$. Then $r(I) < |J|+ \leq |J|+1 = |I|-1+1 = |I|$ contradicting the assumption that I is independent. Thus $r(J) = |J|$ and J is independent. Suppose that (I_1) holds for k, and let $|I|-|J| = k+1$. Let $e \in I$ and $e \notin J$; then $|I|-|J \cup \{e\}| = k$. Then by induction assumption $J \cup \{e\}$ is independent. Hence J is also independent as shown above for the case $k = 1$.

To prove (I_2), suppose that I and J are two independent sets and $|I| = k$, $|J| = k+1$. Suppose that for every element e_i in J, $r(I \cup \{e_i\}) = r(I)$. Then by (r_3), $r(I \cup \{e_i\} \cup \{e_j\}) \leq$ $r(I \cup \{e_i\}) + r(I \cup \{e_j\}) - r((I \cup \{e_i\}) \cap (I \cup \{e_j\})) = r(I)+r(I)-r(I) = k$, since $(I \cup \{e_i\}) \cap (I \cup \{e_j\}) = I$ for $i \neq j$. Since on the other hand $r(I) \leq r(I \cup \{e_i\} \cup \{e_j\})$, we have $r(I \cup \{e_i\} \cup \{e_j\}) = k$. By using induction we can conclude then that $r(I \cup J) = k \geq r(J)$ which contradicts the fact that J is independent. Hence there exists an element $f \in J$ with $r(I \cup \{f\}) = k+1$, that is $I \cup \{f\}$ is independent.

<u>Remark</u>. In a matroid $M = (E,B)$ we define a subset A of E to be an independent set if A is contained in some base of M. And in a matroid $M = (E,I)$ we define a base to be any maximal independent set. Since A is independent if and only if $r(A) = |A|$ it follows that B is a base if and only if $|B| = r(B) = r(E)$ (to mean a maximal independent set).

<u>Example 1</u>. (Matrix). Let E be a finite set of vectors in some vector space V over a field F. We can define a matroid

M on E by taking as independent sets of the matroid those subsets of E which are linear independent in V. The bases of M are then precisely those subsets of E which span the same subspace as E. The rank of any subsets A of E is simply the dimension of the subspace of V spanned by A. The set of columns of a matrix belongs to such class of matroids and the rank of the matroid is the rank of the matrix.

Given a matroid M on a set E, we shall say that M is representable over a field F if there exists a vector space V over F and a map p from E to V with the property that a subset A of E is independent in M if and only if ϕ is one-one on A and $\phi(A)$ is linealy independent in V.

Example 2. (Graphs). A graph is defined to be a pair (V(G),E(G)) consisting of a finite non-empty set V(G) of elements called vertices and another finite family E(G) of unordered pairs of elements of V(G) called edges. Edge of a graph is used to be denoted by its unordered pair of vertices like $\{u,v\}$ or $\{v,v\}$. The edge $\{v,v\}$ is called a loop, and tow edges represented by the same pair $\{u,w\}$ of vertices are called multiple edges. Any graph containing no loops or multiple edges is called a simple graph. A graph, each of whose vertices belongs to V(G) and each of whose edges belongs to E(G), is called a subgraph of G. Two graphs

G and G′ are said to be <u>isomorphic</u> if there is a one-one correspondence between their sets of vertices with the property that the number of edges joining any two vertices of G is equal to the number of edges joining the corresponding vertices of G′.

A <u>path</u> in a graph G is a finite sequence of distinct edges of the form $\{v_0,v_1\},\{v_1,v_2\},\ldots,\{v_{m-1},v_m\}$. A graph is <u>connected</u> if, given any two vertices v and w, there is a path connecting v and w. Any graph which is not connected may be split up into a finite number of connected subgraphs, called <u>components</u>.

A non-empty path in which all vertices are distinct except for v_0 and v_m which are the same, is called a <u>circuit</u>. If $v_0 \neq v_m$ it is called a <u>chain</u>. A <u>cutset</u> of a graph G is a set of edges whose removal increases the number of components of G, and which is mimimal with respect to this property, any such set without minimal property is called a <u>disconnecting set</u>.

A graph which contains no circuits is called a <u>forest</u>, and a connected forest is called a <u>tree</u>. For a tree there are following equivalent statements:

(i) T is a tree with n vertices (i.e. T is connected, and contains no circuits)

(ii) T contains no circuits and has n - 1 edges.

(iii) T is connected and has n - 1 edges.

(iv) T is connected, and every edge is an isthmus, where an isthmus is an edge which itself forms a cutset.

(v) Any two vertices of T are connected by exactly one chain

(vi) T contains no circuits, but the addition of any new edge creates exactly one circuit.

If G is a connected graph, then a <u>spanning tree</u> T of G is a tree which contains every vertex of G and all of whose edges are edges of G. Given any connected graph, we can take a circuit C and remove one of its edges, say e, the resulting graph remains connected, since the vertices of the removed edge are connected by a path $C - \{e\}$. Repeat this procedure with one of the remaining circuits, continuing until there are no circuits left. The graph which remains will be a spanning tree of G.

If G denotes an arbitrary graph with n vertices, m edges and k components, we can carry out the above procedure on each component of G and obtain a <u>spanning forest</u>. The number of edges removed in the process is called the <u>circuit rank</u> of G and it is denoted by $\gamma(G)$. The number of edges in a spanning forest is called the <u>cutset rank</u> and it is denoted by $\kappa(G)$.

Since a tree with h vertices has h - 1 edges, the cutset rank of G (with n vertices, m edges and k components) is $n-k = (n_1-1)+(n_2-1)+\cdots+(n_k-1)$ where n_i is the number of vertices of the i-th component, hence $n = \Sigma n_i$. Hence the circuit rank

$\gamma(G) = m-(n-k) = m-n+k.$

For any graph, it can be shown that

(GB_1) no spanning forest of G contains another spanning forest as a proper subgraph,

(GB_2) if T_1 and T_2 are two spanning forests of G, and e is an edge of T_1, then there exists an edge f of T_2 with the property that $\left(T_1-\{e\}\right) \cup \{f\}$ (the graph obtained from T_1 on replacing e by f) is also a spanning forest of G.

Thus a matroid can be associated in a natural way with any G by letting E be the set of edges of G and taking as bases the sets of edges of the spanning forest of G. This matroid is called the circuit matroid of G and is denoted by $M(G)$.

The independent sets of $M(G)$ are simply those set of edges which contains no circuits, in other words the edge-sets of the forests contained in G. Since a subset of $M(G)$ which is not independent is called dependent, circuits of G is a dependent set of $M(G)$ and every dependent set contains a circuit. Thus a circuit of G is a minimal dependent set of $M(G)$. Since $M(G)$ is also defined by independent sets which are in turns determined by dependent sets, so circuits of G determines independent sets of $M(G)$. Thus the properties of circuits of G would suggest the definition of a matroid in terms of circuits. For any graph G it can be shown that

(GC_1) no circuit properly contains another circuit,

(GC_2) if C_1 and C_2 are distinct circuit of a graph G, each containing an edge e, then there exists a circuit in $C_1 \cup C_2$ which does not contain e.

These properties suggest the following definition of a matroids:

Definition 6.4. A matroid $M = (E,C)$ consists of a non-empty finite set E, together with a collection C of non-empty subsets of E called circuits satisfying the following properties:

(C_1) No circuit contains another circuit.

(C_2) if C_1 and C_2 are distinct circuits each containing an element x, then there exists a circuit in $C_1 \cup C_2$ which does not contain x.

This definition is also equivalent to all definitions given above. In $M = (E,I)$, a minimal dependent set is called a circuit, and in $M = (E,C)$ a set which does not contain a circuit is called independent.

By the equivalence of these definitions, we see that C is a circuit if and only if $r(C) = |C|-1$. Thus in $M(E,r)$, can give the following definitions: A loop of a matroid $M = (E,r)$ is an element e of E satisfying $r(\{e\}) = 0$, and a pair of parallel elements of M is a pair $\{e,f\}$ of elements of E which are not loops and which satisfy $r(\{e,f\}) = 1$. A matroid which contains no loops or pairs of parallel elements is called a <u>simple matroid</u>.

Two manifolds $M_1 = (E_1, T_1)$ and $M_2 = (E_2, I_2)$ are said to be <u>isomorphic</u> if there is a one-one correspondence between the sets E_1 and E_2 which preserves independence.

Although matroid isomorphism preserves circuits, cutsets and the number of edges in a graph, it does not in general preserve connectedness, the number of vertices or their degrees.

Given a matroid M, if there exists a graph G such that M is isomorphic to M(G), then M is called a <u>graphic matroid</u>.

<u>Example 3</u>. Given a graph G, the circuit matroid M(G) is not the only matroid which can be defined on the set of edges of G.

Since a cutset is a disconnecting set, it follow from this minimal property the following:

(GC_1^*) No cutset property contains another cutset.

It can also be shown the following:

(GC_2^*) If C_1^* and C_2^* are distinct cutsets of G, each containing an edge e, then there exists a cutset in $C_1^* \cup C_2^*$ which does not contain e.

Thus the set of all cutsets of a graph satisfies the two conditions of Definition 6.4. of a matroid $M = (E, C)$ by taking circuits of the matroid the cutsets of G.

This matroid is called the <u>cutset matroid</u> of G and is denoted by $M^*(G)$. A set of edges of G is independent on $M^*(G)$

if and only if it contains no cutsets of G.

A matroid M is called <u>cographic</u> if M is isomorphic to the cutset matroid of some graph G.

In order to give lattice theoretic investigation of matroids we give a definition of matroid in terms of closure operation as follows:

<u>Definition 6.5.</u> A matroid $M = (E,c)$ consist of a non-empty finite set E, together with a function $c : P(E) \rightarrow P(E)$ (power set of E), satisfying the following properties:

(c_1) For each subset A of E, $A \subseteq c(A) = c(c(A))$.

(c_2) If $A \subseteq B \subseteq E$, then $c(A) \subseteq c(B)$.

(c_3) If $x \in c\left(A \cup \{y\}\right)$, $x \notin c(A)$ then $y \in c\left(A \cup \{x\}\right)$.

The first two of these conditions express the fact that c is a closure operation on E and the third says that c satisfies the so-called exchange condition.

A matroid defined in terms of its closure operation is sometimes called a <u>pregeometry</u>.

Def. 6.3. $\Rightarrow$ Def. 6.5. If $M = (E,r)$ is a matroid on E defined in terms of its rank function r then the closure (or span) $c(A)$ of a subset A of E is defined to be the set of all those elements x of E which depend on A, that is

$$c(A) = \left\{x \in E : r\left(A \cup \{x\}\right) = r(A)\right\}.$$

if and only if it contains no cutsets of G.

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(c_2) If $A \subseteq B \subseteq E$, then $c(A) \subseteq c(B)$.

(c_3) If $x \in c\left(A \cup \{y\}\right)$, $x \notin c(A)$ then $y \in c\left(A \cup \{x\}\right)$.

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$$c(A) = \left\{x \in E : r\left(A \cup \{x\}\right) = r(A)\right\}.$$

Then we can show that (c_1), (c_2), (c_3) hold:

(c_1). $A \subseteq c(A)$ is obvious. Let $x \in c(c(A))$, then $r\left(c(A) \cup \{x\}\right) = r(c(A))$. Let $c(A) = A \cup \{x_1, \cdots, x_n\}$. Obviously $r\left(A \cup \{x_1, \cdots, x_n\} \cup \{x\}\right) \geq r\left(A \cup \{x\}\right)$. But $r\left(A \cup \{x_1, \cdots, x_n\} \cup \{x\}\right)$ $= r(c(A))$ and it can be shown that $r(c(A)) = r(A) \leq r\left(A \cup \{x\}\right)$. Therefore $r\left(A \cup \{x\}\right) = r(A)$ and $x \in c(A)$. That is $c(c(A)) \subseteq c(A)$.

Now $r(c(A)) = r(A)$ can be shown with respect to s of the number of point such that $c(A) = A \cup \{x_1, \cdots, x_s\}$. Since $c(A) = A \cup \{x_1, \cdots, x_s\}$, $r(A) = r\left(A \cup \{x_1\}\right) = r\left(A \cup \{x_2\}\right)$. Then by R_3, $r\left(A \cup \{x_1, x_2\}\right) = r(A)$. Suppose that $r\left(A \cup \{x_1, \cdots, x_t\}\right) = r(A)$. Then $r\left(A \cup \{x_1, \cdots x_{t+1}\}\right) = r\left(A \cup \{x_1, \cdots, x_t\} \cup \{x_{t+1}\}\right) \leq$ $r\left(A \cup \{x_1, \cdots, x_t\}\right) + r\left(A \cup \{x_{t+1}\}\right) - r(A) = r(A) + r\left(A \cup \{x_{t+1}\}\right) - R(A) =$ $r\left(A \cup \{x_{t+1}\}\right) = r(A)$, since $x_{t+1} \in c(A)$. Obviously $r\left(A \cup \{x_1, \cdots x_t\} \cup \{x_{t+1}\}\right) \geq r(A)$, hence $r\left(A \cup \{x_1, \cdots, x_{t+1}\}\right) = r(A)$.

(c_2). Let $x \in c(A)$ and $A \subseteq B = A \cup \{y_1, \cdots, y_n\}$. Since $x \in c(A)$, we have $r(A \cup \{x\}) = r(A)$. Now suppose $x \notin B$, then $r(B \cup \{x\}) = r(A \cup \{y_1, \cdots, y_n\} \cup \{x\}) \leqslant r(A \cup \{y_1, \cdots, y_n\}) + r(A \cup \{x\}) - r(A) = r(B) + r(A) - r(A) = r(B)$. Obviously $r(B \cup \{x\}) \geqslant r(B)$, which together with the above inequality implies $r(B \cup \{x\}) = r(B)$, that is $x \in c(B)$. Thus $c(A) \subseteq c(B)$.

(c_3). $x \in c(A \cup \{y\})$ implies that $r(A \cup \{y\} \cup \{x\}) = r(A \cup \{y\})$. And $x \notin c(A)$ implies that $r(A \cup \{x\}) = r(A)+1$, since $r(A \cup \{x\}) \neq r(A)$ hence by (R_2), $r(A \cup \{x\}) = r(A)+1$. Now if $y \notin c(A \cup \{x\})$ then $r(A \cup \{x\} \cup \{y\}) = r(A \cup \{x\}) + 1 = r(A)+2$ which contradicts the fact that $r(A \cup \{x\} \cup \{y\}) = r(A \cup \{y\}) \leqslant r(A)+1$.

<u>Remark</u>. If a matroid M is a simple matroid then $c(\phi) = \phi$ and $c(\{x\}) = \{x\}$, since no x satisfies $r(\{x\}) = r(\phi) = 0$, hence $r(\{x\}) = 1$ for every $x \in E$. Moreover, no $y \neq x$ satisfies

$r(\{x\}\cup\{y\}) = r(\{x,y\}) = 1 = r(\{x\})$. Thus if M is simple, then M is a geometry as defined in Crapo and Rota's book.

It is also to be noted that a subset A of E is independent in $M = (E,c)$ if and only if $x \notin c(A-\{x\})$ for each $x \in A$. Since

$r((A-\{x\})\cup\{x\}) = r(A)$, $x \in c(A-\{x\})$ if and only if $r(A-\{x\}) = r(A)$.

But A is independent if and only if $r(A) = |A|$, and for this relation to hold it is necessary and sufficient that $r(B-\{x\}) = |B|-1$ holds for every subset B of A.

If $M = (E,c)$ is a matroid in terms of closure operation c on E defined in Def. 6.5., it can be show the followings:

Theorem 6.1. For any two subsets A, B of E,

(i) $c(A\cap B) \subseteq c(A)\cap c(B)$, and

(ii) $c(A\cup B) = c(c(A)\cup c(B))$.

Proof. From $A \subseteq c(A)$, $B \subseteq c(B)$ it follows that $A\cap B \subseteq c(A)\cap c(B)$ and $A\cup B \subseteq c(A)\cup c(B)$. Then it follows that $c(A\cap B) \subseteq c(c(A)\cap c(B))$ and $c(A\cup B) \subseteq c(c(A)\cup c(B))$. Since $c\left(\prod_\alpha c(A_\alpha)\right) = \prod_\alpha c(A_\alpha)$ holds generally, from the former inclusion relation we obtain (i) $c(A\cap B) \subseteq c(A)\cap c(B)$. To obtain (ii), we note that $A, B \subseteq A\cup B$ imply $c(A), c(B) \subseteq c(A\cup B)$, and hence $c(A)\cup c(B) \subseteq c(A\cup B)$. Thus $c(c(A)\cup c(B)) \subseteq c(c(A\cup B)) = c(A\cup B)$

which together with $c(A \cup B) \leq c(c(A) \cup c(B))$ implies (ii).

Lemma 6.2. For $x \in E$, $c(\{x\}) = c(\phi)$ holds if and only if $x \in c(\phi)$.

Proof. If $c(\{x\}) = c(\phi)$ then $x \in c(\phi)$ since $\{x\} \subseteq c(\{x\})$. Conversely, if $x \in c(\phi)$ then $c(\{x\}) \subseteq c(c(\phi)) = c(\phi)$. On the other hand, since $\phi \subseteq A$ for any subset A, $c(\phi) \subseteq c(A)$. Thus especially $c(\{x\}) \supseteq c(\phi)$. Hence $c(\{x\}) = c(\phi)$.

For lattice-thoretic investigation of $M = (E,c)$, we define a subset of E to be a closed set (or subspace) if $c(A) = A$ as in § 2. The set of all closed subsets of E forms a lattice $L(M)$ with respect to set inclusion. Since $c(\phi) \subseteq c(A)$ for every $c(A) \in L(M)$, $c(\phi)$ is the null-element of the lattice $L(M)$. In $L(M)$, $c(A)+c(B) = c(c(A) \cup c(B))$ and $c(A) \cdot c(B) = c(A) \cap c(B)$.

Lemma 6.3. An element $p \in L(M)$ is an atom if and only if p is represented as $p = c(\{x\})$ with $x \notin c(\phi)$.

Proof. Suppose $x \notin c(\phi)$, then $c(\phi) \subsetneqq c(\{x\})$. Suppose $c(\phi) \subseteq c(A) \subseteq c(\{x\})$ and $c(\phi) \neq c(A)$, then there exists an element $y \in c(A)$, $y \notin c(\phi)$. Then $c(\phi) \subsetneqq c(\{y\}) \subseteq c(A) \subseteq c(\{x\})$.

Since $y \in c(c(\phi) \cup \{x\}) = c(c(\phi) \cup c(\{x\})) = c(c(\{x\})) = c(\{x\})$ and $y \not\in c(\phi)$ by (c_3) we have $x \in c(c(\phi) \cup \{y\}) = c(\{y\})$. Thus $c(\{x\}) \subseteq c(\{y\})$. Therefore $c(\{x\}) = c(\{y\})$ and hence $c(\{y\}) = c(A) = c(\{x\})$. Thus if $c(\phi) \subsetneqq c(\{x\})$ then $c(\{x\})$ covers $c(\phi)$, and it is an atom in $L(M)$.

Conversely, suppose that p is an atom in $L(M)$. Let $p = A = c(A)$ and $A = \{x_1, \dots, x_m\}$. Then $c(A) = c(\{x_1\} \cup \dots \cup \{x_m\})$ $= c(c(\{x_1\}) \cup \dots \cup c(\{x_m\})) = c(\{x_1\}) + \dots + c(\{x_m\})$. Since $p \neq c(\phi) = 0$ there is an i such that $c(\{x_i\}) \neq 0$ (hence $x_i \not\in c(\phi)$). Then $p \geq c(\{x_i\}) > 0$. Since p covers 0, it follows that $p = c(\{x_i\})$.

<u>Lemma 6.4</u>. The lattice $L(M)$ satisfies the exchange property.

<u>Proof</u>. Suppose that p, q are two atoms, then there exist elements $x, y \in E$ such that $x, y \not\in c(\phi)$ and $p = c(\{x\})$, $q = c(\{y\})$. Suppose now that $p \leq a+q$ and $p \not\leq a$. Let $a = c(A)$. Then $c(\{x\}) \leq c(c(A) \cup c(\{y\})) = c(A \cup \{y\})$ and $c(\{x\}) \not\subseteq c(A)$. From the lattice it follows that $x \not\in c(A)$, and from the former we have

$x \in c\left(A \cup \{y\}\right)$. Thus by (c_3), $y \in c\left(A \cup \{x\}\right) = c\left(c(A) \cup c(\{x\})\right)$, hence $c(\{y\}) \subseteq c\left(c(A) \cup c(\{x\})\right)$. That is $q \leqslant a+p$//.

As shown above, every element $a = c(A)$ of $L(M)$ can be represented by $a = c(\{x_1\})+\cdots+c(\{x_m\})$ where $A = \{x_1,\cdots.x_m\}$. Each $c(\{x_i\})$ is either an atom or 0 of the lattice $L(M)$. Thus each lement $a \in L(M)$ is the lattice join of atoms contained in it. Since $a+p = a$ or $a+p$ covers a, every chain from $0 = c(\phi)$ to $a = c(A) = c(\{x_1\})+\cdots+c(\{x_m\})$ is of finite length.

In such lattice, exchange property is equivalent to covering property by Thm. 5.1.. Then by Thm. 5.9. and Thm. 5.10., there exists a rank function $\lambda(x)$ on $L(M)$ which satisfies

$$\lambda(x+y)+\lambda(x\cdot y) \leqslant \lambda(x)+\lambda(y) \quad \text{for } x, y \in L(M).$$

Thus far it is proved that given a matroid on a set E we can define a <u>matroid lattice</u> $L(M)$ whose elements are the closed subsets of E.

By The. 3.4., we know that given a <u>matroid lattice</u> of finite rank we can obtain a <u>simple</u> matroid $M(L)$ defined on the set of points of L, by defining a subset A of E to be independent in $M(L)$ if and only if the join of points in A has (lattice) rank equal to $|A|$.

Def. 6.5. $\Rightarrow$ Def. 6.3. Now in a matroid $M = (E,c)$ we define a rank function r on subsets A of E by $r(A) = \lambda(c(A))$. Then it can be shown that this function r satisfies (r_1), (r_2) and (r_3) in the Def. 6.3. (r_1): Since $0 \leq \lambda(c(A))$, $0 \leq r(A)$. Since $c(A) = c(\{x_1\})+\cdots+c(\{x_m\})$ for $A = \{x_1,\cdots,x_m\}$, and $b + c(\{x\})$ is either equal to b or it covers b, $r(A) = \lambda(c(A)) \leq |A|$. (r_2): If $A \subseteq B \subseteq E$ then $c(A) \subseteq c(B)$, hence $\lambda(c(A)) \leq \lambda(c(B))$ and $r(A) \leq r(B)$. (r_3): For any $A, B \subseteq E$, let $x = c(A)$, $y = c(B)$ then $r(A \cap B) = \lambda(c(A \cap B)) \leq \lambda(c(A) \cap c(B)) = \lambda(x \cdot y)$ and $r(A \cup B) = \lambda(c(A \cup B)) = \lambda(c(c(A) \cup c(B))) = \lambda(x+y)$. Then $r(A \cap B)+r(A \cup B) \leq \lambda(x\cdot y)+\lambda(x+y) \leq \lambda(x)+\lambda(y) = r(A)+r(B)$.

<u>Remark.</u> As remarked above, since Def. 6.3. $\rightleftarrows$ Def. 6.5., M is a simple matroid if and only if $c(\phi) = \phi$ and $c(\{x\}) = \{x\}$ for $x \in E$.

We have only given an introduction to the theory of matroids defined on finite sets. For the theory of matroids defined on infinte sets one should refer to papers of Rado or Brualdi and Scrimger.

§ 7. Linear geometry and descriptive geometry

In § 3 we gave an example-generalized linear geometry — whose associated lattice does not necessarily satisfy the exchange property. Since it contains a remarkable geometry — descriptive geometry, we are now discussing it in details.

A <u>generalized linear geometry</u> was defined there to be a set G of points, in which a 3-term relation (p,q,r) called <u>order</u> is defined such that

01. If (p,q,r) then p, q, r are distinct.

02. If (p,q,r) then (r,q,p).

06′. If $x \in y+r$ and $y \in p+q$, then there exists a point z such that $x \in p+z$ and $z \in q+r$.

Here we denote

$$p+q = \begin{cases} \{x \in G \mid (p,x,q)\} \cup \{p,q\} & \text{if } p \neq q, \\ p & \text{if } p = q. \end{cases}$$

<u>Definition 7.1</u>. If $a \neq b$ the set consisting of a, b and all x for which (x,a,b) or (a,x,b) or (a,b,x) is called a <u>line ab</u> or simply a <u>line</u>, and is denoted ab.

We shall concern with a generalized linear geometry which also satisfies the following postulate:

03. If $a \neq b$ there is a unique line containing a and b.

<u>Definition 7.2</u>. Any sustem which satisfies 01, 02, 06′, 03 and another postulate 04′ on connected point pair is called a

linear geometry, but in this section we are only discussing a generalized linear geometry satisfying O3, and we will call it simply linear geometry (not in the original sense).

The linear geometry can be divided into two different kinds of geometries according as it satisfies which one of the followimg two conditions.

$O7_1$. There exist points a, b, c such that (a,b,c) is true and (b,c,a) is false.

$O7_2$. If (a,b,c) then (c,b,a).

These two conditions are mutually exclusive.

Definition 7.3. We shall call the linear geometry which satisfies $O7_1$ a descriptive geometry and the one which satisfies $O7_2$ a projective geometry.

If G is projective and $a \neq b$, the line ab = a+b, (b,a,c) and (c,b,a).

Definition 7.4. In a linear geometry G, a set of points is said to be additively closed (or convex) if it contains with a, b their join a + b.

Definition 7.5. In a linear geometry G, let S be a set of points such that $a,b \in S$ $(a \neq b)$ implies $S \supset ab$ (line). Then we say S is linearly closed or a linear set. If $a_1,\cdots,a_n$ are points of G then A, the linear set generalized or spanned by $a_1,\cdots,a_n$ denotes the least linear set in G which contains

$a_1, \cdots, a_n$. We also say $a_1, \cdots, a_n$ is a system of generators of A.

Corollary. Any linearly closed set is additively closed, since if $a \neq b$, line $ab \supset a+b$.

As shown in § 3, the set of all additively closed sets of a linear geometry is a complete, upper-continuous (denoted as property (II)) and atomistic (denoted as property (I)) lattice L_G which is also linear, that is

(IV′) (Linear). If a point $p \leq a+b$, $a,b \neq 0$, then there exist $q \leq a$ and $r \leq b$ such that $p \leq q+r$.

Furthermore, it also follows from § 3 that the set of all linear sets forms a complete, upper-continuous and atomistic lattice $\tilde{L}_G$ which is a $(\cdot)$-subband of the lattice L_G by the above corollary.

Let $p_1 \vee \cdots \vee p_n$ be the linear set generated by $p_1, \cdots, p_n$. Then $a_1 \vee \cdots \vee a_n \geq a_1 + \cdots + a_n$. For $n = 2$, this is true since line $pq = p \vee q \geq p+q$. Assume that this is true for k. Let $p \in p_1 + \cdots + p_k + p_{k+1}$. Then by (IV′) there exists $q \in p_1 + \cdots + p_k$ such that $p \in p_{k+1} \subseteq q \vee p_{k+1} \subseteq p_1 \vee \cdots \vee p_k \vee p_{k+1}$, since $q \in p_1 + \cdots + p_k \subseteq p_1 \vee \cdots \vee p_k$ by induction assumption. Thus $p_1 + \cdots + p_n \subseteq p_1 \vee \cdots \vee p_n$.

Now if G is projective then $p_1 + \cdots + p_k$ is a linear set, since $p,q \in p_1 + \cdots + p_k$ implies $p \vee q = p+q \subseteq p_1 + \cdots + p_k$. Then by the definition of $p_1 \vee \cdots \vee p_k$ we have $p_1 + \cdots + p_k = p_1 \vee \cdots \vee p_k$.

Thus if G is projective, its linear sets comprise L_G (that is, $\tilde{L}_G = L_G$). If G is descriptive its linear sets does not comprise L_G.

In order to study more properties of L_G, we recall first that in an arbitrary lattice L, a sequence $p_1,\cdots,p_n$ is called <u>independent</u> if $\left(p_{i_1}+\cdots+p_{i_r}\right)\left(p_{j_1}+\cdots+p_{j_s}\right) = 0$ for every choice of $i_1,\cdots,i_r$, $j_1,\cdots,j_s$ as distinct integers in the range $1,\cdots,n$. In the contrary case we use the term <u>dependent</u>.

<u>Definition 7.6.</u> If $p, p_1,\cdots,p_n$ are points of L such that

$$\left(p+p_{i_1}+\cdots+p_{i_r}\right)\left(p_j+\cdots+p_{j_s}\right) \neq 0,$$

$$\left(p_{i_1}+\cdots+p_{i_r}\right)\left(p_{j_1}+\cdots+p_{j_s}\right) = 0$$

for some choice of $i_1,\cdots,i_r$, $j_1,\cdots,j_s$ in the range $1,\cdots,n$, we say <u>p is dependent on $p_1,\cdots,p_n$</u>.

By this definition with $n = 2$, we see that p is dependent on p_1, p_2 if and only if $p \in$ line $p_1p_2 = p_1 \vee p_2$, since if p is dependent on p_1, p_2 then either $p(p_1+p_2) = 0$ or $(p+p_1)p_2 = 0$, $p_1 \cdot p_2 = 0$ or $(p+p_2)p_1 = 0$, $p_1 \cdot p_2 = 0$, that is either $p = p_1$ or $p = p_2$ or (p_1,p,p_2) or (p,p_2,p_1) or (p,p_1,p_2), hence $p \in$ line p_1p_2, conversely if $p \in$ line p_1p_2, by reversing the above argument we see that p is dependent on p_1, p_2.

Lemma 7.1. In a linear geometry G, L_G satisfies the following:

(III′) p is dependent on $p_1,\cdots,p_n$ if and only if $p \in p_1 \vee \cdots \vee p_n$ the linear set generated by $p_1,\cdots,p_n$.

Proof. Suppose p dependent on $p_1,\cdots,p_n$ in L_G. Then there exist integers $i_1,\cdots,i_r$, $j_1,\cdots,j_s$ in the range $1,\cdots,n$ such that

$$\left(p+p_{i_1}+\cdots+p_{i_r}\right)\left(p_{j_1}+\cdots+p_{j_s}\right) \neq 0$$

$$\left(p_{i_1}+\cdots+p_{i_r}\right)\left(p_{j_1}+\cdots+p_{j_s}\right) = 0.$$

From the first relation we have

$$x \in p+p_{i_1}+\cdots+p_{i_r}$$

$$x \in p_{j_1}+\cdots+p_{j_s}$$

for some point x. If $x = p$ then $p \in p_j+\cdots+p_{j_s} \subseteq p_{j_1} \vee \cdots \vee p_{j_s}$, sinse the last inclusion relation was proved before.

Suppose $x \neq p$. Then $p_i+\cdots+p_{i_r} \neq 0$, since otherwise $x \in p$. Then $x \in p+p_{i_1}+\cdots+p_{i_r}$ implies that x $p+y$ with $y \in p_{i_1}+\cdots+p_{i_r}$. Since $x \in p_j+\cdots+p_{j_s}$, $y \in p_i+\cdots+p_{i_r}$ and $p_j+\cdots+p_{j_s}$,

$p_{i_1}+\cdots+p_{i_r} \subseteq p_1 \vee \cdots \vee p_n$, the line $xy = x \vee y \subseteq p_1 \vee \cdots \vee p_n = P$. We shall show now that p line x y. From $x \in p_{j_1}+\cdots+p_{j_s}$, and $\left(p_{i_1}+\cdots+p_{i_r}\right)\left(p_{j_1}+\cdots+p_{j_s}\right) = 0$ it follows that $x \neq y$. Then from $x \in p+y$ it follows that (p,x,y), since if $p = y$ then $p+y = p = x$ contradictory to $p \neq x$. Thus $p \notin$ line x y follows from the definition of line.

To prove the sufficiency we treat the projective and descriptive cases separately. Suppose G is projective, then $P = p_1 \vee \cdots \vee p_n = p_1+\cdots+p_n$. Hence $p \in P$ inplies $p \in p_1+\cdots+p_n$ so that p is dependent on $p_1,\cdots,p_n$ in L_G.

Now let G be a descriptive geometry, then the sufficiency follows immediately from:

<u>Lemma 7.2.</u> In a descriptive geometry G, let $p_1,\cdots,p_t$ form a minimal system of generators of a linear set A. Then $p \in A$ implies that p is dependent on $p_1,\cdots,p_t$ in L_G.

From a minimal system of generators $p_{k_1},\cdots,p_{k_t}$ of P by deleting superfluous elements from $p_1,\cdots,p_n$ one by one. Then by the above lemma 7.2., $p \in P = p_1 \vee \cdots \vee p_n$ implies that p is dependent on $p_{k_1},\cdots,p_{k_t}$ and so on $p_1,\cdots,p_n$.

The proof of Lemma 7.2., will be given later.

Definition 7.7. Suppose a is an element of an arbitrary lattice such that the relation "p is dependent on $p_1, \cdots, p_n \leq a$" always imply $p \leq a$. Then we say a is a linearly closed element of L.

Since in L the meet of closed elements is closed, if a_i, $i \in I$ is a system of elements of L, there exists in L a unique least closed element containing the a_i.

Definition 7.8. The least closed element containing p_i, $i \in I$ is called the closure of the system p_i, $i \in I$; for a finite system p_i, $1 \leq i \leq n$, it is denoted by $\{p_1, \cdots, p_n\}$.

Since p is dependent on p_1, p_2 if and only if $p \in \text{lin}\, p_1p_2 = p_1 \vee p_2$, any closed element of L_G is a linear set of G, that is an element of $\tilde{L}_G$. Let a be a closed element and let p_1, p_2 be two points with $p_1, p_2 \leq a$. Let p be a point on the line p_1p_2, then p is dependent on p_1, p_2 so $p \leq a$. Thus a is a linear set of G. Conversely, if A is a linear set of G and in L_G, p is dependent on $p_1, \cdots, p_n \in A$. Then by Lemma 7.1., $p \in p_1 \vee \cdots \vee p_n \subseteq A$. Thus A is a closed element of L_G. Thus we have shown:

Lemma 7.3. The closed elements of L_G are identical with the linear sets of G, hence they are identical with elements of $\tilde{L}_G$. Thus $\{p_1,\cdots,p_n\} = p_1 \cdots p_n$.

This yields the following:

<u>Corollary</u>. If G is projective, <u>(VI_1) every element of L_G is closed</u>. If G is descriptive, <u>(VI_2) not every element of L_G is closed</u>.

In view of Lemma 7.1, and Lemma 7.3, we have

<u>Lemma 7.4</u>. In a linear geometry G, L_G satisfies the following:

(III″) p is dependent on $p_1,\cdots,p_n$ if and only if $p \leq \{p_1,\cdots,p_n\}$.

<u>Lemma 7.5</u>. In a complete, upper-continuous, atomistic lattice, two conditions (III) and (III″) are equivalent, where

(III) If p is dependent on $q_1,\cdots,q_m$ and $q_1,\cdots,q_m$ are dependent on $r_1,\cdots,r_n$, then p is dependent on $r_1,\cdots,r_n$.

<u>Proof</u>. (III″) $\Rightarrow$ (III). By (III″) $p \leq \{q_1,\cdots,q_m\}$ and $q_1,\cdots,q_m \leq \{r_1,\cdots,r_n\}$. Hence $\{q_1,\cdots,q_m\} \leq \{r_1,\cdots,r_n\}$ and $p \leq \{r_1,\cdots,r_n\}$. By (III″) again, p is dependent on $r_1,\cdots,r_n$.

(III) $\Rightarrow$ (III″). In L, let

$$\alpha = \{p : \text{point} \mid p \text{ is dependent on } p_1, \cdots, p_n\}.$$

(A) <u>If</u> p <u>is a point</u>, p <u>is dependent on</u> $p_1, \cdots, p_n$ <u>if and only if</u> $p \leq \sup \alpha$.

If p is dependent on $p_1, \cdots, p_n$, then $p \in \alpha$ and $p \leq \sup \alpha$. Conversely, suppose that $p \leq \sup \alpha$. Then by (II) upper-continuity, $p \leq q_1 + \cdots + q_m$ with $q_i \in \alpha$, $i = 1, \cdots, m$. Thus by the definition of dependence of p on a system of points, p is dependent on $q_1, \cdots, q_m$. Each q_i being in α, is dependent on $p_1, \cdots, p_n$ so that by property (III) p is dependent on $p_1, \cdots, p_n$.

Now we show that $\sup \alpha = \{p_1, \cdots, p_n\}$. Since $p_1, \cdots, p_n \in \alpha$, $p_1, \cdots, p_n \leq \sup \alpha$. If x is closed and $p_1, \cdots, p_n \leq x$ then $p \leq x$ for all p dependent of $p_1, \cdots, p_n$. Thus $x \geq p$ for all element p in α. Hence $x \geq \sup \alpha$. Thus to infer $\sup \alpha = \{p_1, \cdots, p_n\}$, it suffices to show $\sup \alpha$ closed. To prove this, suppose q dependent on $r_1, \cdots, r_\ell \leq \sup \alpha$. Then by (A) each r_j is dependent on $p_1, \cdots, p_n$. Hence by (III) q is dependent on $p_1, \cdots, p_n$ and (A) implies $q \leq \sup \alpha$. Thus by Def. 7.7., $\sup \alpha$ is closed and $\sup \alpha = \{p_1, \cdots, p_n\}$ is verified. Conbining (A) and $\sup \alpha = \{p_1, \cdots, p_n\}$ we have the desired result.

<u>Lemma 7.6</u>. In a complete, upper-continuous, atomistic lattice satisfying (III), the two conditions (IV) and (IV′) are equivalent, where

(IV). Let c be closed. Then $a \leq c$ implies $(a+b) = a+bc$, that is $M(bc)$ provided that c is closed.

<u>Proof</u>. (IV) $\Rightarrow$ (IV): (IV) is trivial if a or $b = 0$. Suppose $a, b \neq 0$. Clearly $a+bc \leq (a+b)\cdot c$. To prove the converse inclusion we show that $t \leq (a+b)\cdot c$ implies $t \leq a+bc$. Since $t \leq a+b$, by (IV′) $t \leq p+q$ with $p \leq a$, $q \leq b$. Obviously $p \leq c$ also holds. If $t = p$, then obviously $t = p \leq a+bc$. Suppose $t \neq p$. Then by definition q is dependent on p, t (since $(p+q)\cdot t = 0$, $p\cdot t = 0$). Hence $p, t \leq c$ and c is closed so $q \leq c$. Thus $q \leq bc$ so that $p+q$ $a+bc$ and $t \leq a+b\cdot c$.

Before showing the implication (IV) $\Rightarrow$ (IV′), we show the following:

<u>Lemma 7.7</u>. In a complete, upper-continuous, atomistic lattice satisfying (III″) and (IV), if p, q, r, s are points and $r \neq s$ then $p \vee q \geq r, s$ implies $p \vee q = r \vee s$.

<u>Proof</u>. Since $r \neq s$, one of r, s is distinct from p, say r. We show first that in view of $p \vee q \geq r$ and $p \neq r$ we say "exchange" q for r getting $p \vee q = p \vee r$. Clearly $p \vee q \geq p \vee r$ ($p \vee q \geq r$ implies $p \vee q \geq p \vee r$). By (III″) $p \vee q \geq r$ implies that r is dependent on p, q. Thus either (i) $r(p+q) \neq 0$ or (ii) $(r+p)\cdot q = 0$ and $p\cdot q = 0$ or (iii) $(r+q)\cdot p \neq 0$ and $p\cdot q = 0$.

For the case (i), we have $(p+q)r = r$. By (IV) we have $(p+q)(r \vee p) = p+q(r \vee p)$ isnce $r \vee p$ is closed. If $q(r \vee p) = 0$ then $(p+q)(r \vee p) = p$ contradictory to $p+q \geqslant r$, $r \vee p \geqslant r$ and $r \neq p$. Thus $q(r \vee p) = q$ and $(p+q)(r \vee p) = p+q$. Hence $r \vee p \geqslant p+q \geqslant q$ so $r \vee p \geqslant p \vee q$ and consequently $p \vee q = r \vee p$.

Case (ii): We have $r+p \geqslant q$. Then $r \vee p \geqslant r+p \geqslant q$. Hence $r \vee p \geqslant p \vee q$ and consequently $r \vee p = p \vee q$.

Case (iii): $(r+q)p \neq 0$. Since $r \neq p$ so this means that q is dependent on p, r and $q \leqslant p \vee r$, hence $p \vee q \leqslant p \vee r$ and $r \vee p = p \vee q$.

Hence $r \vee p = p \vee q \geqslant s$ and $r \neq s$, we may, by the above argument "exchange" p for s in $r \vee p$ getting $r \vee p = r \vee s$. Thus $p \vee q = r \vee s$ and the proof is complete.

<u>Proof of Lemma 7.6 (contnued)</u> (IV) $\Rightarrow$ (IV'). Let $p \leqslant a+b$ and $a,b \neq 0$. First consider the case in which <u>a</u> is a point. If $p = a$ or $p \leqslant b$ the result is trivial, since each non-zero element contains a point by (I) the atomisticity. Suppose $p \neq a$, $p \nleqslant b$. Then we have by (IV)

$$p \leqslant (a+b)(a \vee p) = a+b(a \vee p) = a+b',$$

where $b' = b(a \vee p)$ and $\{a,p\}$ is also denoted by $a \vee p$.

Let α be the set of points contained in b', and α_1 be the set formed by adjoining a to α. Then by (I), $b' = \sup \alpha$ so that $p \leqslant a+b = a+\sup \alpha = \sup \alpha_1$. Hence by (II), p is contained in the join of a finite subset of α_1. Since $p \not\leq b$, this finite subset of α_1 must contain a (assume that a is not contained in this set, then p is contained in b', that is $p \leq b(a \vee p)$ so it is also contained in b contradictory to the supposition $p \not\leq b$). Thus

$$(1) \qquad p \leq a+r_1+\cdots+r_m$$

where r_i are points, $r_i \leq b'$, $1 \leq i \leq m$. In (1), $m \geq 1$ since otherwise $p = a$ contradictory to our supposition. We may assume that superfluous r's have been deleted in (1). Thus the r_i are distinct.

We show $m = 1$. Suppose $m > 1$. Then there are at least two $r_1, r_2 \leq b' \leq a\ p$. Since $r_1 \neq r_2$, by Lemma 7.7. we have $r_1 \vee r_2 = a \vee p \geq a$. Thus by (III″) a is dependent on r_1, r_2. Then either $a(r_1+r_2) \neq 0$ or $(a+r_1)r_2 \neq 0$ or $(a+r_2)r_1 \neq 0$, that is one of the points a, r_1, r_2 is contained in the join of the other two. This is not true for a (that is, $a \not\leq r_1+r_2$), since $a \leq r_1+r_2$ implies the redundance of a in (1) so that $p \leq b$ contradictory to our supposition. Hence $(a+r_1)r_2 \neq 0$ or $(a+r_2)r_1 \neq 0$, that is r_2 or r_1 is superfluous in (1). This contradiction implies $m = 1$, and the proposition holds with $q = a$ and $r = r$.

Now consider the general case with no restriction on a.

By (I) and (II) as used to derive (1), we can show

$$(2) \qquad p \leqslant q_1+\cdots+q_n+r_1+\cdots+r_m = q_1+\cdots+q_n+b,$$

where q_i, r_j are points $q_i \leqslant a$, $1 \leqslant i \leqslant n$, $n \geqslant 1$; $r_j \leqslant b$, $1 \leqslant j \leqslant m$. Applying to (2) the "restricted" result with q_1 playing the role of a, we have

$$(3) \qquad p \leqslant q_1+p_1, \quad p_1 \leqslant q_2+\cdots+q_n+b$$

for some p_1. Similarly, if $n > 1$ the second relation in (3) implies

$$p_1 \leqslant q_2+p_2, \quad p_2 \leqslant q_3+\cdots+q_n+b$$

for some point p_2. Continuing in this way we obtain a sequence of points $p_0 = p, p_1, \cdots, p_n$ such that

$$p_i \leqslant q_{i+1}+p_{i+1}, \quad (0 \leqslant i \leqslant n-1), \quad p_n \leqslant b.$$

Eliminating $p_1, \cdots, p_{n-1}$ successively from these relations we have

$$(4) \qquad p \leqslant q_1+\cdots+q_n+p_n.$$

Applying to (4) "restricted" result with p_n playing the role of a, we have $p \leqslant q+p_n$ for some point $q \leqslant q_1+\cdots+q_n \leqslant a$ and the proof is complete.

We now summarize the results of the above discussion concering L_G as

Theorem 7.1. For any linear geometry, L_G is complete and satisfies (I),...,(IV). If G is projective (descriptive) L_G satisfies in addition (VI_1) ((VI_2)).

Given a complete lattice L which satisfies (I), (II), (IV′) we have constructed an "associated" geometry G_L which is generalized linear geometry such that L is isomorphic to the lattice of additively closed sets of the geometry G_L. If L satisfies (III), we shall now prove that G_L is a linear goemtry. For this aim we need only show 03 (01, 02, 06′ were proved in § 3). To aid in the verification of 03 we deduce a criterion that a point belong to a line, in G_L; line in G_L being determined by Def. 7.1. in terms of order.

Lemma 7.8. Suppose a, b, p are points of G_L and $a \neq b$. Then $p \in$ line ab if and only if in L, $p = \{a,b\}$.

Proof. By Def. 7.1., Def. 7.6. and the definition of 3-term order relation in G_L, $p \in$ line ab is equivalent in L to "p is dependent on a, b". By (III″), this is equivalent to $p \leq \{a,b\}$.

Proof of 03 in G_L. To show this it suffices to prove: if p, q, r, s are points of G_L, $r \neq s$ and line pq $\ni$ r,s then line pq = line rs. Assume the hypothesis of this proposition. Then in L, by Lemma 7.8., $\{p,q\} \geq r,s$ and the lemma 7.7. implies $\{p,q\} = \{r,s\}$. Hence by Lemma 7.8. each point in line pq is in line rs and vice versa. Thus line pq = line rs and O3 holds in G_L.

Thus we have

Theorem 7.2. Let L be a complete lattice satisfying (I), ...,(IV). Then its associated geometry G_L is a linear geometry.

By Theorem 3.4 and Theorem 3.5, we also have

Theorem 7.3. Let L be a complete lattice satisfying (I), ...,(IV). Let L* be the associated lattice of the linear geometry G_L. Then L is isomorphic to L*.

We impose further restrictions on L in order to distinguish the cases in which G_L is projective and descriptive. Suppose L satisfies (VI_1) that every element is closed. Then by Thm. 7.3. L* the associated lattice of G_L, also satisfies (VI_1). This is due to the fact that the isomorphism preserves the dependence of a point on a sequence of points, and hence maps closed element into closed element. Thus G_L must be projective, since otherwise it would be descriptive and L* would satisfy (VI_2), the contradictory of (VI_1). Similarly, if L satisfies (VI_2) instead of (VI_1), G_L must be descriptive. Thus we have

Theorem 7.4. The lattice of linear spaces (convex sets) of a projective (descriptive) geometry is characterijed by the properties (I),...,(IV) completeness and modularity (non modularity).

In the above discussion we have assumed the validity of Lemma 7.2. If it would be able to prove it using only 01, 02, 06′ and 03 then Thm. 7.1, Thm. 7.2 and Thm. 7.3 would give the character-ization of the lattice of additively closed sets of the generalized lineat geometry satisfying 03. It might be done, but in the paper of Prenowitz its proof depends also on the following pastulate:

<u>04</u>′. If a, b are points, then a is projective to b.

In the statement of this postulate we have used the

<u>Definition</u> <u>7.9</u>. If a, b are points such that (abc) for some c or a = b, we say a <u>is perspective to</u> b. If a, a′ are points such that there exists a sequence $a = a_1, a_2, \cdots, a_n = a'$ in which each term is perspective to its success or, we say a <u>is projective to</u> a′.

Then corresponding to Thm. 7.1, Thm. 7.2 and Thm. 7.3, we have the followings.

<u>Theorem</u> <u>7.1′</u>. For any linear geometry (01, 02, 06′, 03, 04′) L_G is complete and satisfies (I),$\cdots$,(IV) and

(V) There exist only trivial complete congruence relations.

If G is projective (descriptive) L_G satisfies in addition $(VI_1)((VI_2))$.

<u>Definition</u> <u>7.10</u>. Let ≡ be an equivalence relation in lattice L such that a ≡ a′, b ≡ b′ implies a+b ≡ a′+b′ and a·b ≡ a′·b′. Then we call ≡ a <u>congruence</u> <u>relation</u> in L. Suppose in addition

that $a_i \equiv a'_i$, where i ranges over an arbitrary set I, implies $\Sigma_i a_i = \Sigma_i a'_i$ whenever the members of this relation exist, and likewise for $\Pi_i a_i \equiv \Pi_i a'_i$. Then $\equiv$ is a complete congruence relation in L. Any lattice L admits the trivial congruence relations (which are complete) defined by (1) $a \equiv b$ if $a = b$; (2) $a \equiv b$ if a, b are in L. L is smiple if it admits only trivial congruence relations.

Lemma 7.8. L_G admits only trivial complete congruence relations.

Proof. Let $\equiv$ be a non-trivial congruence relation in L_G. Then there exist A, B in L_G such that $A \equiv B$ and $A \neq B$, Hence there exists a point p which is contained in just one of A, B, say A. (that is $p \in A$, $p \notin B$). Then $p \equiv 0$, $p = p \cdot A \equiv p \cdot B = 0$. We show $x \equiv 0$, for each x of L_G. First suppose $x \neq p$ and p is perspective to x. Then in G we have $(p\ x\ y)$ for some point y. Thus $x \neq y$ and $x \in p+y$. (hence $p+x \leq p+y$). Since $p \equiv 0$ we have $p+x \equiv x$ and $p+y \equiv y$. Hence $x \equiv p+x = (p+x)(p+y) \equiv x \cdot y = 0$. By induction we see that $x \equiv 0$ merely if p is projective to x. Hence by O4′ $x \equiv 0$ for each point x.

Now suppose $\equiv$ is a non-trivial complete congruence relation in L_G. Since each element of L_G is expressible as join of points,

$\equiv$ annihilates in L_G every join of points and so every element. This contradicts the supposition that $\equiv$ is non-trivial.

Theorem 7.2′. Let L be a complete lattice satisfying (I), ..., (IV) and (V). Then its associated geometry G_L is a linear geometry (satisfying 01, 02, 06′, 03, 04′).

Lemma 7.9. If a, b are points of G_L then a is projective to b.

Proof. Let a be an arbitrary chosen point of G_L, and let α be the set of points of G_L to which a is projective. In L we define $x \equiv y$ to mean that any point which is contained in exactly one of x, y is an element of α. We show that $\equiv$ is a complete congruence relation in L. It is easily seen that $\equiv$ is an equivalence. Obviously $x \equiv x$, since no point is contained in exactly one side of $\equiv$. It is also obvious that $x \equiv y$ implies $y \equiv x$. Suppose that $x \equiv y$ and $y \equiv z$, and suppose that a point $p \leqslant x$ but $p \nleqslant z$. If $p \nleqslant y$ then $p \in \alpha$ since $x \equiv y$. If $p \leqslant y$, since $p \nleqslant z$, $p \in \alpha$ as $y \equiv z$.

We suppose $x \equiv x'$, $y \equiv y'$ and we infer $x+y \equiv x'+y'$. To do this, we assume $p \leqslant x+y$, $p \nleqslant x'+y'$ for an arbitrary point p, and we show that $p \in \alpha$. First consider the case $p \leqslant x$. By definition $x \equiv x'$ implies $p \leqslant x'$ or $p \in \alpha$. The former contradicts $p \nleqslant x'+y'$ and so the latter holds. Likewise $p \leqslant y$

implies $p \in \alpha$. Now suppose $p \nleq x,y$. This with $p \leq x+y$ implies $x,y \neq 0$. Since L is linear, $p \leq x+y$ implies $p \leq q+r$ where q, r are points and $q \leq x$, $r \leq y$. Since $x \equiv x'$, $y \equiv y'$, $q \leq x'$ or $q \in \alpha$, and $r \leq y'$ or $r \in \alpha$. If $q \leq x'$ and $r \leq y'$ then $p \leq q+r \leq x'+y'$ contradicts $p \nleq x'+y'$. Hence one of q, r let us say q is in α. If $q = r$ then by $p \leq q+r$ we have $p = q \in \alpha$. If $p = r$ then $p = r \leq y'$ contrary to supposition $p \nleq x'+y'$. Thus we need consider only the case p, q, r distinct. In this case we have from $p \leq q+r$ (qpr) by definition. Since $q \in \alpha$, a is projective to q. By (q,p,r), q is perspective to p. Hence a is projective to p, and $p \in \alpha$. An identical argument holds if we interchange x, y and x', y' respectively and assume $p \nleq x+y$, $p \leq x'+y'$. Thus by definition $x+y \equiv x'+y'$. An easy induction shows that $x_i \equiv x_i'$, $1 \leq i \leq n$, implies $x_1+\cdots+x_n \equiv x_1'+\cdots+x_n'$.

Now suppose $x_i \equiv x_i'$ for each i in I, an arbitrary set of indices. We show $\Sigma_i x_i \equiv \Sigma_i x_i'$, $\Pi_i x_i = \Pi_i x_i'$. Suppose point $p \leq \Sigma_i x_i$. By (I) and (II) we have $p \leq x_{i_1}+\cdots+x_{i_n}$ where i's are in I. $x_i \equiv x_i'$ implies $x_{i_1}+\cdots+x_{i_m} \equiv x_{i_1}'+\cdots+x_{i_m}'$ so that $p \in \alpha$ or $p \leq x_{i_1}'+\cdots+x_{i_m}' \leq \Sigma_i x_i'$. Similarly point $p \leq \Sigma_i x_i'$ implies $p \in \alpha$ or $p \leq \Sigma_i x_i$ and $\Sigma_i x_i \equiv \Sigma_i x_i'$ is verified.

To justify $\Pi_i x_i \equiv \Pi_i x_i'$ suppose point $p \leq \Pi_i x_i$ then $p \leq x_i$ for each i of I. Suppose p not in α. Then $p \leq x_i'$ for each i of I and $p \leq \Pi_i x_i'$. Thus $\Pi_i x_i \equiv \Pi_i x_i'$ follows, and $\equiv$ is a complete congruence relation in L.

Observe that <u>point</u> $p \equiv 0$ <u>if and only if</u> p <u>is in</u> α. Since a is in α, $a \equiv 0$. By (V) this relation $\equiv$ is trivial and annihilates every element z (i.e. $z \equiv 0$). Especially every point point $p \equiv 0$. Thus α contains all point of G_L. Thus a is projective to every point of G_L.

<u>Theorem 7.3'</u>. Let L be a complete lattice satisfying (I), ...,(IV) and (V). Let L^* be the associated lattice of the linear geometry G_L (01, 02, 06',03, 04'). Then L is isomorphic to L^*.

<u>Descriptive geometry</u>.

We have defined above a linear geometry to be a system G which satisfies 01, 02, 03, 04', 06' and called a linear geometry which satisfies 07' a descriptive geometry.

Veblen's classical postulate system for descriptive geometry is in a slightly modified form 01, 03, 04, 05, 06, 07, to which it is equivalent. The postulates not yet appeared before are listed in the followings.

04. If a, b are points, then a is perspective to b.

05. There exist distrinct points a, b, c which do not colline.

06. If a, b, c do not colline and if (bcd), (cea) then there exists f such that (def) and (afb).

07_1. If (abc) then (bca) is false (this postulate is denoted by 02 in Veblen's original notation).

It is proved by Prenowitz that in the face of 05, our definition of descriptive geometry (01, 02, 03, 04', 06', 07_1) is equivalent to that of Veblen (01, 03, 04, 05, 06, 07_1).

We are now giving some postulates which are convenient calculation:

In a descriptive geometry we define

$$a \dot{+} b = \begin{cases} \{x \mid (axb)\} & \text{if } a \neq b \\ a & \text{if } a = b. \end{cases}$$

Here we agree to identify element a and the set $\{a\}$ whose only one element is a. There si no inconsistency in the definition, since in the familiar formula of elementary analytic geometry for the point which divides segment ab in a given ratio r, $0 < r < 1$, if we set $a = b$, we get point a itself. The agreement that $a \dot{+} a = a$ is essential for the unrestricted validity of the following J3.

From the axiom system of Veblen, the following postulates can be obtained.

J1. If $a, b \in G$, $a \dotplus b$ is a uniquely determined non-empty subset of G. [See H. G. Forder, The foundations of euclidean geometry. Cambridge Univ. Press. (1927) p. 50 Theorem 7].

J2. If $a, b \in G$, $a \dotplus b = b \dotplus a$. [Forder p. 45, Theorem 2].

Definition 7.11. Let A, B be non-empty subsets of G, then $A \dotplus B$ is the set union $\bigcup_{a \in A, b \in B}(a \dotplus b)$. For an arbitrary subset A of G we define $A+0 = 0+A = A$.

J3. If $a, b, c \in G$, $(a \dotplus b) \dotplus c = a \dotplus (b \dotplus c)$. [$(a \dotplus b) \dotplus d = a \dotplus (b \dotplus d)$ is a restatement of 06].

J4. If $a \in G$, $a \dotplus a = a$. [above definition].

J5. If $a, b \in G$, the relation $b \dotplus x \ni a$ has a solution x in G. [restatement of 04 for the existence of x such that (bax)].

Definition 7.12. Suppose $a, b \in G$. Then $a \dot{-} b$ is the set of x for which $b \dotplus x \ni a$.

J6. If $a \in G$, $a \dot{-} a = a$. [This signifies that a segment $a \dotplus b$ does not contains its end points and is essentially a form of 01. (That is, $a \dotplus x \ni a$ implies $x = a$, since otherwise $a \dotplus x \not\ni a$)]

Definition 7.13. Let A, B be subsets of G. Then $A \approx B$ means that A and B have common element, that is the set product $A \cap B \neq 0$.

J7. Suppose $a, b, c, d \in G$. Then $a \dot{-} b \approx c \dot{-} d$ implies $a \dot{+} d \approx b \dot{+} c$. [Forder p.55, Thm. 11.6. $f \dot{-} a \approx c \dot{-} d$ implies that there is a point $b \in (f \dot{-} a) \cap (c \dot{-} d)$ whcih in turn implies that $(a+b)$ and (bcd). If a, b, c do not colline then by Thm. 11.6 there is a point e such that (cea) and (def). Thus $c \dot{+} a \approx d \dot{+} f$. If a, b, c are collinear, then $c \dot{+} a \approx d \dot{+} f$ holds in descriptive geometry].

It can be shown that postulate O3 is independent of J1, ⋯, J7 and that J1, ⋯, J7 together imply O1, O7₁, O4, O6 [Prenowitz]. Thus the system J1, ⋯, J7 plus O3 ahd O5 is equivalent to O1, O7₁, O4, O6, O3, O5, the postulate system of Veblen for descriptive geometry.

From the above postulates and definitions, we can derive the following algebraic formulas and formal properties constantly used in the sequel:

(7.2.1). Suppose $0 \neq A' \subset A$ and $0 \neq B' \subset B$. Then $A' \dot{+} B' \subset A \dot{+} B$.

(7.2.3). a) $A \dot{+} B = B \dot{+} A$, b) $(A \dot{+} B) \dot{+} C = A \dot{+} (B \dot{+} C)$, c) $A \subset A \dot{+} A$.

Definition 7.14. If $A, B \neq 0$, then $A \dot{-} B$ denotes the set union $\bigcup_{a \in A, b \in B} (a \dot{-} b)$. For arbitrary A, we define $A \dot{-} 0 = A$, $0 \dot{-} A = 0$.

(7.2.4). $A \dot{-} B \approx C$ implies $A \approx B \dot{+} C$. Conversely, $A \approx B \dot{+} C$ implies $A \dot{-} B \approx C$ provided $C \neq 0$.

Proof. Suppose $A \dot{-} B \approx C$. If $B = 0$, then $A = A \dot{-} B \ (= 0) \approx C$

so certainly $A \approx B\,(=0)\dot{+}C = C$. Suppose $B \neq 0$. Let $c \in A\dot{-}B, C$. Then $A \neq 0$ (since otherwise $A(=0)\dot{-}B = 0$) and by definition $c \in a\dot{-}b$ where $a \in A$, $b \in B$. Hence $a \in b\dot{+}c \subset B\dot{+}C$ so that $A \approx B\dot{+}C$. Conversely suppose $A \approx B\dot{+}C$ and $c \neq 0$. If $B = 0$ then $A \approx B(=0)\dot{+}C = C$ and $A\dot{-}B(=0) = A \approx C$ is trivial. Suppose $b \neq 0$. Let $a \in A$, $B \dot{+} C$ then $a \in b\dot{+}c$ with $b \in B$, $c \in C$. Hence $c \in a\dot{-}b \subset A\dot{-}B$ so that $A\dot{-}B \approx C$.

(7.2.5). $A\dot{-}B \approx C\dot{-}D$ implies $A\dot{+}D \approx B\dot{+}C$.

<u>Proof</u>. If $B = D = 0$ then $A\dot{-}B = A$, $C\dot{-}D = C$ and $A\dot{-}B \approx C\dot{-}D$ turns out to be $A \approx C$. This together with $A\dot{+}D = A$, $B\dot{+}C = C$ imply $A\dot{+}D \approx B\dot{+}C$. Suppose only one of $B, D = 0$, say $B = 0$. Then from the assumption $A \approx C\dot{-}D$ so that by (7.2.4) $A\dot{+}D \approx C = B\dot{+}C$ and the theorem holds. Now suppose $B, D \neq 0$. Let $x \in A\dot{-}B, C\dot{-}D$. Certainly $A, C \neq 0$ since otherwise $A = 0$, $C = 0$ implies $A\dot{-}B = 0$, $C\dot{-}D = 0$. Hence by definition $x \in a\dot{-}b, c\dot{-}d$ where $a, b, c, d \in A, B, C, D$ respectively. Thus by definition $a\dot{-}b \approx c\dot{-}d$ and by J7 $a\dot{+}d \approx b\dot{+}c$. The conclusion is immediate since $A\dot{+}D \supset a\dot{+}d$ and $B\dot{+}C \supset b\dot{+}c$.

(7.2.6). $a\dot{-}(b\dot{+}c) = (a\dot{-}b)\dot{-}c$; $A\dot{-}(B\dot{+}C) = (A\dot{-}B)\dot{-}C = (A\dot{-}C)\dot{-}B$.

(7.2.7). $a\dot{-}(b\dot{-}c) \subset (a\dot{+}c)\dot{-}b$; $A\dot{-}(B\dot{-}C) \subset (A\dot{+}C)\dot{-}B$.

(7.2.8). $a+(b-c) \subset (a+b)-c$; $A+(B-C) \subset (A+B)-C$ provided $B \neq 0$.

(7.2.9). $a-(a-b) \ni b$; $A-(A-B) \supset B$ provided $A \neq 0$.

(7.2.9′). $a \dot{-} (a \dot{-} b) \supset a \dotplus b$; $A \dot{-} (A \dot{-} B) \supset A \dotplus B$ provided $A \neq 0$.

(7.2.10). Suppose $A, B \neq 0$. Then $(A \cup B) \dot{\pm} C = (A \dot{\pm} C) \cup (B \dot{\pm} C)$ and $C \dot{-} (A \cup B) = (C \dot{-} A) \cup (C \dot{-} B)$.

We can also define the <u>additively closed</u> (or <u>convex</u>) <u>set</u> S to be the set having the property that $a, b \in S$ implies $a \dotplus b \subset S$. Then we have

(7.4.1). A is additively closed (convexP if and only if a). $A \supset A \dotplus A$ or b). $A = A \dotplus A$.

(7.4.1′). Let $A \supset B$ where A is additively closed. Then $A \supset A \dotplus B$.

(7.4.2). Let A, B be additively closed. Then $A \cap B$, $A \dotplus B$, $A \dot{-} B$ are also additively closed (convex).

<u>Proof.</u>

$$\begin{aligned}
(A \dot{-} B) \dotplus (A \dot{-} B) &\subset ((A \dot{-} B) \dotplus A) \dot{-} B && \text{by } (7.2.8) \\
&= (A \dotplus (A \dot{-} B)) \dot{-} B && \text{by } (7.2.3) \\
&\subset ((A \dotplus A) \dot{-} B) \dot{-} B && \text{by } (7.2.8) \\
&= (A \dot{-} B) \dot{-} B && \text{by } (7.4.1) \\
&= A \dot{-} (B \dotplus B) && \text{by } (7.2.6) \\
&= A \dot{-} B && \text{by } (7.4.1)
\end{aligned}$$

and the conclusion by (7.4.1).

(7.4.2′) $a_1 \dotplus \cdots \dotplus a_n$ is additively closed.

(7.4.3′) Let $[S]$ be the additively closed set generated by S. Then $[a_1, \cdots, a_n] = a_1 \dotplus \cdots \dotplus a_n \cup a_2 \dotplus \cdots \dotplus a_n \cup a_1 \dotplus a_3 \dotplus \cdots \dotplus a_n \cup \cdots \cup a_1 \cup \cdots \cup a_n$.

Proof. $[a_1,a_2] = a_1+a_2 = a_1\dot{+}a_2 \cup a_1 \cup a_2$, hence the formula holds for $n = 2$. Assume that the formula holds for $n-1$. Let $p \in [a_1,\cdots,a_n] = (a_1+\cdots+a_{n-1})+a_n$, then $p \in q+a_n$ with $q \in a_1+\cdots+a_{n-1}$. Then either $p = q$ or $p = a_n$ or $p \in q\dot{+}a_n$, $p \neq q, a_n$. If $p = q$ then $p \in a_1\dot{+}\cdots\dot{+}a_{n-1} \cup \cdots \cup a_{n-1}$. If $p \in q\dot{+}a_n$, $p \neq q, a_n$, then $p \in (a_1\dot{+}\cdots\dot{+}a_{n-1})\dot{+}a_n \cup (a_2\dot{+}\cdots\dot{+}a_{n-1})\dot{+}a_n \cup \cdots \cup a_{n-1}\dot{+}a_n$. Thus $p \in a_1\dot{+}\cdots\dot{+}a_n \cup \cdots \cup a_n$. Thus $[a_1,\cdots,a_n] \subset a_1\dot{+}\cdots\dot{+}a_n \cup \cdots \cup a_n$. Obviously $[a_1,\cdots,a_n] \supset a_1\dot{+}\cdots\dot{+}a_n \cup \cdots \cup a_n$. Thus the proof is complete.

(7.4.3). $[S]$ is the set union of all expressions of the form $a_1\dot{+}\cdots\dot{+}a_n$ where the a's are in S.

Proof. Due to the property (II) of L_G $[S]$ is the set union of $[a_1,\cdots,a_n]$, $a_i \in S$, $i = 1,\cdots,n$. Thus by (7.4.3′) $[S]$ is the set union of the forms $a_1\dot{+}\cdots\dot{+}a_n$.

(7.4.4.) Let A be additively closed set which has a minimal set of additive generators. Then A has a unique minimal set of additive generators.

(7.4.4′). An additively closed set with a finite set of additive generators has a unique minimal set of additive generators.

If A has a finite set of additive generators, S′, we can delet redundant element in S′, one by one eventually yielding a minimal set of additive generators of A.

(7.4.5). Let $A \neq 0$ be additively closed. Then $A \dot{-} (\dot{-} B) = (A \dot{+} B) \dot{-} A$.

We have defined the <u>linear set</u> or <u>linear subspace</u> to be the set S having the property that if $a, b \in S$ then line $ab \subset S$. Since line ab can be represented by the set $\{a\} \cup \{b\} \cup a \dot{-} b \cup a \dot{+} b \cup b \dot{-} A$, the linear subspace can also be defined by the condition that $a, b \in S$ implies $a \dot{+} b \subset S$ and $a \dot{-} b \subset S$. By this reason, if we call an abstract system $(G, \dot{+})$ satisfying J1,···,J7 a multigroup, then a linear subspace can also be called a <u>subgroup</u>. By the <u>linear subspace of</u> G <u>determined by</u> S, denoted $\{S\}$, we mean the least linear subspace of G which contains S. If $\{S\} = A$ we say S is a set of generators of the linear subspace A. Similarly, if $S_1, \cdots, S_n \subset G$, we define <u>the linear subspace of</u> G <u>generted by</u> $S_1, \cdots, S_n$ to be the least linear subspace which contains $S_1, \cdots, S_n$ and we denote it $\{S_1, \cdots, S_n\}$.

(7.5.1). A is a linear subspace of G if and only if a) A is closed under $\dot{-}$. or b) $A \supset A \dot{-} A$, or c) $A = A \dot{-} A$.

<u>Proof</u>. The necessity of c) is trivial. To prove its sufficiency, suppose $A = A \dot{-} A$. We need show merely that A is closed

under $\dotplus$. By (7.2.9′) $A \dotplus A \subset A \dot{-} (A \dot{-} A) = A \dot{-} A = A$, and the result follows by (7.4.1).

(7.5.2′). $x \in \{S\}$ if and only if $x \in \{a_1, \cdots, a_n\}$ where $a_i \in S$, $1 \leqslant i \leqslant n$. [Due to the upper-continuity of the lattice $\tilde{L}_G$]

(7.5.3). $\{S\} = [S] \dot{-} [S]$.

<u>Proof.</u> Any linear subspace of G which contain S contains $[S]$, hence also contains $[S] \dot{-} [S]$. Moreover, by (7.2.3) c) $[S] \dot{-} [S] \supset [S] \supset S$. Thus we have to show that $[S] \dot{-} [S]$ is a lineat subspace of G. Letting $A = [S]$ we have

$$
\begin{aligned}
(A \dot{-} A) \dot{-} (A \dot{-} A) &= (A \dot{-} (A \dot{-} A)) \dot{-} A && \text{by } (7.2.6) \\
&\subset ((\underline{A \dotplus A}) \dot{-} A) \dot{-} A && \text{by } (7.2.7) \\
&= (A \dot{-} A) \dot{-} A && \text{by } (7.4.1) \\
&= A \dot{-} (A \dotplus A) && \text{by } (7.2.6) \\
&= A \dot{-} A && \text{by } (7.4.1)
\end{aligned}
$$

and the result follows by (7.5.1).

(7.5.3′). $\{S\}$ is the set union of all expressions of the form $(a_1 \dotplus \cdots \dotplus a_n) \dot{-} (b_1 \dotplus \cdots \dotplus b_m)$, involving elements of S. [Since $[S]$ is the set union of the expressions of the form $a_1 \dotplus \cdots \dotplus a_n$ by (7.4.3)].

(7.5.4). $\{S,T\} = \{S \dot{+} T\}$.

Proof. If S or T = 0 the theorem is trivial. Suppose $S,T \neq 0$. Clearly $\{S,T\} \supset S\dot{+}T$, so that $\{S,T\} \supset \{S\dot{+}T\}$. Thus we need prove merely $\{S\dot{+}T\} \supset S,T$. We have

$$\{S\dot{+}T\} \supset (S\dot{+}T)\dot{-}(S\dot{+}T)$$

$$= ((S\dot{+}T)\dot{-}S)\dot{-}T \quad \text{by } (7.2.6)$$

$$\supset (S\dot{-}(S\dot{-}T))\dot{-}T \quad \text{by } (7.2.7),\ (7.2.1)$$

$$\supset T\dot{-}T \quad \text{by } (7.2.9),\ (7.2.1)$$

$$\supset T \quad \text{by } (7.2.3)\ c).$$

By symmetry $\{S\dot{+}\} \supset S$ and the proof is complete.

(7.5.4′). Let $S_1,\cdots,S_n$ be additively closed. Then $\{S_1,\cdots,S_n\} = (S_1\dot{+}\cdots\dot{+}S_n)\dot{-}(S_1\dot{+}\cdots\dot{+}S_n)$. Especially $\{a_1,\cdots,a_n\} = (a_1\dot{+}\cdots\dot{+}a_n)\dot{-}(a_1\dot{+}\cdots\dot{+}a_n)$.

By (7.4.2), $S_1\dot{+}\cdots\dot{+}S_n$ is additively closed. By (7.5.4) $\{S_1,\cdots,S_n\} = \{S_1\dot{+}\cdots\dot{+}S_n\}$ which is equal to $(S_1\dot{+}\cdots\dot{+}S_n)\dot{-}(S_1\dot{+}\cdots\dot{+}S_n)$ by (7.5.3).

(7.5.5.). Let A, B be linear subspaces of G, $A \cap B = A \cdot B \neq 0$. Then $\{A,B\} = A \dot{-} B$. Especially $\{S,T\} = \{\{S\},\{T\}\} = \{S\} \dot{-} \{T\}$ provided $\{S\} \cdot \{T\} \neq 0$.

Proof. $\{A,B\} = \{A \dot{+} B\} = (A \dot{+} B) \dot{-} (A \dot{+} B)$ by (7.5.4), (7.4.2), (7.5.3)

$= ((A \dot{+} B) \dot{-} A) \dot{-} B$ by (7.2.6)

$= (A \dot{-} (A \dot{-} B)) \dot{-} B$ by (7.4.5)

$= (A \dot{-} B) \dot{-} (A \dot{-} B)$ by (7.2.6)

$\subset ((A \dot{-} B) \dot{+} B) \dot{-} A$ by (7.2.7)

$\subset (A \dot{-} B) \dot{-} A$ by (7.4.1'), since $A \dot{-} B$ is additively closed and $A \dot{-} B \supset B$

$\subset (A \dot{-} A) \dot{-} B$ by (7.2.6)

$= A \dot{-} B$ by (7.5.1).

Thus $$\{A,B\} \subset A \dot{-} B.$$

It is obvious that $\{A,B\} \supset A \dot{-} B$. Hence $\{A,B\} = A \dot{-} B$. It remains to show that $A \dot{-} B \supset B$. Since $B = B \dot{-} B \supset A \cdot B \dot{-} B$ and $A \supset A \cdot B$, we have $A \dot{-} B \supset A \cdot B \dot{-} (A \cdot B \dot{-} B) \supset B$ by (7.2.9) as $A \cdot B \neq 0$.

(7.5.5'). If A, B are linear subspaces and $A \cdot B \neq 0$. Then

$$A \dot{-} B = B \dot{-} A \quad \text{and} \quad A \dot{-} (A \dot{-} B) = A \dot{-} B.$$

Proof. $A \dot{-} B = \{A,B\} \supset A \dot{-} (A \dot{-} B) = A \dot{-} \{A,B\} \supset A \dot{-} B$ since

$\{A,B\} = A \dot{-} B \supset B$. Hence $A \dot{-} B = A \dot{-} (A \dot{-} B)$.

(7.5.6). Let A, B, C be linear subspaces of G, $A \cdot B \neq 0$. Then $A \subset C$ implies $\{A,B\} \cdot C = \{A, B \cdot C\}$.

<u>Proof</u>. Obviously $\{A,B\} \cdot C \supset \{A, B \cdot C\}$ or $(A \dot{-} B) \cdot C \supset A \dot{-} B \cdot C$. To establish the converse inclusion, let $x \in (A \dot{-} B) \cdot C$. Then since $0 \neq A \cdot B \subset A,B$ we have $x \in a \dot{-} b$ with point $a \in A$, point $b \in B$ by definition, in addition $x \in C$. Now $x \in a \dot{-} b$ means $x \approx a \dot{-} b$, hence $x \dot{+} b \approx a$ by (7.2.4). By (7.2.4) again we have $b \approx a \dot{-} x$ that is, $b \in a \dot{-} x \subset C$, since $A \subset C$ and C is a linear subspace. Thus $b \in B \cdot C$ so that $x \in a \dot{-} b$ implies $x \in A \dot{-} B \cdot C$. Hence $(A \dot{-} B) \cdot C \subset A \dot{-} B \cdot C$. Since $A \cdot B \neq 0$, $\{A,B\} = A - B$. Furthermore $A \cdot (B \cdot C) = (A \cdot B) \cdot C = A \cdot B \neq 0$, since $A \cdot B \subset A \subset C$, hence $\{A, B \cdot C\} = A \dot{-} B \cdot C$. Thus $\{A,B\} C \subset \{A, B \cdot C\}$.

(7.5.6′). Let A, B, C be linear subspaces of G, $B \cdot C \neq 0$. Then $A \subset C$ implies $(B \dot{-} A) \cdot C = B \cdot C \dot{-} A$.

Let A, B be linear subspaces of G. Then we say that A covers B if in $\tilde{L}_G$ A covers B.

(7.5.7). Let A, B, C be linear subspaces of G such that A and B cover $C \neq 0$ and $A \neq B$. Then $\{A,B\}$ covers A, B.

<u>Proof</u>. We show $\{A,B\}$ covers A. Obviously $\{A,B\} \supset A$. Suppose $\{A,B\} = A$. Then $A \supset B \supset C$ which implies, since A covers C, that $B = A$ or $B = C$ contrary to the hypothesis. Hence $\{A,B\} \neq A$. Let X be a linear subspace satisfying $\{A,B\} \supset X \supset A$. It sufficies to show that $X = \{A,B\}$ or $X = A$. Now $B = B\cdot\{A,B\} \supset B\cdot X \supset B\cdot A \supset C$ implies that $B\cdot X = B$ or $B\cdot X = C$ since B covers C. Suppose $B\cdot X = B$. Then $X \supset B$. Now since X is a linear subspace containing A and B; $X \supset \{A,B\}$ which together with $\{A,B\} \supset X$ gives $X = \{A,B\}$. Next, if $B\cdot X = C$ then $C \neq 0$ implies $BX = BA = C \neq 0$. Then

$$X = X\cdot\{A,B\} \quad \text{since } \{A,B\} \supset X$$

$$= \{A, B\cdot X\} \quad \text{by (7.5.6) since } B\cdot A \neq 0$$

$$= \{A,C\} \quad \text{since } BX = C$$

$$= A \quad \text{since } A \supset C.$$

Thus $X = A$ and the proof is complete.

(7.9.0). If points $a,b \in G$ and $a \neq b$, then $\{a,b\}$ covers a.

This can be seen as the case $C = 0$ not covered by (7.5.7). For this case the hypothesis of (7.5.7) reduces to: A, B are distinct elements of G.

Since every linear subspace containing a, b $(a \neq b)$ contains line ab, line $ab \subset \{a,b\}$. On the other hand line ab is a linear subspace. Let $c,d \in$ line ab then line cd also contains c, d so by 03 line ab = line cd. Thus line $ab = \{a,b\}$.

Now obviously $a \neq \{a,b\}$. Let $a \leq x \leq \{a,b\}$ for a linear subspace x. If $a \neq x$ then there is a point $c \neq a$ such that $c \leq x$. Then $a \leq \{a,c\} \leq x \leq \{a,b\}$. By 03 we have $\{a,c\} = \{a,b\}$ so that $x = \{a,b\}$. This shows that $\{a,b\}$ covers a.

(7.9.1′). Let A, B be linear subspaces and A, B cover $A \cdot B$ then $\{A,B\}$ covers A, B. That is the lattice $\tilde{L}_G$ satisfies condition (V) of Theorem 5.1.

<u>Proof</u>. If linear subspaces A, B cover $C \neq 0$ then $A,B \supset C$ and $A \cdot B \supset C$. Hence $A \supset A \cdot B \supset C$ and $B \supset A \cdot B \supset C$. Now if $A \neq A \cdot B$ then $A \cdot B = C$ since A covers C. If $A = A \cdot B$ then $B \supset A$. Since $B \neq A$, $B \neq B \cdot A = A$. Hence $A \cdot B = C$ since B covers C. Then from (7.5.7) and (7.9.0) we get (7.9.1′).

(7.9.1). If A is a linear subspace of G and $b \not\in A$ then $\{A, b\}$ covers A. Thus $\tilde{L}_G$ is a matroid lattice.

(7.9.1′) is equivalent to (7.9.1) as asserted in Theorem 5.1.

<u>Definition 7.16</u>. Let $N \neq 0$ be a linear subspace. Then $N \dot{-} (N \dot{-} a) \equiv (a)_N$ is called the <u>half-space with edge N determined by</u> a. For any set A, let $(A)_N = \{(a)_N | a \in A\}$.

(7.7.1). Let $N \neq 0$ be a linear subspace of G. Then the half-spaces with edge N are convex, disjoint and exhaust G.

(7.7.1′). $(a)_N \ni a$, $b \in (a)_N$ implies $(a)_N = (b)_N$, and $(a)_N = (b)_N$ if and only if $N \dot{-} a = N \dot{-} b$.

<u>Definition 7.17</u>. If $(a)_N = (b)_N$ we write $a \equiv b(\text{mod } N)$. In general. if for each $a \in A$ there is a $b \in B$ such that $a \equiv b(\text{mod } N)$ and vice vessa, we write $A \equiv B(\text{mod } N)$. Clearly this is equivalent to $(A)_N = (B)_N$.

(7.7.2). The relation congruence modulo N has the following properties: (a) $a \equiv a(\text{mod } N)$; (b) $a \equiv b(\text{mod } N)$ implies $b \equiv a(\text{mod } N)$; (c) $a \equiv b(\text{mod } N)$, $b \equiv c(\text{mod } N)$ imply $a \equiv c(\text{mod } N)$, (d) $a \equiv a'(\text{nod } N)$, $b \equiv b'(\text{mod } N)$ imply $a \dot{+} b \equiv a' \dot{+} b'(\text{mod } N)$; (e) $a \dot{+} n \equiv a(\text{mod } N)$ for $n \in N$.

<u>Definition 7.18</u>. Let G/N denote the set of all half-spaces (cosets) with edge N determined by elements of G. We define

addition in G/N thus $(a)_N \tilde{+} (b)_N = (a \dot{+} b)_N$. We call G/N with addition so defined the <u>factor group of</u> G <u>with respect to</u> N. The <u>order</u> of the factor group G/N is the cardinal number of the set G/N.

(7.7.2′). Addition of half-spaces (cosets) in G/N is independent of the elements of G which determine the cosets.

It is easily seen that $\tilde{+}$ in G/N is associative, commutative, idempotent. In regard to substruction, G/N is exactly analoguous to an abelian group and is conveniently studied in the familiar manner.

<u>Definition 7.19</u>. Let I be an element of G/N such that $A \tilde{+} I = I \tilde{+} A = A$ for each A in G/N. Then we say that I is an <u>identity element of</u> G/N.

(7.7.3). G/N has a unique identity element, namely, N. If $n \in N$ then $(n)_N = N$.

(7.7.4). In G/N, for each element A there exists a unique element X satisfying $A \tilde{+} X \supset N$.

<u>Definition 7.20</u>. In G/N the unique solution of the relation $A \tilde{+} X \supset N$ is called the <u>inverse</u> of A or the <u>half-space oppositive to</u> A, and is denoted by $\tilde{-} A$. Similarly if α is a subset of G/N, $\tilde{-} \alpha$ denotes the set of inverses of the elements in α.

(7.7.4′). In G/N, $\tilde{-}(\tilde{-}A) = A$. $\tilde{-}(a)_N = (a\ )_N$ if and only if (1) $a \dot{+} a' \approx N$ or (2) $a' \in N \dot{-} a$. $\tilde{-}(a)_N = N \dot{-} a$ and $a \dot{+} a' \approx N$ implies $(a)_N = N \dot{-} a'$.

(7.7.5). In G/N, $\tilde{-}(A \tilde{+} B) = (\tilde{-}A) \tilde{+} (\tilde{-}B)$.

(7.7.6). In G/N, $A \tilde{-} B = A \tilde{+} (\tilde{-}B)$.

(7.7.7). Let $N \neq 0$ be a proper subgroup of G. Then the order of G/N is greater than or equal to 3.

<u>Definition 7.21</u>. Let A, B be linear subspaces of G such that the order of A/B is 3. Then we say B separates A.

(7.7.8). Let B separate A. Then A is decomposed into B, S, S' where S, S' are mutually inverse cosets of B.

(7.7.8'). Let B separate A. Suppose $a \in A$, $a \notin B$, $a + a' \approx B$. Then $A = B \cup B \dot{-} a \cup B \dot{-} a'$, where the addends are disjoint.

If G is a descriptive space of finite dimension and let $N \neq 0$, G be a linear space. Then the elements of G/N are half-spaces with edge N. Let $(a)_N$, $(b)_N$ be half-space which forms an angle aNb. Then sum as elements of G/N, is the set of half-spaces in the angle aNb [see Fader pp. 69-72].

Now we relate definition 7.21., to the more familiar idea of separation in the foundations of geometry. Suppose B separates A. Then A is decomposed into B, S, S' where S, S' are mutually inverse cosets of B. Hence the following properties hold:

(1) $A = B \cup S \cup S'$ and B, S, S' are disjoint; (2) $S, S' \neq 0$ are

convex sets (7.7.1); (3) the join of any element of S to any element of S' meets B. (7.7.4'). Properties (1), (2), (3) are essentially the criteria of the foundations of geometry for the separation of a linear space A by a linear subspace B.

Conversely suppose that A, B are linear spaces for which sets S, S' exist satisfying (1), (2), (3) above. Suppose $a \in S$. Then by (3), $x \in S'$ implies $x+a \approx B$ and $x \in B \dot{-} a$. Thus $S' \subset B \dot{-} a$. Similarly $S \subset B \dot{-} a'$, where $a' \in S'$. By (3), $a \dot{+} a' \approx B$ and we can apply (7.7.4'), getting $B \dot{-} a' = (a)_B$, $B \dot{-} a = (a')_B$. Hence in view of (1), $A = B \cup (a)_B \cup (a')_B$. The cosets B, $(a)_B$, $(a')_B$ are distinct since B, S, S' are distinct and $(a)_B$, $(a')_B$ are mutually inverse by (7.7.4'). Hence A/B has order 3 and B separates A in the sense of Def. 7.21. Thus our definition is equivalent to the familiar geometric notion of separation of linear spaces.

(7.7.9). n separate the line na.

Proof. By (7.7.7) $\{n,a\}/n$ has order greater than or equal to 3. Thus $\{n,a\} \supset (n \dot{-} a) \cup n \cup (n \dot{-} a)$, where $a \dot{+} a' \approx n$ that is (ana'). On the other hand $\{n,a\} = n \dot{-} a \cup n \cup a \cup n \dot{+} a \cup a \dot{-} n$. Obviously $(a'na)$, so that $n-a' \ni a$. From which we obtain $(n \dot{-} a') \dot{-} n \supset a \dot{-} n$.

Since $(n \dot{-} a') \dot{-} n = (n \dot{-} n) \dot{-} a' = n \dot{-} a'$, we have $n \dot{-} a' \supset a \dot{-} n$. Furthermore $n \dot{+} (n \dot{-} a') \supset n \dot{+} a$. Since $n \dot{+} (n \dot{-} a') \subset (n \dot{+} n) \dot{-} a' = n \dot{-} a'$ we also obtain $n \dot{-} a' \supset n \dot{+} a$. Thus $\{n, a\} \subset (n \dot{-} a) \cup n \cup (n \dot{-} a')$, hence $\{n, a\} = (n \dot{-} a) \cup n \cup (n \dot{-} a')$. Thus $\{n, a\}/n$ has order 3, that is n separates the <u>line</u> na.

(7.8.2). Let A, B be linear subspaces of G; $A \cdot B \neq 0$. Then $\{A, B\} = \bigcup_{b \in B} (b)_A$.

(7.8.3). Let A, B be subgroups (linear subspaces) of G, $A \cdot B \neq 0$. Then $\{A, B\}/A$ is isomorphic to $B/A \cdot B$.

(7.10.1). Any factor group A/B is isomorphic to a factor group of the form A'/b.

<u>Proof of (7.10.1)</u>. By the property of relative complementation valid in any matroid lattice, for given $b \in B \subset A$ there exists a linear subspace A' such that $\{A', B\} = A$ and $A' \cdot B = b$. Hence by (7.8.3)

$$A/B = \{A' \cdot B\}/B \cong A'/A' \cdot B = A'/b,$$

and the proof is complete.

<u>Definition 7.22</u>. A <u>subgroup</u> of a factor group is a <u>non-empty</u> subsystem closed under $\tilde{+}$ and $\tilde{-}$. It easily follows that the

identity element constitutes a subgroup of every factor group. If the only proper subgroup of a factor groups is the identity group, we say it is <u>simple</u>.

(7.10.2). A/B is simple if and only if A covers B and $B \neq 0$.

(7.10.2′). Let A/B be simple. Then $A/B \cong \{a,b\}/b$ where $a \neq b$.

(7.10.3). Suppose $B \neq 0$ is a subgroup of A and $x \notin A$. Then $\{A,x\}/\{B,x\} \cong A/B$.

(7.10.4). All simple groups A/B are isomorphic (O5 is assumed). Thus all simple factor groups A/B have the same order.

Since $\{n,a\}/n$ has order 3, every simple factor group has order 3. Thus

(7.10.5). A/B is simple if and only if it has order 3.

Proof of (7.10.5). Suppose A/B has order 3. Any subgroups of A/B distinct from the dentity group must contain the identity I and element $X \neq I$ and $-X$, the inverse of X. Since there are distinct the subgroup is identical with A/B (as A/B has order 3) and A/B is simple. The converse is easily proved. Since for the line $\{a,b\}/b$ has order 3, it is simple as we have just seen. Since O5 is assumed all simple factor groups have the same order, hence all simple factor group has order 3.

(7.10.6). B separates A if and only if A covers B and $B \neq 0$.

(7.10.6′) In G let A cover B. Suppose $a \in A$, $a \notin B$, $a \dotplus a' \approx B$. Then $A = B \cup B \dot{-} a \cup B \dot{-} a'$, and the addends are disjoint.

(7.10.6″). Let B be a subgroup of G. Suppose $a \dotplus a' \approx B$. Then

$$\{a,B\} = B \cup B \dot{-} a \cup B \dot{-} a' = B \cup B \dot{-} a \cup B \dot{-} (B \dot{-} a).$$

Moreover the addends are disjoint provided $a \notin B$.

<u>Proof of (7.10.6″)</u>. If $a \in B$ then $B \dot{-} a \in B$, $B \dot{-} (B \dot{-} a) \subset B$ since B is a linear subspace. Hence $\{a,B\} = B = B \cup B \dot{-} a \cup B \dot{-} a'$, holds trivially. If $a \notin B$ then $\{a,B\}$ covers B. Since $a \in \{a,B\}$, $a \notin B$, $a + a' \approx B$ by (7.10.6′) we have $\{a,B\} = B \cup B \dot{-} a \cup B \dot{-} a' = B \cup B \dot{-} a \cup B \dot{-} (B \dot{-} a)$ where the addends are disjoint.

(7.11.2′). $a \dot{-} (a \dot{-} b) = b \dotplus a \cup b \dot{-} a \cup b$; and $a \dot{-} (a \dot{-} B) = B \dotplus a \cup B \dot{-} a \cup B$.

<u>Proof</u>. If $a = b$, this is trivial. Suppose $a \neq b$. By (7.10.6″), $\{a,b\} = a \cup a \dot{-} b \cup a \dot{-} (a \dot{-} b)$ where the addends are disjoint.

On the other hand, by definition $\{a,b\} = a \cup a \dot{-} b \cup b \cup b \dot{-} a \cup b \dotplus a$ and the addends are also disjoint, since the contrary supposition implies $a = b$. For example, $a \dot{-} b \approx b \dotplus a$ implies $a \approx (b \dotplus a) \dotplus b = (b \dotplus b) \dotplus a = b \dotplus a$, hence $a \dot{-} a \approx b$ and $a \approx b$ that is $a = b$. Comparing these two expressions for $\{a,b\}$ we obtain $a \dot{-} (a \dot{-} b) = b \dot{+} a \cup b \dot{-} a \cup b$. This is easily generalized to yield another identity.

(7.11.ℓ). $(a \dot{+} B) \dot{-} (a \dot{+} C) = (a \dot{+} B) \dot{-} C \cup B \dot{-} (a \dot{+} C) \cup B \dot{-} C$.

<u>Proof</u>.
$$\begin{aligned} (a \dot{+} B) \dot{-} (a \dot{+} C) &= ((a \dot{+} B) \dot{-} a) \dot{-} C && \text{by } (7.2.6) \\ &= (a \dot{-} (a \dot{-} B)) \dot{-} C && \text{by } (7.5.5) \\ &= (B \dot{+} a \cup B \dot{-} a \cup B) \dot{-} C && \text{by } (7.11.2') \\ &= (B \dot{+} a \dot{-} C) \cup (B \dot{-} a) \dot{-} C \cup B \dot{-} C && \text{by } (7.2.10) \\ &= (a \dot{+} B) \dot{-} C \cup B \dot{-} (a \dot{+} C) \cup B \dot{-} C && \text{by } (7.2.6) \end{aligned}$$

(7.11.3). $\{a_1, \cdots, a_n\}$ is the set union of all expressions of the form $\left(a_{i_1} \dot{+} \cdots \dot{+} a_{i_r}\right) \dot{-} \left(a_{i_{r+1}} \dot{+} \cdots \dot{+} a_{i_s}\right)$, where $1 \leq i_j \leq n$, and $i_j \neq i_k$, if $j \neq k$.

<u>Proof</u>. By (7.5.4')

$$\{a_1, \cdots, a_n\} = (a_1 \dot{+} \cdots \dot{+} a_n) \dot{-} (a_1 \dot{+} \cdots \dot{+} a_n).$$

We apply (7.11.ℓ) to the right member in order to eliminate repetition of the tetter a_1 getting

$$\begin{aligned} \{a_1, \cdots, a_n\} = &(a_1 \dot{+} \cdots \dot{+} a_n) \dot{-} (a_1 \dot{+} \cdots \dot{+} a_n) \\ &\cup (a_2 \dot{+} \cdots \dot{+} a_n) \dot{-} (a_1 \dot{+} \cdots \dot{+} a_n) \\ &\cup (a_2 \dot{+} \cdots \dot{+} a_n) \dot{-} (a_2 \dot{+} \cdots \dot{+} a_n). \end{aligned}$$

Similarly we eliminate repetitions of the letter a_2 in each addend of this set union, for example we reduce

$$(a_1 \dotplus \cdots \dotplus a_n) \dot{-} (a_2 \dotplus \cdots \dotplus a_n)$$

to

$$(a_1 \dotplus \cdots \dotplus a_n) \dot{-} (a_3 \dotplus \cdots \dotplus a_n)$$
$$\cup (a_1 \dotplus a_3 \dotplus \cdots \dotplus a_n) \dot{-} (a_2 \dotplus \cdots \dotplus a_n)$$
$$\cup (a_1 \dotplus a_3 \dotplus \cdots \dotplus a_n) \dot{-} (a_3 \dotplus \cdots \dotplus a_n).$$

Continuing to eliminate repeated letters in this way we eventually get an expression for $\{a_1, \cdots, a_n\}$ in which all addends have the desired form. Hence, since $\{a_1, \cdots, a_n\}$ contains every expression of this form, it is the set union of all such expressions.

We have given a definition of linear dependence earlier in this section. Now we are giving a second definition:

<u>Second definition of linear independence</u>. Suppose $S \subset G$, and suppose $\{S-x\} \not\ni x$ for each $x \in S$. Then we say S is linearly independent or simply <u>independent</u>.

Then we can show easily the followings.

(1) Any subset of an independent set is also independent.

(2) A set is independent proveded each of its finite subsets is independent. (By the upper-continuity).

(3) Let S be a set of generators of <u>linear subspace</u> G,

then S is independent if and only if S is a minimal set of generators of G.

Theorem 7.5. S is independent if and only if the sets $a_1\dotplus\cdots\dotplus a_n$, where the a's are in S and $a_i \neq a_j$ for $i \neq j$, are disjoint.

Proof. Suppose S independent. Let $a_1\dotplus\cdots\dotplus a_n$, $a_1'\dotplus\cdots\dotplus a_m'$ be sets of type described, satisfying

$$a_1\dotplus\cdots\dotplus a_n \approx a_1'\dotplus\cdots\dotplus a_m'.$$

If a letter appears in only one member of the above relation, we can solve this relation for this letter, which is therefore "dependent" on the other letters in the relation. Suppose a_1 appears in only one member, then by (7.2.4) $a_1\dotplus(a_2\dotplus\cdots\dotplus a_n) \approx a_1'\dotplus\cdots\dotplus a_n'$ implies $a_1 \approx (a_1\dotplus\cdots\dotplus a_m)\dot{-}(a_2\dotplus\cdots\dotplus a_n) \subset \left\{S-a_1\right\}$, since $a_1'\dotplus\cdots\dotplus a_m'$, $a_2\dotplus\cdots\dotplus a_n \subset S-a_1$. This contradict our supposition. Hence the members of the above relation are identical except posssibly for the order of the letters, and the necessity of the condition is established.

To prove its sufficiency, suppose S satisfies the given condition. Assume S not independent. Then $a \in \left\{S-a\right\}$ for some $a \in S$. By (7.5.2′) and (7.5.4′)

$$a \in \{a_1, \cdots, a_n\} = (a_1 \dotplus \cdots \dotplus a_n) \dot{-} (a_1 \dotplus \cdots \dotplus a_n),$$

where $a_i \in S-a$. Hence

$$a \dotplus a_1 \dotplus \cdots \dotplus a_n \approx a_1 \dotplus \cdots \dotplus a_n$$

contrary to our supposition. Thus S is independent and the proof is complete.

<u>Theorem 7.6.</u> $\{a_1, \cdots, a_n\}$ is the set union of all expression of the form $\left(a_{i_1} \dotplus \cdots \dotplus a_{i_r}\right) \dot{-} \left(a_{i_{r+1}} \dotplus \cdots \dotplus a_{i_s}\right)$, where $1 \leqslant i_j \leqslant n$, and $i_j \neq i_k$ if $j \neq k$. Furthermore if $a_1, \cdots, a_n$ are distinct and form an independent set the addends are disjoint.

<u>Proof.</u> The proof of the first part of the theorem is given in (7.11.3). To prove the second part, let $a_1, \cdots, a_n$ be distinct and form an independent set. Suppose

$$\left(a_{i_1} \dotplus \cdots \dotplus a_{i_r}\right) \dot{-} \left(a_{i_{r+1}} \dotplus \cdots \dotplus a_{i_s}\right)$$

$$\approx \left(a_{j_1} \dotplus \cdots \dotplus a_{j_t}\right) \dot{-} \left(a_{j_{s+1}} \dotplus \cdots \dotplus a_{j_s}\right)$$

holds, where the i's are distinct and the j's are distinct. We show the member of this relation identical. By (7.25), this relation implies

$$a_{i_1} \dotplus \cdots \dotplus a_{i_r} \dotplus a_{j_{t+1}} \dotplus \cdots \dotplus a_{j_n} \approx a_j \dotplus \cdots \dotplus a_{j_t} \dotplus a_{i_{r+1}} \dotplus \cdots \dotplus a_{i_s}.$$

By Theorem 7.5, this latter relation is an equality and the same letters are present in both members of this relation. Since i's are distinct, each a_{i_p}, $1 \leqslant p \leqslant r$ is an a_{j_q}, $1 \leqslant q \leqslant t$ and vice versa. Thus $a_{i_1} \dotplus \cdots \dotplus a_{i_r} = a_{j_1} \dotplus \cdots \dotplus a_{j_t}$. Similarly $a_{i_{r+1}} \dotplus \cdots \dotplus a_{i_s} = a_{j_{t+1}} \dotplus \cdots \dotplus a_{j_u}$, so that the former relations becomes an equality. Hence the addends in the expansion of $\{a_1, \cdots, a_r\}$ are disjoint.

Using these theorems, we can prove the equivlence of the two definitions of lineat dependence.

<u>If S is linearly independent in the second definition, then it is linearly independent in the first definition.</u>

<u>Proof</u>. Suppose the contrary, then there exist disjoint subsets I, J of K such that $\left(a_{i_1} + \cdots + a_{i_r}\right)\left(a_{j_1} + \cdots + a_{j_s}\right) \neq 0$. By deleting the redundant elements we can assume that $\left(a_{i_1}, \cdots, a_{i_r}\right)$ is a minimal set of generators of $a_{i_1} + \cdots + a_{i_r}$ and similarly $\left(a_{j_1}, \cdots, a_{j_s}\right)$ is a minimal set of generators of $a_{j_1} + \cdots + a_{j_s}$. Then by property (1) of the linear independence in the second

definition, $\left(a_{i_1},\cdots a_{i_r}\right)$ and $\left(a_{j_1},\cdots,a_{j_s}\right)$ are independent sets respectively. Then the addends of $[a_{i_1},\cdots,a_{i_r}] = a_{i_1}+\cdots+a_{i_r} =$ $a_{i_1}+\cdots+a_{i_r} \cup \cdots \cup a_i \cup \cdots \cup a_{i_r}$ are disjoint by Theorem 7.5. Then $\left(a_{i_1}+\cdots+a_{i_r}\right)\left(a_{j_1}+\cdots+a_{j_s}\right) \neq 0$ implies that $a \in a_{i_{r_1}} \dotplus\cdots\dotplus a_{i_{r_t}}$, $a_{j_{s_1}}+\cdots+a_{j_{s_k}}$ which contradicts the Theorem 7.6 as we assumed that S is independent.

<u>If S is independent under the first definition, then it is also independent under the second definition.</u>

Assume the contrary, then there is a point x such that $x \in \left\{S-x\right\}$ then $x \in \left\{a_1,\cdots,a_n\right\}$ where $a_i \in S-x$, $1 \leqslant i \leqslant n$. Without loss of generality, we can assume that the set $(a_1,\cdots,a_n)$ is linearly independent under the second definition.

Since $x \in \left\{a_1,\cdots,a_n\right\}$, by the theorem 7.6 we have $x \in \left(a_i \dotplus\cdots\dotplus a_{i_r}\right) \dot{-} \left(a_{i_{r+1}} \dotplus\cdots\dotplus a_{i_s}\right)$. Hence $x \dotplus a_{i_{r+1}} \dotplus\cdots\dotplus a_{i_s} \approx$ $a_{i_1} \dotplus\cdots\dotplus a_{i_r}$, that is $\left(x+a_{i_{r+1}} \dotplus\cdots\dotplus a_{i_s}\right)\left(a_i \dotplus\cdots\dotplus a_{i_r}\right) \neq 0$. Hence $\left(x+a_{i_{r+1}}+\cdots+a_{i_s}\right)\left(a_{i_1}+\cdots+a_{i_r}\right) \neq 0$. On the other hand, since $(a_1,\cdots,a_n)$ is linearly independent under the second definition it is linearly independent under the first definition, hence

$$\left(a_{i_{r+1}}+\cdots+a_{i_s}\right)\left(a_{i_1}+\cdots+a_{i_r}\right)=0.$$

Thus x is dependent on $(a_1,\cdots,a_n)$ under the first definition, hence x is dependent on $S - x$. Thus S is dependent under the first definition contradictory to our assumption.

Thus the two definitions of linear independence are equivalent.

In the above proof, it is also shown at the same time that <u>if</u> $x \in \{a_1,\cdots,a_n\}$ <u>and</u> $(a_1,\cdots,a_n)$ <u>is the minimal set of generators of</u> $\{a_1,\cdots,a_n\}$, <u>then</u> x <u>is dependent on</u> $(a_1,\cdots,a_n)$ <u>under the first definition</u>.

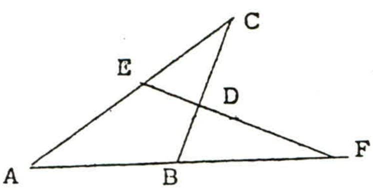
§ 8. Projective geometry

In § 7 a projective geometry is defined to be a system satisfying 01, 02, 03, 04′, 06′ and 07_2. It is proved by Prenowitz that such a system is equivalent to a system satisfying the following set of postulates by Veblen and Young:

<u>A1</u>. If A and B are distinct points, there is at least one line on both A and B.

<u>A2</u>. If A and B are distinct points, there is not more than on line on both A and B.

<u>A3</u>. If A, B, C are points not all on the same line, and D and E (D $\neq$ E) are points such that B, C, D are on a line and C, E, A are on a line, there is a point F such that A, B, F are on a line and also E, E, F are on a line.



<u>EO</u>. There are at least three points on every line.

It is also proved that the lattice of linear spaces of a projective geometry is characterized by the properties of completeness, atomisticity, upper-continuity and modularity.

Since the lattice-theoretic treatments of geometry started with projective geometry and it is a model to the analoguous investigation for other geometries and also for its generalization, we like to discuss this lattice again starting from the set of axioms of Veblen and Young. We shall call the axioms A1 and A2 put together as axiom P1, axiom A3 as P2 and EO as P3.

Thus we understand a generalized projective geometry to consist of a set m of elements called points and another set of elements called lines each of which is a set of points such that P1 and P2 are satisfied. A subset A of m is called a subspace if $P, Q \in A$ implies that $PQ \subset A$, where PQ is the line determined by P, Q. For any subset $B \subset m$, we define as usual $\bar{B}$ to be the least subspace containing B, then $C : B \rightarrow \bar{B}$ is a closure operation and $\langle mC \rangle$ is easily seen to be a merely finitary geometry as defined before. Thus by the Theorem 3.4., the set of all subspace of a generalized projective geometry G forms a complete, upper-continuous, atomistic lattice $\mathcal{L}(G)$.

This lattice $\mathcal{L}(G)$ has several more special properties:

By (P1), it is obvious that the subspace $\overline{\{P, Q\}}$ is the line PQ determined by two distinct points P, Q. Thus if p, q are distinct points in $\mathcal{L}(G)$, $p + q$ is the line determined by the two points.

Theorem 8.1. (The join theorem) The lattice $\mathcal{L}(G)$ is linear, that is, for any non-zero element $x, y \in \mathcal{L}$ and point $r \leq x+y$ there exist point $p \leq x$ and $q \leq y$ such that $r \leq p+q$.

Proof. It is assumed that $x \neq 0$, $y \neq 0$. If $x = y = \{p\}$ it is trivial. Let S be the point set defined by $S = \{r \leq p+q \mid p \leq x \text{ and } q \leq y\}$. By definition $x + y$ is the least linear subspace x and y, so it is sufficient to show the following three items considering $x+y$, x, y as point sets:

(1) $S \subseteq x+y$,

(2) $x \subseteq S$, $y \subseteq S$

(3) S is a linear subspace.

(1), (2) are obvious, we need merely show (3), that is, show that S contains every point u of the line (that is $u \leq s+t$) along with two distinct points s, t of S.

Special case. At least one of s, t is contained in x or y, say $s \leq x$. For $t \in S$, there exist $p \leq x$ and $q \leq y$ such that $t \leq p+q$, $p \neq q$ by definition of S. Since $u \leq s+t$, $s \neq t$; $t \leq p+q$, $p \neq q$ there exists, by (P2), a point v such that $v \leq p+s$ and $v \leq u+q$.

Case 1. Now if $p = s$, then $u \leq p+t \leq p+(p+q) = p+q$, hence $u \in S$.

Case 2. If $p \neq s$, then $v \leq p+s \leq x$ since $p, s \leq x$. Now, if $v \neq q$, then from $v \leq u+q$ we have $u \leq v+q$ where $v \leq x$ and $q \leq y$. Hence $u \in S$. If $v = q$ then $q \leq x$, hence

$t \leq p+q \leq x$ and x is a subspace so $t \leq x$. Hence $u \leq s+t \leq x$ and $u \in S$.

General situation. Neither s nor t are contained in x or y. For t in S there exist points $p \leq x$, $q \leq y$ such that $t \leq p+q$ and $p \neq q$. Then from $t \leq p+q$, $u \leq s+t$, $s \neq t$ it follows by (2) that there exists a point v such that $v \leq p+s$ andv $\leq q+u$. From $v \leq p+s$ it follows that $v \in S$ by the above spacial case, since $p \leq x$, $s \nleq x$ and hence $p \neq s$. Again, by the above special case we have $u \leq q+v$ and $u \in S$ provided $q \neq v$. If $q = v$, then $t \leq p+q$, $p \neq q$ and $q \leq p+s$ implies $s \leq p+q$. Thus we have $t,s \leq p+q$ and $s+t = p+q$ by (P1). Hence $u \leq s+t = p+q$ hence $u \in S$.

Theorem 8.2. The lattice $\mathcal{L}(G)$ is modular.

Proof. Since $(a+b)c \geq a+bc$ holds in general lattice, we need only to show that if $a \leq c$ then $(a+b)c \leq a+bc$. It is trivial when $a = 0$ or $b = 0$ or $a = b$.

Now let point $r \leq (a+b)c$, then $r \leq a+b$ and $r \leq c$. By thejoin theorem, there exist $p \leq a$, $q \leq b$ such that $r \leq p+q$. If $r = p$ then $r \leq a$ and $r \leq a+bc$. If $r \neq p$, then $q \leq p+r \leq c$ since $p \leq a \leq c$ and $r \leq c$. Hence $q \leq bc$ and $r \leq p+q \leq a+bc$ what is to be proved.

Since a complete, upper-continuous, atomistic modular lattice is a matroid lattice by Theorem 5.1, we have by Theorem 5.3:

Corollary 8.1. $\mathcal{L}(G)$ is relatively complemented.

Definition 8.1. A lattice is said to be (general) projective if and only if it is isomorphic to the lattice of all subspaces of a (general) projictive geometry.

Definition 8.2. A projective geometry is a general projective geometry which also satisfies

(PO) On each line lie at least two distinct points.

Theorem 8.3. In the lattice $\mathcal{L}(G)$ of a projective geometry G the hyperatom (that is an element which covers an atom) coincide with the point set $[g]$ (the set of points on the line g).

Proof. We show first that to each hyperatom x there is a line g such that $[g] = x$. For this we need not use (PO). Let p, q be any two distinct points we claim first that $\{p\}+\{q\} = [pq]$ the set of point on the line pq. $[pq]$ is obviously a linear subspace which contains $\{p\}$ and $\{q\}$, since if $r,s \in [pq]$ $r \neq s$ then line pq = line rs hence $[rs] = [pq]$. It remains to be shown that $[pq]$ is contained in every subspace x which contains $\{p\}$, $\{q\}$, but this follows from the definition of a linear subspace. Thus we have shown that $\{p\}+\{q\} = [pq]$.

By definition a hyperatom x covers an atom $\{p\}$,

$\dot{x}$ contains a point $q \neq p$ as its element. Therefore $\{p\} \nless \{p\}+\{q\} = [pq] \leq x$. Since x covers $\{p\}$, we have $x = [pq]$, that is each hyperatom can be written in the form $[g]$. We shall show now that $[g]$ is a hyperatom. On each line g there are at least two distinct points by (P0). Therefore $[g]$ is neither the zero element nor an atom. Thus it suffices to show that there is no linear subspace x such that $\{p\} \lneqq x \lneqq [g]$. If x is such an element, then x contains a second point q ($\neq p$). By (P1) it follows that $g = pq$ and $[g] = [pq] = \{p\}+\{q\} \leq x$ which contradicts $x \lneqq [g]$.

We can assign a geometry $G_{\mathcal{L}}$ to a given geometric lattice $\mathcal{L}$. If $\mathcal{L}$ is modular, then $G_{\mathcal{L}}$ is generalized projective geometry. We take atoms of $\mathcal{L}$ as points of $G_{\mathcal{L}}$ and hyperatoms of $\mathcal{L}$ as lines of $G_{\mathcal{L}}$ and say that a point p lies on a line g if $p \leq q$ in L. For the proof of (P1) and (P2) we shall use the dimension concept. We have already obtained that an atom of a modular lattice has dimension 1 and a hyperatom has dimiension 2.

(P1) Two distinct points p and q always lie on a unique line.

Proof. Since p, q are distinct $p \cdot q = 0$ so p covers $p \cdot q$, by Theorem 5.1, $p + q$ covers q. Thus $p + q$ is of dimension 2 hence a line. Since $p, q \leqslant p+q$, p, q be on the line $p + q$. We need moreover to show that every line on which p, q lie coincides with $p + q$, that is the uniqueness property. Since $p \leqslant q$, $q \leqslant g$ it follows that $p+q \leqslant g$. Since they have the same dimension 2, $p+q = g$.

For the proof of (P2) we start from points p, q, r, p′, q′ for which there exist lines g and h so that p′, q and r are on g; p, q′ and r are on h. We need to show that there is a points r′ such that p, q and r′ lie on a lines; and p′, q′, r′ lie on a line.

Case (1). $q = q'$ (no matter if $p = q$ or not). Let us take $r' = r$. Since p, q′, r are on a line h and $q = q'$, $r = r'$, it follows that p, q, r′ are on the line h. Since p′, q, r are on a line g and $q = q'$, $r = r'$ it follows that p′, q′, r′ are on the line g.

Case (2). $q \neq q'$, $p = q$. It follows from these assumption that $p \neq q'$. Take $r' = q'$. Since $p = q$, obviously p, q, r′ are on a line. Moreover $q' = r'$ implies that p′, q′, r′ are on a line. The case $p' = q'$ can be shown similarily.

Case (3). $q \neq q'$, $p \neq q$ and $p' \neq q'$. In this case both $p + q$ and $p' + q'$ are lines. Thus we need merely show that

$\lambda((p+q)\cdot(p'+q')) \geq 1$, since this implies the existence of a poine r' which lies on both $p + q$ and $p' + q'$. Since p', q, r lie on the line g, $p'+q+r \leq g$, hence $\lambda(p'+q+r) \leq 2$. Similarly $(p+q+r) \leq 2$. Then

$\lambda(p'+q+r+p+q'+r) = \lambda(p'+q+r)+\lambda(p+q'+r)-\lambda((p'+q+r)(p+q'+r)) \leq 2+2-1 = 3$, Since $r \leq (p'+q+r)(p+q'+r)$ and $\lambda((p'+q+r)(p+q'+r)) \geq \lambda(r) = 1$. Then $\lambda((p+q)(p'+q')) = \lambda(p+q)+\lambda(p'+q')-\lambda(p+q+p'+q') \geq 2+2-3 = 1$, since $\lambda(p+q) = \lambda(p'+q') = 2$ and $\lambda(p+q+p'+q') \leq \lambda(p'+q+r+p+q'+r) \leq 3$.

If the modular lattice $\mathcal{L}$ is complemented the corresponding geometry $G_{\mathcal{L}}$ also satisfies the axiom (PO). Thus we have to show that there are at least two points on each line.

Since g is a hyperatom, it covers a point p. Thus we have a point p on g. Since any complemented modular lattice is relatively complemented, p has a relative complement q in $[Og]$. Then $q \leq g$, hence $\lambda(q) \leq 2$. Now $\lambda(q) \neq 0$, since otherwise $\lambda(q) = 0$ implies $q = 0$ and $p+q = p \neq q$. $\lambda(q) \neq 2$, since otherwise $q = g$ and $p\cdot q \neq 0$. Therefore $\lambda(q) = 1$, hence q is a point on g and $p \neq q$, hence $p+q = g$. Thus g has at least two points.

<u>Theorem</u> <u>8.4</u>. If $\mathcal{L}$ is a complemented modular lattice the associated geometry $G_{\mathcal{L}}$ is a projective geometry.

<u>Theorem</u> <u>8.5</u>. A lattice $\mathcal{L}$ is (general) projective if and only if it is complete, upper-continuous, atomistic and modular.

Proof. It is shown before that L is isomorphic to the lattice of subspaces of a merely finitary geometry, so we need merely show that this merely finitary geometry is a (generalized) projective geometry and the subspaces of this merely finitary geometry is the subspaces of the projective geometry. We define the points and lines of associated geometry as above, as shown above (P1), (P2) are satisfied. Since L is atomistic every hyperatom contains at least two atoms, hence (P0) is satisfied.

As remarked in p. 28, to show that the subspaces of the merely finitary geometry coincide with subspaces of the projective geometry, we need merely show that the point set $p_1 + \cdots + p_m$ is actually the subspace in projective geometry generated by these points. This can be done by induction on m. For $m = 2$, by definition $p_1 + p_2$ is the line determined by p_1 and p_2. Obviously $p_1 + \cdots + p_m$ is a subspace, so it contains the subspace x generated by $p_1, \cdots, p_m$. Assume that the $p_1 + \cdots + p_m \leqslant x$ holds for $m - 1$, then for every $p \leqslant (p_1 + p_2 + \cdots + p_{m-1}) + p_m$ there is a point $q \leqslant p_1 + \cdots + p_{m-1}$ such that $p \leqslant q + p_m$. Thus by induction assumption q is contained in the subspace y generated by $p_1, \cdots, p_{m-1}$. Since $q \leqslant y \leqslant x$ and $p_m \leqslant x$, $p \leqslant q + p_m \leqslant y + p_m \leqslant x$. Thus $p_1 + \cdots + p_m \leqslant x$. This together with $p_1 + \cdots + p_n \geqslant x$ mensioned above give $x = p_1 + \cdots + p_m$.

Thus we have proved that $\mathcal{L}(\mathcal{G}(L)) \cong L$. We have also $\mathcal{G}(\mathcal{L}(G)) \cong G$ for a projective geometry G, where the isomorphism of two geometries is defined by:

Definition 8.3. Two projective geometries are said to be isomorphic if there is a one-valued invertible mapping ϑ of the set of points of one geometry to the set of points of another geometry, and also a one-valued invertible mapping Θ of the set of lines of one geometry to the set of lines of another geometry so that α lies on g if $\vartheta(\alpha)$ lies on $\Theta(g)$.

To prove that if G is a projective geometry then $\mathcal{G}(\mathcal{L}(G))$ is isomorphic to G, for point p and line g of G we define $\vartheta(p) = \{p\}$ and $\Theta(g) = [g]$. As we have seen before ϑ is a one-one invertible mapping between the set of points of G and the set of atoms of $\mathcal{L}(G)$ and therefore between the set of points of G and the set of points of $\mathcal{G}(\mathcal{L}(G))$. By Theorem 8.3., hyperatoms is $\mathcal{L}(G)$ coincide with point sets $[g]$ of lines of G. Thus Θ is a one-one invertible mapping of the set of line of G onto the set of hyperatoms of $\mathcal{L}(G)$ and hence onto the set of lines of $\mathcal{G}(\mathcal{L}(G))$. And finally

point p on line g $\leftrightarrow$ $p \in [g]$ (definition of $[g]$)

$\leftrightarrow \{p\} \leq [g]$ (inclusion in $\mathcal{L}(G)$)

$\leftrightarrow \{p\}$ on $[g]$ (definition of $\mathcal{G}(L)$)

$\longleftrightarrow \vartheta(p)$ on $\Theta(g)$ (definition of ϑ and Θ).

Thus the isomorphism between G and $\mathcal{G}(\mathcal{L}(G))$ is established.

Since $\mathcal{L}(G)$ of a projective geometry is modular, hence it is a matroid lattice. Thus $\mathcal{L}(G)$ is irreducible if and only if any two points of L are perspective to each other. Now point p is perspective to point q if there exists an element $x \in \mathcal{L}(G)$ such that $q \leq p+x$ and $qx = 0$. Let $p \neq q$ then $x \neq 0$. Thus by Theorem 8.1., there exists a point $r \leq x$ such that $q \leq p+r$. From $qx = 0$ it follows that $q \neq r$. Furthermore $p \neq r$ since otherwise $q \leq p+r = p$ contradictory to $p \neq q$. Thus on the line pq there exist at least three distinct points. Conversely, if every line has at least three distinct points, then for every pair of distinct points p, q, they are perspective to each other. Thus $\mathcal{L}(G)$ is irreducible if and only if the projective geometry G satisfies

(P3). There exist at least three distinct points on each line.

By Theorem 5.8., any projective lattice is a direct union of irreducible sublattices. Let G be a projective geometry, then each sublattice of L_G is also projective, since for the set of atoms in a sublattice (P0), (P1), (P2) and also (P3) (for, any two points of a sublattice are perspective). Thus each sublattice is the projective lattice of an irreducible projective space which is defined as follows:

Definition 8.4. If a line contains only two points, it well be called degenerate. If a projective space contains any degenerated lines it will be called reducible, otherwise it will be called irreducible.

Definition 8.5. By the direct union D of any given collection K of mutually disjoint projective spaces, we mean a projective space whose points are all the points of the different spaces that make up the collection; the lines of D consists of all the lines of these spaces, and in addition all degenerate lines which can be formed by taking two points, one from each of two distinct, and hence disjoint, spaces of the collection K. That D is a projective space can be easily shown by checking (P1), (P2) and (P0).

Then from the above argument it follows

Theorem 8.6. Every projective space is either irreducible or is the direct union in a unique way, of irreducible projective spaces.

We shall now construct an irreducible projective space $PS(\alpha,D)$ of dimension α which is any cardinal number, with coordinates from an arbitrary division ring D. Let T be a set of elements of cardinal number α, and let t be a variable ranging over the set T.

We shall call a function $d(t)$ a coordinate function if

(1) The function $d(t)$ is defined over the set T.

(2) The function values of $d(t)$ are in the ring D.

(3) The function $d(t)$ is not identically zero, but $d(t) = 0$ except for a finite set of values of t.

Here the symbol 0 stands for the zero element of the ring D. If $d(t)$ is a coordinate function, and r is any non-zero element of the ring D, then $rd(t)$, obtained by multiplying all the function values of $d(t)$ on the left by r is also a coordinate function.

If $d(t)$ is a coordinate function, we define the <u>point</u> P of the space $PS(\alpha,D)$ belonging to the function $d(t)$ to be the set of all coordinate functions $rd(t)$, where $r \in D$ and $r \neq 0$. Since D is a ring, it is clear that P belong likewise to all the other coordinate functions of the set P, and that P is determined by any one of its coordinate functions.

Three distinct P, P', P'' of $PS(\alpha,D)$ will be called collinear if they have coordinate functions $d(t)$, $d'(t)$, $d''(t)$ respectively such that $d(t)+d'(t)+d''(t) = 0$.

<u>Theorem</u> <u>8.7</u>. The space $PS(\alpha,D)$ is an irreducible projective space.

<u>Proof</u>. If we define a line x_0 be the set of all points collinear with two distinct points, it is easy to see that (P1) is satisfied. Actually $rd(t)+r'd'(t)$ for $r,r' \in D$ is the line through $P(d(t))$ and $P'(d'(t))$. Let the line $P''P'''$: $r''d''(t)$ $+r'''d'''(t)$ be any line through P, P' then $d(t) = r''_0 d''(t)+r'''_0 d'''(t)$

and $d''(t) = r''_1 d''(t) + r'''_1 d'''(t)$ therefore $rd(t)+r'd'(t) = (rr''_0+r'r''_1)d(t) + (rr'''_0+r'r'''_1)d\ (t)$. From these two equations we have that line $pp' \subseteq$ line $p''p'''$. On the other hand, from the expressions of $d(t)$ and $d'(t)$ we have $r''_1 d(t) - r''_0 d'(t) = (r''_1 r'''_0 - r''_0 r'''_1) d'''(t)$ where $(r''_1 r'''_0 - r''_0 r'''_1) \neq 0$ since otherwise $r''_1 d(t) - r''_0 d'(t) = 0$ contradictory to $p \neq p'$. Similarly $r'''_1 d(t) - r'''_0 d'(t) = (r'''_1 r''_0 - r'''_0 r''_1) d''(t)$ and $(r'''_1 r''_0 - r'''_0 r''_1) \neq 0$. Thus p'' and p''' are also contained in the line pp', hence line $p''p''' \subseteq$ line pp'. Therefore line pp' = line $p''p'''$.

To show that the postulate (P2) is satisfied, let p, p', p'' be three non collinear points with coordinate functions $d(t)$, $d'(t)$, $d''(t)$ respectively, and let Q with coordinate function $rd(t)+r'd'(t)$ be on the line pp', Q' with coordinate function $r'd'(t) + r''d''(t)$ be on line $p'p''$. If Q and Q' are distinct, then r and r'' are not both zero. Then $rd(t) - r''d''(t)$ is not identically zero and is therefore the coordinate function of a point Q'' which is on the line pp'' and also on the line QQ' (since $rd(t)-r''d''(t) = rd(t)+r'd'(t)-(r'd'(t)+r''d''(t))$).

The projective space $PS(\alpha,D)$ is irreducible, since the line joining the distinct points p and p' with coordinate functions $d(t)$ and $d'(t)$ always contains at least a third point p'' with coordinate function $d(t) - d'(t)$.

We now consider a sort of converse of this theorem. Is every irreducible projective space isomorphic to a coordinate space $PS(\alpha,D)$?

Obviously not, since it is well known that there exist projective planes which can not be coordinatized by rings (M. Hall; Projective plane, Trans. Amer. Math. Soc. 54 (1943) pp. 229-277).

With this single exception, however, a converse of the above theorem can be proved:

Theorem 8.8. Every irreducible projective space is either a projective plane, or is isomorphic to a projective coordinate space $PS(\alpha,D)$.

Suppose S is an irreducible projective space which is not a projective plane. In order to construct a coordinate system in S we need to have a maximal independent set of points of S, to serve as the vertices of a coordinate simplex.

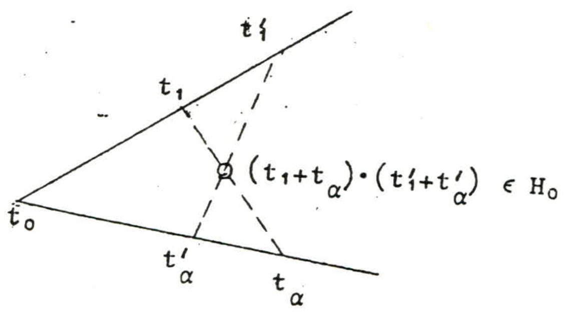
It follows form Zorn's lemma that such a maximal independent set of points of S exists. For consider the collection K of all independent sets of S. The set union of any linearly ordered subcollection H of the collection K is clearly an independent set (every finite subset of the set union of H is contained in a set of H, hence is independent and is therefore in K. Since every finite subset is independent, the set union of H is independent). The hypothesis of Zorn's lemma is verified, hence there exists a maximal set T in K, that is, a maximal independent set T of points of S.

Let α be the cardinal number of the set T. We shall call

α the projective dimension of the space S. If α if finite, it is one greater than the ordinary dimension of S. Select a particular point t_0 of T. The set complement $T - t_0$ has a join H_0 in the lattice of subspaces of S. It is easily verified that H_0 is a complement of the subspace t_0 in this lattice. ($H_0 = \Sigma(T-t_0)$, $t_0 \not\leq \Sigma(T-t_0)$ since T is independent, hence $t_0 \cdot H_0 = 0$. Obviously $t_0+H_0 \geq$ every point of S, hence $t_0+H_0 = \Sigma S$.) Furthermore, H_0 is a maximal element of the lattice, and is called a hyperplane, Since the points of T are to be the vertices of a coordinate simplex, we wish the line $t_0 + t_\alpha$ joining t_0 to the other point t_α of T to be coordinate axis.

Hence it is necessary to set up a scale on each such lines. We first select arbitrarily on each line $t_0 + t_\alpha$ a third point t'_α distinct from t_0 and t_α. This is possible since the space S is irreducible. Call t'_α the unit point of the line $t_0 + t_\alpha$. Let t_1 be a particular point of T distinct from t_0. The plan is to set up a scale on line $t_0 + t_1$ and then to project this scale onto the lines $t_0 + t_\alpha$ so that unit points correspond and so that t_1 corresponds to point t_α. Let t'_1 be the unit point of line $t_0 + t_1$. We may now define a division ring D whose elements are all the points of the line $t_0 + t_1$ except point t_1 in such a way that t_0 is the zero element of D, t'_1 is the unit element of D, while t_1 is the "infinite point" of the line. This can be done by the method of von Staudt (Veblen & Young, Projective geometry, vol. 1, pp. 141-167) provided S contains a projective three-space S_3 containing the line $t_0 + t_1$. Except in the trivial cases where S consist of a single line or of a single point, S will contain such a three-space, since by assumption S is irreducible and is not a projective plane. This division ring is the ring D of the space $PS(\alpha, D)$ we are to construct. It determines a coordinate system on the line $t_0 + t_1$, whereby to each point P of the line $t_0 + t_1$ other than t_1 there is assigned as coordinates the point P it self, considered as an element of the ring D.

We now set up on each other coordinate axis $t_0 + t_\alpha$ a scale or coordinate system by means of a perspectivity projecting the points of $t_0 + t_1$, and hence the elements of the ring D, onto the points of the line $t_0 + t_\alpha$ so that the point t_1 goes into the point t_α while the unit point t_1' goes into the unit point t_α'. This perspectivily has its center at the point $(t_1+t_\alpha)\cdot(t_1'+t_\alpha')$ which is in the hyperplane H_0. Now let p be any point of S not in H_0.



By the merely finitary property, P is in the join of a finite minimal independent subset T_n of the points of T which we shall call the points $t_0, t_1, \cdots, t_n$. We wish to assign a coordinate function $d(t)$ to the point p, where t ranges over the set T, and the function $d(t)$ takes values in the ring D. We define $d(t)$ to be zero if t is not one of the points $t_0, \cdots, t_n$, and define $d(t_0) = 1$, the unit element of the ring D. If i is one of the subscripts $1, 2, \cdots, n$, we define $d(t_i)$ to be the coordinates on $t_0 + t_i$ of the point $(t_1+t_2+\cdots+t_{i-1}+p+t_{i+1}+\cdots+t_n)\cdot(t_0+t_i)$ $(\lambda((t_1+\cdots+t_{i-1}+p+t_{i+1}+\cdots+t_n)\cdot(t_0+t_i) = \lambda(t_1+\cdots+t_{i-1}+p+t_{i+1}+\cdots+t_n)+\lambda(t_0+t_i)-\lambda(t_1+\cdots+t_{i-1}+p+t_{i+1}+\cdots+t_n+t_0+t_i) = n+2-(n+1) = 1$ since p is not in H_0 so $t_1,+\cdots+t_{i-1}p, t_{i+1}\cdots t_n$ are independent, so that $\lambda(t_1+\cdots+t_{i-1}+p+t_{i+1}+\cdots+t_n) = n$ and $t_1+\cdots+t_{i-1}+p+t_{i+1}+\cdots+t_n+t_0+t_i = t_0+t_1+\cdots+t_n$ so that $\lambda(t_1+\cdots+t_{i-1}+p+t_{i+1}+\cdots+t_n+t_0+t_i) = n+1)$. In other word, we assign to each point in the finite-dimensional projective space determined by the finite set T_n a set of coordinates in the usual way.

On the other hand, if p is any point of the hyperplane H_0, then P is in the join of a finite independent set of points $t_1, \cdots, t_n$ of T other than t_0. Let Q be any point of the line $t_0 + p$ other than t_0 or p. Since Q is not in H_0, a coordinate function $d(t)$ has already been assigned to it. In term of this coordinate function $d(t)$ of the point Q; we define a coordinate function $d'(t)$ for the point p so that $d'(t_0) = 0$ while $d'(t) = d(t)$ if t is distinct from t_0. In this way every point of S has a coordinate function assigned to it. By multiplying each such coordinate function $d(t)$ on the left by all non zero element r of the ring D; we get a set of coordinate functions of the form $rd(t)$, which set is by definiton a point of the projective coordinate space $PS(\alpha,D)$. Thus each point of the space S has been set into correpondence with a point of the space $PS(\alpha,D)$. It remains to show that this correspondence between projective spaces is an isomorphism.

To show this, let P, Q and R be three distinct points of the space S. They lie in the join J of a finite number of points $t_1,\cdots,t_n$ of the maximal independent set T. This subspace J is an ordinary projective space of finite projective dimension n, and our coordinate functions were assigned to point of J in the same manner that ordinary projective coordinates would be assigned to these points, considered as points of J rather than of S. Hence the condition, in terms of coordinates, that the three points P, Q and R be collinear in S is the same as the condition that the corresponding points of $PS(\alpha,D)$ be collinear. Likewise distinct points of J have linearly independent coordinates in J

and hence have disjoint sets of coordinate functions assigned to them in $PS(\alpha,D)$. This shows that the spaces S and $PS(\alpha,D)$ are isomorphism, and complete the proofif the theorem.

Let us now study if the duality principle holds. Here we are concerned with the lattice-theoretic formulation of the axioms as given above. Some of the assumption of such lattice L are self-dual. For example, the completeness assumption and the modularity and the complementedness are self-dual ones. The modularity is self-dual, since if replace the dual concept in the proposition "$a \leqslant c$ implies $(a+b)c = a+bc$" we have "$a \geqslant c$ implies $a \cdot b+c = a(b+c)$" which is again the modularity.

The dual of atomicity is anti-atomicity, which say that to each $a \neq 1$ there is an anti-atom x_0 which satisfies $a \leqslant x_0$. An anti-atom is an element which is covered by 1. We shall show that L is also anti-atomic. Let $a \neq 1$, then a has a complement $b \neq 0$ since $a \neq 1$. Since $b \neq 0$, there is an atom $p \leqslant b$. Let $A = \{x \in L \mid a \leqslant x \text{ and } p \cdot x = 0\}$, then $A \neq \phi$ since $a \in A$. By using Zorn's lemma we see that there is a maximal element x_0 in A and that x_0 is a complement of p. Since p covers $p \cdot x_0 = 0$, so $1 = p+x_0$ covers x_0 by theorem 5.1. Thus x_0 is an anti-atom and $x_0 \geqslant a$.

The upper-continuity is stated as follows: For any set $\{y_\alpha\}$

directed above, and any element of L, $x(\Sigma y_\alpha) = \Sigma(x\cdot y_\alpha)$. The dual proposition of this can be read as follows: For any set $\{y_\alpha\}$ directed below, and any element of L, $x+(\pi y_\alpha) = \pi(x+y_\alpha)$. we shall show that in L the dual of upper-continuity does not hold. Let K be a field and $M = \{0,1,2,\cdots,n,\cdots\}$. Consider $PS(\omega K)$ as above. Using $\bar{\varphi}$ to denote the point represented by the coordinate function φ. let the subspaces y_n and x are defined as follows:

For $n = 0,1,2,\cdots$, let

$$\bar{\varphi} \leq y_n \longleftrightarrow \varphi(0) = \varphi(1) = \cdots = \varphi(n) = 0 \quad \text{and}$$

$$\bar{\varphi} \leq x \longleftrightarrow \sum_{m=0}^{\infty} \varphi(m) = 0.$$

It is easy to see that these definitions are independent of the representation. Since $y_{n+1} \leq y_n$ the set of all y_n is a set directed below. Clearly $\pi y_n = 0$ since there is no point whose representation has all 0 components. It follows that $x+\pi y_n = x$. If φ_0 is defined by $\varphi_0(0) = 1$ and $\varphi_0(m) = 0$ for $m \geq 1$, then $\bar{\varphi}_0 \not\leq x+\pi y_n = x$. But it can be shown that $\bar{\varphi}_0 \in \pi(x+y_n)$. Actually $\bar{\varphi}_0 \leq x+y_n$ for every n. For, if φ_{n+1} is defined by $\varphi_{n+1}(n+1) = -1$ and $\varphi_{n+1}(m) = 0$ for $m \neq n+1$ then $\bar{\varphi}_{n+1} \leq y_n$. If φ^*_{n+1} is defined by $\varphi^*_{n+1}(0) = -1$, $\varphi^*_{n+1}(n+1) = 1$ and $\varphi^*_{n+1}(m) = 0$ for

$m \neq 0$, $n+1$ and $\overline{\varphi^*_{n+1}} < x$ since $\sum_{m=0}^{\infty} \varphi^*_{n+1}(m) = 0$. Moreover $\overline{\varphi}_{n+1}$ and $\overline{\varphi^*_{n+1}}$ are distinct. Since $\varphi_0 + \varphi_{n+1} + \varphi^*_{n+1} = 0$, $\overline{\varphi}_0$, $\overline{\varphi}^*_{n+1}$ and $\overline{\varphi^*_{n+}}$ are collinear. Thus $\overline{\varphi}_0 \leqslant \overline{\varphi}_{n+1} + \overline{\varphi^*_{n+1}} \leqslant y_n + x$ (for every n). Thus $x + \pi y_n \neq \pi(x + y_n)$. Thus in the projective geometry introduced above the duality principle does not hold.

Consider a complete, modular, complemented lattice of finite length (that is, every chain connecting two elements is of finite length). Since $0 \in L$ every element of L has a finite dimension, there dimensions are bounded, that is $\lambda(x) \leqslant \lambda(1)$ for every $x \in L$. Thus we can say such a lattice a finite dimensional complete, modular complemented lattice. We call a projective geometry finite dimensional if L_G is finite dimensional. <u>In a finite dimensional projective geometry the principle of duality holds.</u>

We have shown above that the decomposition of L_G corresponds to a partitioning of S into subspaces S_i in such a way that two distinct points belong to the same subspace if and only if they determine a non-degenerate line. Some of these components may be trivial, consisting of just one point or of just one line and others may be non-Arguesian planes. With these exceptions, we can associate with each component S_i a division ring and introduce coordinates in the manner of classical geometry, with the sole difference that number of coordinates may be infinite. This bring us to the subject of Desargue's law:

Definition 8.6. A projective geometry is said to be Arguesian if for any points p_0, p_1, p_2, q_0, q_1, q_2 the condition

$$\ell(p_1,q_1) \cap \ell(p_2,q_2) \subseteq \ell(p_0,q_0)$$

implies that

$$\ell(p_1,p_2) \cap \ell(q_1,q_2) \leq \ell((\ell(p_0,p_1) \cap \ell(q_0,q_1)) \cup (\ell(p_0 p_2) \cap \ell(q_0 q_2))),$$

where $\ell(p_0,q_0)$ denotes the linear subspace determined by points p_0, q_0.

Definition 8.7. A lattice is said to be Arguesian if and only if it is isomorphic to the lattice of all subspaces of an Arguesian projective geometry.

The formulation of Desargues' law in Def. 8.6 differs from classical version in that no restriction is placed on the six points involved (such as that they be distinct, or that the three pairs p_i, q_i $i = 0,1,2,$ lie on three distinct but concunent lines). However, the two formulations are actually equivalent, for some of the special cases that are normally excluded are actually valid in all projective geometries, while the remaining cases follows: from the classical Desargues' law.

Theorem 8.9. If A is a geometric lattice, then the following conditions are equivalent:

(i) A is Augusian.

(ii) A is modular and, for any atoms $p_0, p_1, p_2, q_0, q_1, q_2$ of A, the condition $(p_1+q_1)(p_2+q_2) \leqslant p_0+q_0$ implies that $(p_1+p_2)(q_1+q_2) \leqslant (p_0+p_1)(q_0+q_1)+(p_0+p_2)(q_0+q_2)$.

(iii) For any element $a_0, a_1, a_2, b_0, b_1, b_2 \in A$, the condition $(a_0+b_0)(a_1+b_1) \leqslant a_2+b_2$ implies that $(a_0+a_1)(b_0+b_1) \leqslant (a_0+a_2)(b_0+b_2)+(a_1+a_2)(b_1+b_2)$.

(iv) For any elements $a_0, a_1, a_2, b_0, b_1, b_2 \in A$, if

$$y = (a_0+a_1)(b_0+b_1)[(a_0+a_2)(b_0+b_2)+(a_1+a_2)(b_1+b_2)]$$

then
$$(a_0+b_0)(a_1+b_1)(a_2+b_2) \leqslant a_0(a_1+y)+b_0(b_1+y).$$

Observe that in (iii) and (iv) we do not assume the modular law, it turns out to be a consequence of the given conditions. In fact, in any lattice A, (iv) implies (iii) and (iii) in turn implies that A is modular.

In terms of the decomposition discussed above, a projective lattice A is Arguesian if and only if none of its indecomposable factors is isomorphic to the lattice of all subspaces of a non Arguesian projective space.