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MONOTONE EMPIRICAL BAYES TEST FOR A TRUNCATION PARAMETER DISTRIBUTION USING LINEX LOSS

BY

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Abstract

This paper deals with a monotone empirical Bayes test δ_n^* for a truncation parameter distribution using the linex loss. The asymptotic optimality of δ_n^* is investigated. Under very mild conditions, it is shown that δ_n^* is asymptotically optimal with a rate of order $n^{-2/3}$. This rate improves the empirical Bayes test δ_n^{XS} of Xu and Shi (2004) in the sense that faster convergence rate is achieved under conditions relatively weaker than that assumed in Xu and Shi (2004).

1. Introduction

Let X be a random variable arising from a truncation parameter distribution with a pdf $f(x|\theta)$ of the following form

$$f(x|\theta) = u(x)A(\theta)I[\theta \le x \le m\theta], \tag{1.1}$$

where $m > 1$ is a constant, $u(x) \ge B_1 > 0$ for some constant B_1, and integrable over $[a, b]$ for every $0 < a < b < \infty$ and $A(\theta) = [\int_\theta^{m\theta} u(x)dx]^{-1}$. Let θ_0 be a known positive constant. We are interested in testing $H_0 : \theta \le \theta_0$ against $H_1 : \theta > \theta_0$. Let d_i denote the action deciding in favor of H_i, $i = 0,\ 1$.

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The following asymmetric linex loss function is employed:

$$L(\theta, d_0) = l(\theta)I(\theta > \theta_0) \quad \text{and} \quad L(\theta, d_1) = l(\theta)I(\theta \leq \theta_0) \tag{1.2}$$

where $l(\theta) = e^{b(\theta_0 - \theta)} - b(\theta_0 - \theta) - 1$, $b \neq 0$. The constant b determines the shape of the loss function. Varian (1975) and Zellner (1986) have discussed the behavior of the loss function and their various applications. Huang (1995) studied empirical Bayes tests for a one-sided truncation parameter distribution using an asymmetric loss, and Huang and Liang (1997) investigated empirical Bayes estimation problem based on linex error loss. In this paper, we consider the case that $b = 1$.

Assume that the parameter θ is a realization of a positive random variable Θ having a pdf g over $(0, \infty)$ and satisfies that

$$0 < c_0 \leq g(\theta) \leq c_1 < \infty \text{ for } \theta \text{ in } [\theta_0/m, 2m\theta_0]. \tag{1.3}$$

Thus, X has a marginal pdf $f_G(x) = \int_0^\infty f(x|\theta)g(\theta)d\theta = u(x)\int_{x/m}^x A(\theta)g(\theta)d\theta$ $= u(x)v_G(x)$ where $v_G(x) = \int_{x/m}^x A(\theta)g(\theta)d\theta$.

Let χ be the sample space of X. A test δ is defined to be a mapping from χ into $[0, 1]$ such that $\delta(x) = P\{\text{accepting} H_0 | X = x\}$ is the probability of accepting H_0 when $X = x$ is observed. Let $R(G, \delta)$ denote the Bayes risk of a test δ. Thus,

$$\begin{aligned} R(G, \delta) &= \int_\theta \int_x \{L(\theta, d_0)\delta(x) + [1 - \delta(x)]L(\theta, d_1)\} f(x|\theta)dG(\theta)dx \\ &= \int_0^\infty u(x)\delta(x)Q_G(x)dx + c_G, \end{aligned} \tag{1.4}$$

where $c_G = \int l(\theta)I(\theta < \theta_0)dG(\theta)$,

$$\begin{aligned} Q_G(x) &= \int l(\theta)I(\theta > \theta_0)I_{[\theta, m\theta]}(x)A(\theta)g(\theta)d\theta \\ &\quad - \int l(\theta)I(\theta < \theta_0)I_{[\theta, m\theta]}(x)A(\theta)g(\theta)d\theta \\ &\equiv Q_{G1}(x) - Q_{G2}(x). \end{aligned}$$

For $x \leq \theta_0$, $I(\theta > \theta_0)I_{[\theta, m\theta]}(x) = 0$. Thus $Q_{G1}(x) = 0$. So, $Q_G(x) = -Q_{G2}(x) \leq 0$.

For $x \geq m\theta_0$, $I(\theta < \theta_0)I_{[\theta, m\theta]}(x) = 0$. Thus $Q_{G2}(x) = 0$. So, $Q_G(x) = Q_{G1}(x) \geq 0$.

For $\theta_0 < x < m\theta_0$, under condition (1.3), $Q_G(x) = \int_{\theta_0}^{x} l(\theta)A(\theta)g(\theta)d\theta - \int_{\theta_{x/m}}^{\theta_0} l(\theta)A(\theta)g(\theta)d\theta$ is strictly increasing in x. Hence, we see that: $Q_G(x) \leq 0$ for $x \leq \theta_0$, $Q_G(x) \geq 0$ for $x \geq m\theta_0$, and $Q_G(x)$ is continuous and strictly increasing in $[\theta_0/m, 2m\theta_0]$. Thus, there exists a point, say a_G, between θ_0 and $m\theta_0$ such that $Q_G(a_G) = 0$, and $Q_G(x) < 0$ for $x < a_G$ and $Q_G(x) > 0$ for $x > a_G$. Therefore, a Bayes test δ_G, which minimizes the Bayes risks among all tests, can be obtained as follows:

$$\delta_G(x) = I[x < a_G]. \tag{1.5}$$

Thus, δ_G is a monotone test and a_G is called a critical point of the Bayes test δ_G. The minimum Bayes risk of this testing problem is:

$$R(G, \delta_G) = \int_0^\infty u(x)\delta_G(x)Q_G(x)dx + c_G. \tag{1.6}$$

Note that the Bayes test δ_G heavily depends on the prior distribution G. When G is unknown, it is impossible to implement the Bayes test δ_G. Assuming a sequence of past data is available, Xu and Shi (2004) treated this testing problem via the empirical Bayes approach. They proposed an empirical Bayes test δ_n^{XS} for this testing problem. However, they imposed too strong assumptions on the unknown prior distribution G, and the obtainable rate of convergence is relatively slow. Also, their paper contains certain errors.

In this paper, we shall study this testing problem through the empirical Bayes approach. The paper is organized as follows. In Section 2, a monotone empirical Bayes test δ_n^* is constructed. We study the asymptotic optimality of δ_n^* in Section 3. It is shown that under certain mild conditions, δ_n^* achieves a rate of convergence of order $n^{-2/3}$, where n is the number of past data available when the current testing problem is studied. Examples are provided for demonstrating the performance of δ_n^*. Comparison between the performance of δ_n^* and δ_n^{XS} is also made. The theoretic proof of asymptotic optimality is given in Appendix.

2. Construction of A Monotone Empirical Bayes Test

In the empirical Bayes framework, we let (X_j, Θ_j), $j = 1, 2, \ldots$ be iid copies of the random vector (X, Θ), where X_j are observable, but Θ_j are not observable. At the present stage, say stage $n+1$, let $\boldsymbol{X(n)} = (X_1, \ldots, X_n)$ denote the n past data and $X = X_{n+1}$ stands for the present random observation. Let θ_{n+1} be a realized value of the present random parameter Θ_{n+1}. We are interested in testing $H_0 : \theta_{n+1} \leq \theta_0$ against $H_1 : \theta_{n+1} > \theta_0$ using the linex loss $L(\theta_{n+1}, d_i)$ of (1.2).

An empirical Bayes test δ_n is a probability function defined on the presently observed value $X \equiv X_{n+1} = x$ and the past data $\boldsymbol{X(n)}$, such that $\delta_n(x) \equiv \delta_n((x, \boldsymbol{X(n)})$ is the probability of accepting $H_0 : \theta_{n+1} \leq \theta_0$. Let $R(G, \delta_n | \boldsymbol{X(n)})$ denote the Bayes risk of an empirical Bayes test δ_n conditioning on $\boldsymbol{X(n)}$, and $R(G, \delta_n) = E_n \; R(G, \delta_n | \boldsymbol{X(n)})$ the Bayes risk of δ_n, where the expectation E_n is taken with respect to the probability measure generated by $\boldsymbol{X(n)}$.

The minimum Bayes risk $R(G, \delta_G)$ would be achieved if the prior distribution G were known and the Bayes test δ_G had been applied. Thus, $D(G, \delta_n) = R(G, \delta_n) - R(G, \delta_G) \geq 0$ for all n. This non-negative difference $D(G, \delta_n)$, called as the regret of the empirical Bayes test δ_n, is often used as a measure of performance of δ_n. δ_n is said to be asymptotically optimal, relative to the prior distribution G, at a rate $O(\varepsilon_n)$ if $D(G, \delta_n) = O(\varepsilon_n)$ where $\{\varepsilon_n\}$ is a sequence of positive, decreasing numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$.

Note that as the concerned testing problem is a monotonic decision problem, the class of monotone tests is essentially complete. Thus, one desires that the concerned empirical Bayes tests be monotone. For this purpose, we need an estimator for the critical point a_G. Assume a_n is an estimator of a_G. Then we may define a monotone test δ_n as follows:

$$\delta_n(x) = I(x < a_n). \tag{2.1}$$

Note that a_G is the point such that $Q_G(a_G) = 0$. Also, a_G can be expressed as:

$$a_G = \theta_0 + \int_{\theta_0}^{m\theta_0} I[Q_G(x) < 0] dx. \tag{2.2}$$

The form (2.2) provides us a motivation for the construction of an estimator for a_G. Thus, we need to find an estimator for $Q_G(x)$ first. For this purpose, we consider an alternative expression of $Q_G(x)$.

An Alternative Expression of $Q_G(x)$

From Xu and Shi (2004), $A(\theta)G(\theta) = -\sum_{j=1}^{\infty} dv_G(m^j\theta)/d\theta$. For $\theta_0 < x < m\theta_0$, substituting this relation into $Q_{G1}(x)$ and $Q_{G2}(x)$, and applying the integration by part technique, we can obtain:

$$\begin{aligned}
Q_{G1}(x) &= \int_{\theta_0}^{x} [e^{\theta_0-\theta} - (\theta_0-\theta) - 1]\Big[-\sum_{j=1}^{\infty} \frac{d}{d\theta} v_G(m^j\theta)\Big] d\theta \\
&= \sum_{j=1}^{\infty} \int_{m^j\theta_0}^{m^j x} \frac{[1-e^{\theta_0-t/m^j}]}{u(t)m^j} f_G(t)dt \\
&\quad -\sum_{j=1}^{\infty} [e^{\theta_0-x} - (\theta_0-x) - 1] v_G(m^j x) \\
&\equiv \sum_{j=1}^{\infty} \psi_j(x) - \sum_{j=1}^{\infty} \beta_j(x);
\end{aligned}$$

$$\begin{aligned}
Q_{G2}(x) &= \int_{x/m}^{\theta_0} [e^{\theta_0-\theta} - (\theta_0-\theta) - 1]\Big[-\sum_{j=1}^{\infty} \frac{d}{d\theta} v_G(m^j\theta)\Big] d\theta \\
&= \sum_{j=1}^{\infty} \int_{m^{j-1}x}^{m^j\theta_0} \frac{[1-e^{\theta_0-t/m^j}]}{u(t)m^j} f_G(t)dt \\
&\quad +\sum_{j=1}^{\infty} [e^{\theta_0-x/m} - (\theta_0-x/m) - 1] v_G(m^{j-1} x) \\
&\equiv \sum_{j=1}^{\infty} \alpha_j(x) + \sum_{j=1}^{\infty} \eta_j(x),
\end{aligned}$$

where

$$\begin{aligned}
\psi_j(x) &= \int_{m^j\theta_0}^{m^j x} \frac{[1-e^{\theta_0-t/m^j}]}{u(t)m^j} f_G(t)dt, \\
\alpha_j(x) &= \int_{m^{j-1}x}^{m^j\theta_0} \frac{[1-e^{\theta_0-t/m^j}]}{u(t)m^j} f_G(t)dt,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
\beta_j(x) &= [e^{\theta_0-x} - (\theta_0-x) - 1] v_G(m^j x), \\
\eta_j(x) &= [e^{\theta_0-x/m} - (\theta_0-x/m) - 1] v_G(m^{j-1} x).
\end{aligned}$$

Thus,

$$\begin{aligned} Q_G(x) &= \sum_{j=1}^{\infty}\psi_j(x) - \sum_{j=1}^{\infty}\alpha_j(x) - \sum_{j=1}^{\infty}\beta_j(x) - \sum_{j=1}^{\infty}\eta_j(x) \\ &\equiv \psi_G(x) - \alpha_G(x) - \beta_G(x) - \eta_G(x). \end{aligned} \tag{2.4}$$

Empirical Bayes estimation of $Q_G(x)$

Let $h = n^{-1/3}$. For each $l = 1, \ldots, n$, $j = 1, 2, \ldots$ and x in $[\theta_0, m\theta_0]$, define

$$\begin{aligned} \psi_{nlj}(x) &= \frac{[1 - e^{\theta_0 - X_l/m^j}]}{m^j u(X_l)} I[m^j\theta_0 < X_l \le m^j x], \\ \alpha_{nlj}(x) &= \frac{[1 - e^{\theta_0 - X_l/m^j}]}{m^j u(X_l)} I[m^{j-1}x < X_l \le m^j\theta_0], \\ \beta_{nlj}(x) &= [e^{\theta_0 - x} - (\theta_0 - x) - 1]/(u(X_l)h) I[m^j x < X_l \le m^j x + h], \\ \eta_{nlj}(x) &= [e^{\theta_0 - x/m} - (\theta_0 - x/m) - 1] \\ &\quad /(u(X_l)h) I[m^{j-1}x < X_l \le m^{j-1}x + h]. \\ \psi_n(x) &= \sum_{j=1}^{\infty}\frac{1}{n}\sum_{l=1}^{n}\psi_{nlj}(x), \\ \alpha_n(x) &= \sum_{j=1}^{\infty}\frac{1}{n}\sum_{l=1}^{n}\alpha_{nlj}(x), \\ \beta_n(x) &= \sum_{j=1}^{\infty}\frac{1}{n}\sum_{l=1}^{n}\beta_{nlj}(x), \\ \eta_n(x) &= \sum_{j=1}^{\infty}\frac{1}{n}\sum_{l=1}^{n}\eta_{nlj}(x), \\ Q_n(x) &= \psi_n(x) - \alpha_n(x) - \beta_n(x) - \eta_n(x). \end{aligned} \tag{2.5}$$

We use $Q_n(x)$ to estimate $Q_G(x)$.

Monotone empirical Bayes test δ_n^*

Motivated by the form (2.2) and based on $Q_n(x)$. we define an estimate

a_n^* for a_G as follows:

$$a_n^* = \theta_0 + \int_{\theta_0}^{m\theta_0} I[Q_n(x) < 0]dx. \tag{2.6}$$

Accordingly, we propose a monotone empirical Bayes test δ_n^* as follows:

$$\delta_n^*(x) = I[x < a_n^*]. \tag{2.7}$$

The Bayes risk of δ_n^* is:

$$R(G, \delta_n^*) = \int_0^\infty u(x) E_n \delta_n^*(x) Q_G(x) dx + c_G. \tag{2.8}$$

3. Asymptotic Optimality

We assume the following assumptions hold:

[A1] $u(x) \geq B_1$ for all $x > 0$, and $u(x) \leq B_2$ for x in $[\theta_0, m\theta_0]$.
[A2] The prior pdf g satisfies $0 < c_0 \leq g(\theta) \leq c_1 < \infty$ for θ in $[\theta_0/m, 2m\theta_0]$.
[A3] $\sum_{j=0}^\infty v_G^*(m^j x) \leq M < \infty$ and $\sum_{j=0}^\infty v_G^{(1)^*}(m^j x) \leq M < \infty$ for all x in $[\theta_0, m\theta_0]$.

where $v_G^*(x) = \sup_{0\leq y\leq 1} v_G(x+hy)$, and $v_G^{(1)^*}(x) = \sup_{0\leq y\leq 1} |v_G^{(1)}(x+hy)|$.

We shall investigate the asymptotic optimality of the empirical Bayes test δ_n^* under the [A1]$-$[A3] conditions. By (1.5)$-$(1.6) and (2.7)$-$(2.8), the regret of δ_n^* can be written as

$$D(G, \delta_n^*) = E_n\Big[\int_{a_G}^{a_n^*} u(x) Q_G(x) dx\Big]. \tag{3.1}$$

By [A1],

$$B_1 \int_{a_G}^{a_n^*} Q_G(x)dx \leq \int_{a_G}^{a_n^*} u(x) Q_G(x) dx \leq B_2 \int_{a_G}^{a_n^*} Q_G(x)dx. \tag{3.2}$$

Since $Q_G(a_G) = 0$, by taking Taylor's series expansion of order 2,

$$\int_{a_G}^{a_n^*} Q_G(x)dx = Q_G^{(1)}(x^*)(a_n^* - a_G)^2/2 \tag{3.3}$$

for some value x^* between a_n^* and a_G, where

$$Q_G^{(1)}(x) = l(x)A(x)g(x) + l(x/m)A(x/m)g(x/m)/m.$$

Under assumptions [A1]-[A2],

$$\begin{aligned} Q_{G^*}^{(1)} &= \inf\{Q_G^{(1)}(x)|\theta_0 \le x \le m\theta_0\} > 0 \quad \text{and} \\ Q_G^{(1)^*} &= \sup\{Q_G^{(1)}(x)|\theta_0 \le x \le m\theta_0\} < \infty. \end{aligned} \tag{3.4}$$

Combining (3.1)−(3.4) yields that

$$B_1 Q_{G^*}^{(1)} E_n(a_n^* - a_G)^2/2 \le D(G, \delta_n^*) \le B_2 Q_G^{(1)^*} E_n(a_n^* - a_G)^2/2. \tag{3.5}$$

The inequality of (3.5) provides an equivalence in convergence rates between the empirical Bayes testing problem and the problem of estimating the critical point a_G. Thus, it suffices to study the asymptotic behavior of $E_n(a_n^* - a_G)^2$.

We have the following theorem.

Theorem 3.1. *Suppose that assumptions* [A1]−[A3] *hold. Then*

(a) $E_n(a_n^* - a_G)^2 = O(n^{-2/3})$, *and*
(b) *The empirical Bayes test* δ_n^* *is asymptotically optimal and* $D(G, \delta_n^*) = O(n^{-2/3})$.

Proof. Under [A2], $Q_G(\theta_0) < 0 = Q_G(a_G) < Q_G(m\theta_0)$. So, for sufficiently large n, there exist points $a_{G1}(n)$ and $a_{G2}(n)$ such that $\theta_0 < a_{G1}(n) < a_G < a_{G2}(n) < m\theta_0$, and $Q_G(a_{G1}(n)) = -4hc_4$ and $Q_G(a_{G2}(n)) = 4hc_4$, where $c_4 = 2e^{\theta_0}M$. From (2.2) and (2.6), the difference $a_n^* - a_G$ can be expressed as:

$$\begin{aligned} a_n^* - a_G &= -\int_{\theta_0}^{a_{G1}(n)} I[Q_n(x) \ge 0]dx - \int_{a_{G1}(n)}^{a_G} I[Q_n(x) \ge 0]dx \\ &\quad + \int_{a_G}^{a_{G2}(n)} I[Q_n(x) < 0]dx + \int_{a_{G2}(n)}^{m\theta_0} I[Q_n(x) < 0]dx \\ &\equiv -A_1(n) - A_2(n) + A_3(n) + A_4(n). \end{aligned}$$

Thus,

$$E_n(a_n^* - a_G)^2 \le 4\sum_{i=1}^{4} E_n A_i^2(n). \tag{3.6}$$

Hence, we need to evaluate $E_n A_j^2(n)$ for each $j = 1, 2, 3$ and 4. From Appendix, we have:

$$\begin{aligned} E_n A_1^2(n) &\le \frac{d_1}{n^2h^4}, & (3.7)\\ E_n A_2^2(n) &\le d_2 h^2, & (3.8)\\ E_n A_3^2(n) &\le d_2 h^2, & (3.9)\\ E_n A_4^2(n) &\le \frac{d_1}{n^2h^4}, & (3.10) \end{aligned}$$

where d_1 and d_2 are some finite positive values defined in Appendix. Combining (3.6)$-$(3.10) leads to

$$E_n(a_n^* - a_G)^2 \le \frac{2d_1}{n^2h^4} + 2d_2h^2 = O(n^{-2/3}) \text{ since } h = n^{-1/3}.$$

The result of part (b) also follows immediately. □

4. Examples and Comparison

We use the following examples to demonstrate the performance of the empirical Bayes test δ_n^*.

Example 4.1. Let $f(x|\theta) = \theta^{-1}I[\theta \le x \le 2\theta]$, $g(\theta) = \theta e^{-\theta}I[\theta > 0]$. Thus, $u(x) = 1$, $A(\theta) = \theta^{-1}$, $\theta > 0$. Then, $v_G(x) = e^{-x/2} - e^{-x}$, $|v_G^{(1)}(x)| \le 2e^{-x/2}$. We can verify that for each $\theta_0 > 0$, assumptions [A1]-[A3] hold. Thus, by Theorem 3.1, the empirical Bayes test δ_n^* is asymptotically optimal and $D(G, \delta_n^*) = O(n^{-2/3})$.

Example 4.2. Let $f(x|\theta) = u(x)A(\theta)I[\theta \le x \le 2\theta]$, where $B_1 = u(0) > 0$ and $u(x)$ is an increasing function of x for $x > 0$. $g(\theta) = \theta^{-2}I[\theta \ge 1]$. Thus, $A(\theta) = [\int_\theta^{2\theta} u(x)dx]^{-1} \le \frac{1}{B_1\theta}$.

$$v_G(x) = \int_{x/m}^{x} A(\theta)g(\theta)d\theta \le \int_{x/m}^{x} \frac{1}{B_1\theta^3}d\theta = \frac{1.5}{B_1x^2}.$$

$$|v_G^{(1)}(x)| \le A(X)g(X) + A(X/2)g(X/2)/2 \le \frac{1.5}{B_1 x^3}.$$

It is easy to verify that for each $\theta_0 > 0$, assumptions [Al]-[A3] hold. Thus. by Theorem 3.1, the empirical Bayes test δ_n^* is asymptotically optimal and $D(G, \delta_n^*) = O(n^{-2/3})$.

Example 4.3. $f(x|\theta) = u(x)A(\theta)I[\theta \le x \le m\theta]$, where $u(x) = [x+1]$ and $[y]$ denotes the largest integer not greater than y. For each $\theta_0 > 0$, the prior pdf g satisfies that $0 < c_0 \le g(\theta) \le c_1 < \infty$ for θ in $[\theta_0/m, 2m\theta_0]$ and $g(\theta) = 0$ for $\theta > k\theta_0$ for some $k > 2m$. Thus, $v_G(x) = 0$ for $x > k\theta_0$. Hence, we see that assumptions [A1]-[A3] hold. Thus, by Theorem 3.1, the empirical Bayes test δ_n^* is asymptotically optimal and $D(G, \delta_n^*) = O(n^{-2/3})$.

Xu and Shi (2004) have studied the same testing problem via the empirical Bayes approach. They proposed an empirical Bayes test δ_n^{XS} and studied its associated asymptotic optimality. Under certain stronger conditions, they showed that δ_n^{XS} is asymptotically optimal at a rate of order $O(n^{-\varepsilon s/(2s+1)})$ where $0 < \varepsilon < 1$ and s in an integer pertaining to their assumptions. It should be noted that δ_n^{XS} is not a monotone test. When s is small, the obtainable rate of δ_n^{XS} is slow, and the best possible rate is $O(n^{-\varepsilon/2})$, which is obtained when s tends to ∞. Therefore, we see that our proposed empirical Bayes test δ_n^* improves δ_n^{XS} in the sense of a faster rate $O(n^{-2/3})$ under weaker conditions.

One of the assumptions required in Xu and Shi (2004) is that the marginal pdf $f_G(x)$ be s-times differentiable. In Example 4.3, the marginal pdf $f_G(x)$ is not a continuous function. Thus, the condition that "$f_G(x)$ is s-times differentiable" is not satisfied. Therefore, the empirical Bayes test δ_n^{XS} cannot be applied. However, in such a situation, our proposed empirical Bayes test δ_n^* remains working well.

Appendix

We first investigate certain properties relating to the estimator $Q_n(x)$. For each $l = 1, \dots, n$, let

$$Q_{nl}(x) = \sum_{j=1}^{\infty} [\psi_{nlj}(x) - \alpha_{nlj}(x) - \beta_{nlj}(x) - \eta_{nlj}(x)].$$

Note that $Q_{nl}(x)$ is a random function of the random variable X_l. Thus, $Q_{nl}(x)$ are iid. We have:

$$Q_n(x) = \frac{1}{n}\sum_{l=1}^{n} Q_{nl}(x). \tag{A.1}$$

Since

$$\begin{aligned}
\psi_{nlj}(x) &= \frac{[1-e^{\theta_0 - X_l/m^j}]}{m^j u(X_l)} I[m^j\theta_0 < X_l \le m^j x],\\
\alpha_{nlj}(x) &= \frac{[1-e^{\theta_0 - X_l/m^j}]}{m^j u(X_l)} I[m^{j-1}x < X_l \le m^j\theta_0],\\
\beta_{nlj}(x) &= [e^{\theta_0 - x} - (\theta_0 - x) - 1]/(u(X_l)h) I[m^j x < X_l \le m^j x + h],\\
\eta_{nlj}(x) &= [e^{\theta_0 - x/m} - (\theta_0 - x/m) - 1]\\
&\quad /(u(X_l)h) I[m^{j-1}x < X_l \le m^{j-1}x + h],
\end{aligned}$$

we can obtain:

$|\sum_{j=1}^{\infty}\psi_{nlj}(x)| \le 1/B_1$, $|\sum_{j=1}^{\infty}\alpha_{nlj}(x)| \le e^{\theta_0}/B_1$, $|\sum_{j=1}^{\infty}\beta_{nlj}(x)| \le e^{\theta_0}/[hB_1]$ and $|\sum_{j=1}^{\infty}\eta_{nlj}(x)| \le e^{\theta_0}/[hB_1]$. Thus, $|Q_{nl}(x)| \le 4e^{\theta_0}/[hB_1]$ and

$$|Q_{nl}(x) - E_n Q_{nl}(x)| \le 8e^{\theta_0}/[hB_1] = c_3/h, \tag{A.2}$$

where $c_3 = 8e^{\theta_0}/B_1$.

Lemma A.1. *Under assumptions* [A1]-[A3]*, the following results hold.*

(a) $E_n\psi_{nlj}(x) = \psi_j(x)$.
(b) $E_n\alpha_{nlj}(x) = \alpha_j(x)$.
(c) $|E_n\beta_{nlj}(x) - \beta_j(x)| \le he^{\theta_0}v_G^{(1)^*}(m^j x)$, *where* $v_G^{(1)^*}(y) = \sup\{|v_G^{(1)}(t)|;$ $y \le t < y+h\}$.
(d) $|E_n\eta_{nlj}(x) - \eta_j(x)| \le he^{\theta_0}v_G^{(1)^*}(m^{j-1}x)$.
(e) $|E_nQ_{nl}(x) - Q_G(x)| \le 2he^{\theta_0}\sum_{j=1}^{\infty} v_G^{(1)^*}(m^{j-1}x) \le 2he^{\theta_0}M = hc_4$, *where* $c_4 = 2e^{\theta_0}M$.
(f) $|E_nQ_n(x) - Q_G(x)| \le hc_4$.

Proof. We provide proof for part (c) only. Parts (e) and (f) are the results of parts (a)−(d) and an application of [A3].

By the definition of $\beta_{nlj}(x)$ and $\beta_j(x)$,

$$\begin{aligned} |E_n\beta_{nlj}(x) - \beta_j(x)| &\le e^{\theta_0}\Big|\int_{m^j x}^{m^j x+h} \frac{[v_G(t) - v_G(x)]}{h} dt\Big| \\ &\le e^{\theta_0}\int_{m^j x}^{m^j x+h} \Big|\frac{(t-x)v_G^{(1)}(t^*)}{h}\Big| dt \le h e^{\theta_0} v_G^{(1)^*}(m^j x) \end{aligned}$$

where t^* is some value between x and t. □

Lemma A.2. *Under assumptions* [A1]−[A3]*, the following results hold.*

(a) $E_n[\psi^2_{nlj}(x)] \le 1/[m^{2j}B_1^2]$.

(b) $E_n[\alpha^2_{nlj}(x)] \le e^{2\theta_0}/[m^{2j}B_1^2]$,

(c) $E_n[\beta^2_{nlj}(x)] \le e^{2\theta_0} v_G^*(m^j x)/[hB_1]$, *where* $v_G^*(y) = \sup\{v_G(t);$ $y \le t < y+h\}$.

(d) $E_n[\eta^2_{nlj}(x)] \le e^{2\theta_0} v_G^*(m^{j-1} x)/[hB_1]$,

(e) $\mathrm{Var}\,(Q_{nl}(x)) \le c_5/h$, *where* $c_5 = \frac{2e^{2\theta_0}M}{B_1} + \frac{8e^{2\theta_0}}{B_1^2}\sum_{j=1}^{\infty}\frac{1}{m^{2j}}$.

(f) $\mathrm{Var}\,(Q_n(x)) \le c_5/(nh)$.

Proof. We provide proof for part (e) only. Let p_{nlj} and q_{nlk} be any two of the functions ψ_{nlj}, α_{nlj}, β_{nlj} and η_{nlj}. We see that $p_{nlj}(x)q_{nlk}(x) = 0$ if $j \ne k$. Thus,

$$\begin{aligned} Q^2_{nl}(x) &= \sum_{j=1}^{\infty}[\psi_{nlj}(x) - \alpha_{nlj}(x) - \beta_{nlj}(x) - \eta_{nlj}(x)]^2 \\ &\le 4\sum_{j=1}^{\infty}[\psi^2_{nlj}(x) + \alpha^2_{nlj}(x) + \beta^2_{nlj}(x) + \eta^2_{nlj}(x)]. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Var}\,(Q_{nl}(x)) &\le E_n[Q^2_{nl}(x)] \\ &\le 4\sum_{j=1}^{\infty}[E_n\psi^2_{nlj}(x) + E_n\alpha^2_{nlj}(x) + E_n\beta^2_{nlj}(x) + E_n\eta^2_{nlj}(x)] \\ &\le 4\sum_{j=1}^{\infty}\frac{1}{m^{2j}B_1^2} + 4\sum_{j=1}^{\infty}\frac{e^{2\theta_0}}{m^{2j}B_1^2} + \frac{2e^{2\theta_0}}{hB_1}\sum_{j=0}^{\infty} v_G^*(m^j x) \end{aligned}$$

$$\leq \ 8\sum_{j=1}^{\infty}\frac{e^{2e\theta_0}}{m^{2j}B_1^2}+\frac{2e^{2\theta_0}M}{hB_1}$$
$$\leq \ c_5/h, \qquad \text{where } c_5=\frac{2e^{2\theta_0}M}{B_1}+\frac{8e^{2\theta_0}}{B_1^2}\sum_{j=1}^{\infty}\frac{1}{m^{2j}}.$$

□

Lemma A.3.

(a) $a_G - a_{G1}(n) \leq 4hc_4/Q^{(1)}_{G^*}$; (b) $a_{G2}(n) - a_G \leq 4hc_4/Q^{(1)}_{G^*}$, *where* $Q^{(1)}_{G^*} = \inf\{Q^{(1)}_G(x)|\theta_0 \leq x \leq m\theta_0\} > 0$.

Proof. We provide proof for part (a) only.

By the definitions of the points a_G and $a_{G1}(n)$, and an application of mean-value theorem, $4hc_4 = Q_G(a_G) - Q_G(a_{G1}(n)) = Q^{(1)}_G(x^*)(a_G - a_{G1}(n))$ for some value x^* between a_G and $a_{G1}(n)$. Thus, $a_G - a_{G1}(n) = 4hc_4/Q^{(1)}_G(x^*) \leq 4hc_4/Q^{(1)}_{G^*}$. □

Lemma A.4.

(a) *For* x *in* $[\theta_0, a_{G1}(n)]$, $E_nQ_n(x) < Q_G(x)/2 < 0$.
(b) *For* x *in* $[a_{G2}(n), m\theta_0]$, $E_nQ_n(x) > Q_G(x)/2 > 0$.

Proof. We provide proof for part (a) only. From (A.1), $Q_G(x) - hc_4 \leq E_nQ_n(x) \leq Q_G(x) + hc_4$. Since $Q_G(x)$ is increasing in x and $Q_G(a_{G1}(n)) = -4hc_4$, for x in $[\theta_0, a_{G1}(n)]$,

$$\begin{aligned} E_nQ_n(x) \ &\leq \ Q_G(x)/2 + Q_G(x)/2 + hc_4 \leq Q_G(x)/2 + Q_G(a_{G1}(n))/2 + hc_4 \\ &= \ Q_G(x)/2 - 4hc_4 + hc_4 < Q_G(x)/2 < 0. \end{aligned}$$

□

Proof of (3.7). By Hölder inequality,

$$\begin{aligned} A_1(n) \leq &\Big\{\int_{\theta_0}^{a_{G1}(n)}\frac{I[Q_n(x)\geq 0]}{[-Q^3_G(x)]Q^{(1)}_G(x)}dx\Big\}^{1/2} \\ &\times\Big\{\int_{\theta_0}^{a_{G1}(n)} I[Q_n(x)\geq 0][-Q^3_G(x)]Q^{(1)}_G(x)dx\Big\}^{1/2} \end{aligned}$$

where

$$\begin{aligned}\int_{\theta_0}^{a_{G1}(n)} \frac{I[Q_n(x) \geq 0]}{[-Q_G^3(x)]Q_G^{(1)}(x)}dx &\leq \frac{1}{[Q_{G^*}^{(1)}]^2}\int_{\theta_0}^{a_{G1}(n)} \frac{Q_G^{(1)}(x)}{[-Q_G^3(x)]}dx \\ &\leq \frac{1}{[Q_{G^*}^{(1)}]^2 Q_G^2(a_{G1}(n))} \leq \frac{1}{[Q_{G^*}^{(1)}]^2 16c_4^2h^2}.\end{aligned}$$

Thus,

$$E_nA_1^2(n) \leq \frac{1}{[Q_{G^*}^{(1)}]^2 16c_4^2h^2}\int_{\theta_0}^{a_{G1}(n)} P_n\{Q_n(x) \geq 0\}[-Q_G^3(x)]dQ_G(x).$$

By Lemma A.4, (A.1)−(A.2), an application of Bernstein's inequality and Lemma A.2(e), we can obtain: for each x in $[\theta_0, a_{G1}(n)]$,

$$\begin{aligned}P_n\{Q_n(x) \geq 0\} &\leq P_n\{Q_n(x) - E_nQ_n(x) \geq -Q_G(x)/2\} \\ &\leq \exp\Big\{-\frac{n[Q_G(x)/2]^2/2}{\text{Var}\,(Q_{nl}(x)) + \frac{c_3}{3h}\times|\frac{Q_G(x)}{2}|}\Big\} \\ &= \exp\Big\{-\frac{6nh}{8}\times\frac{Q_G^2(x)}{6h\text{Var}\,(Q_{nl}(x)) + c_3|Q_G(x)|}\Big\} \\ &\leq \exp\Big\{-\frac{6nh}{8}\frac{Q_G^2(x)}{6h\times\frac{c_5}{h} + c_3|Q_G(x)|}\Big\} \\ &\leq \exp\Big\{-\frac{6nh}{8}\times\frac{Q_G^2(x)}{6c_5 + c_3|Q_G(\theta_0)|}\Big\} \\ &\leq \exp\{-nh\tau Q_G^2(x)\}\end{aligned}$$

where $\tau = \frac{6}{8}\times\frac{1}{6c_5 + c_3\max(|Q_G(\theta_0)|, Q_G(m\theta_0))}$.

Therefore,

$$\begin{aligned}&\int_{\theta_0}^{a_{G1}(n)} P_n\{Q_n(x) \geq 0\}[-Q_G^3(x)]dQ_G(x) \\ &\leq \int_{\theta_0}^{a_{G1}(n)} \exp\{-nh\tau Q_G^2(x)\}[-Q_G^3(x)]dQ_G(x) \leq \frac{1}{n^2h^2\tau^2}.\end{aligned}$$

Hence,

$$E_n A_1^2(n) \leq \frac{1}{[Q_{G^*}^{(1)}]^2 16 c_4^2 h^2} \times \frac{1}{n^2 h^2 \tau^2} = \frac{d_1}{n^2 h^4} \quad \text{where } d_1 = \frac{1}{[Q_{G^*}^{(1)}]^2 16 \tau^2 c_4^2}.$$

□

Proof of (3.8). By Lemma A.3(a), $A_2(n) = \int_{a_{G1}(n)}^{a_G} I[Q_n(x) \geq O]dx \leq a_G - a_{G1}(n) \leq 4hc_4/Q_{G^*}^{(1)}$. Hence, $E_n A_2^2(n) \leq d_2 h^2$, where $d_2 = 16c_4^2/[Q_{G^*}^{(1)}]^2$. □

Proofs of (3.9) and (3.10). The proofs of (3.9) and (3.10) are similar to that of (3.8) and (3.7), respectively. The details are thus omitted. □

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