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# A CONSTRUCTIVE METHOD FOR THE CYCLOIDAL NORMAL FREE SUBGROUPS OF FINITE INDEX OF HECKE GROUPS $H(\sqrt{2})$ AND $H(\sqrt{3})$

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**Abstract**

Cycloidal subgrups of the modular group are studied in [8]. Here cycloidal free normal subgroups of Hecke groups are considered. It is found that when $q \equiv 2 \pmod 4$, $H(\lambda_q)$ has no such subgroups. In all other cases the signatures of these subgroups are constructed by means of $q$-gons and their signatures are given.

## 1. Introduction

Hecke groups $H(\lambda_q)$ are the discrete subgroups of $PSL(2,\mathbb{R})$ generated by the linear fractional transformations

$$R(z) = -\frac{1}{z} \text{ and } T(z) = z + \lambda_q,$$

where $\lambda_q = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}$, $q \geqslant 3$. We put $S = RT$, i.e.

$$S(z) = -\frac{1}{z+\lambda_q}.$$

For $q = 3$, the obtained group is the modular group $\Gamma = PSL(2,\mathbb{Z})$.

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Other well-known Hecke groups are $H(\sqrt{2})$, $H(\frac{1+\sqrt{5}}{2})$ and $H(\sqrt{3})$, obtained for $q = 4, 5$ and $6$, respectively.

$H(\lambda_q)$ is isomorphic to the free product of two cyclic groups of orders 2 and $q$, [1]. It has signature $(0; 2, q, \infty)$ as a Fuchsian group of the first kind. That is, it acts on a Riemann surface of genus 0, and generated by two elliptic elements $R$ and $S$ of order 2 and $q$ with their product $T$, being parabolic, is of infinite order. All parabolic elements in $H(\lambda_q)$ are conjugate to $T$, so that the parabolic class number of $H(\lambda_q)$ is one.

A subgroup $N$ of $H(\lambda_q)$ having only one parabolic conjugacy class is called cycloidal. Cycloidal subgroups of the modular group are studied in [8] by Millington. Cycloidal normal subgroups of $H(\lambda_q)$ of finite index are studied in [3], and a formula for the number of them was given.

Here we give a constructive method of obtaining cycloidal free normal subgroups of $H(\sqrt{2})$ and we prove that $H(\sqrt{3})$ has no such subgroups.

## 2. Background

Let $N$ be a normal cycloidal free subgroup of $H(\lambda_q)$ with finite index $\mu$. Let the genus of $N$ be $g$. $g$ is also the genus of the underlying Riemann surface $H(\lambda_q)/N$.

By the Riemann-Hurwitz formula, we have

$$g = \frac{1}{2}(1 + \mu\frac{q-2}{2q}).$$

As $g$ depends only on $q$ and $\mu$, and $q$ is a given fixed number, we need to determine the values of $\mu$ by means of the above equation. In this way, we find the possible $N$'s. Then we give permutation representation of $N$ to guarantee its existence. This representation is obtained by means of $\mu/q$ $q$-gons.

The following result which we recall from [9] will be referred as the permutation method during the paper.

**Lemma 2.1.** *Let $\Gamma$ be a Fuchsian group of the first kind with signature $(g; m_1, \ldots, m_k)$. Let $\Lambda$ be a normal subgroup of $\Gamma$ with finite index $\mu$ and let* *the exponent of $x_i$ modulo $\Lambda$ be $l_i$. Then the signature of $\Lambda$ is*

$$(g'; \left(\frac{m_1}{l_1}\right)^{\frac{\mu}{l_1}}, \ldots, \left(\frac{m_k}{l_k}\right)^{\frac{\mu}{l_k}})$$

*where $\left(\frac{m_i}{l_i}\right)^{\frac{\mu}{l_i}}$ means that the period $\left(\frac{m_i}{l_i}\right)$ occurs $\mu/l_i$ times. Here, periods equal to 1 are omitted and the genus $g'$ can be computed by the Riemann-Hurwitz formula.*

The idea of this result is to map the given group onto a finite subgroup of the symmetric group on $\mu$ symbols, and then to obtain the periods of the subgroup by dividing the orders of the generators to the length of each cycle in the permutation representation of this generator. When a generator is of infinite order, then there will be as many infinities, corresponding to this generator as the number of cycles in the permutation representation.

The number of cycloidal free subgroups $N$ of $H(\lambda_q)$ with finite quotient $H(\lambda_q)\ /N \cong G$, is equal to the number of homomorphisms from $H(\lambda_q)$ onto $G$. Different homomorphisms are obtained by taking different cycles. To have a normal subgroup one must have cycles of equal length. As a result of this, the cycle length of the cycles in the image of $H(\lambda_q)$ must divide the index $\mu$ of $N$ in $H(\lambda_q)$.

## 3. Cycloidal Free Normal Subgroups of $H(\sqrt{2})$ of Finite Index

Let $N$ be a cycloidal free normal subgroup of $H(\sqrt{2})$ of finite index $\mu$. It is well-known that $\mu = n.t$ where $t$ denotes the parabolic class number of $N$ and $n$ is the level, which is the least positive integer so that $T^n$ belongs to $N$. Since $N$ is free, $2 \mid \mu$ and $4 \mid \mu$ implying $\mu = 4k$, $k \in \mathbb{N}$. Also $N$ has no element of finite order and as it is cycloidal, $t = 1$. Therefore $N$ has the signature $(g; \infty)$. We only need to determine the genus $g$. By the Riemann-Hurwitz formula,

$$g = \frac{1}{8}(\mu + 4) = \frac{1}{8}(4k + 4) = \frac{k+1}{2}.$$

Then for even values of $k$, we cannot find integer values of $g$; hence $k$ must be odd.

We are now in a state to know for which values of $\mu$, we can have a cycloidal free normal subgroup of $H(\sqrt{2})$. It then remains to construct these subgroups.

**Theorem 3.1.** *Let* $\mu = 4(2m+1)$, $m \in \mathbb{N}$. *Then* $H(\sqrt{2})$ *has a cycloidal free normal subgroup* $N$ *of index* $\mu$ *given by the signature* $(m+1, \infty)$.

*Proof.* We already know that $N$ has genus $m+1$. We only need to show the existence of such a subgroup. Let $\Theta$ be the homomorphism from $H(\sqrt{2})$ onto the image group $G$ induced by taking $R$ to a product of transformations and $S$ to a product of 4-cycles (note that $R$ is of of order 2 and $S$ is fo order 4). Then $G$ acts transitively on $\mu = 8m+4$ objects. We define the permutations corresponding to $R$ as follows:

$$R \longrightarrow \left(\frac{\mu}{2} \;\; \frac{\mu}{2}-2\right)\left(\frac{\mu}{2}+1 \;\; \frac{\mu}{2}+3\right)(2k \;\; 2k+3) \pmod{\mu}$$

for $k = 1, 2, \ldots, \frac{\mu}{4}-2, \frac{\mu}{4}+1, \ldots, \frac{\mu}{2}$. Also $S$ goes to the following product:

$$S \longrightarrow (1 \;\; 2 \;\; 3 \;\; 4)(5 \;\; 6 \;\; 7 \;\; 8) \ldots (\mu-3 \;\; \mu-2 \;\; \mu-1 \;\; \mu)$$

Then, after some calculations, one finds that $R.S = T$ goes to one $\mu-$cycles, implying the subgroup obtained is cycloidal. □

**Example 3.1.** Let $\mu = 20$. Then we have the following permutations:

$$\begin{aligned}
R &\rightarrow (8 \;\; 10)(11 \;\; 13)(2 \;\; 5)(4 \;\; 7)(6 \;\; 9)(12 \;\; 15)(14 \;\; 17)(16 \;\; 19)(18 \;\; 1)(20 \;\; 3)\\
S &\rightarrow (1 \;\; 2 \;\; 3 \;\; 4)(5 \;\; 6 \;\; 7 \;\; 8)(9 \;\; 10 \;\; 11 \;\; 12)(13 \;\; 14 \;\; 15 \;\; 16)(17 \;\; 18 \;\; 19 \;\; 20)\\
T &\rightarrow (1 \;\; 19 \;\; 13 \;\; 12 \;\; 16 \;\; 20 \;\; 4 \;\; 8 \;\; 11 \;\; 14 \;\; 18 \;\; 2 \;\; 6 \;\; 10 \;\; 5 \;\; 3 \;\; 17 \;\; 15 \;\; 9 \;\; 7)
\end{aligned}$$

We choose the transformations for $R$ by means of $\frac{\mu}{4}$ 4-gons(squares) each having vertices coresponding to the numbers in the permutations of $S$ as shown in Figure 1. To give a general formula for the permutations of $R$, we choose the lines connecting the vertices as symmetrical as possible. There are many other choices for these permutations. Each will give another isomorphism of $H(\sqrt{2})$ onto a group of order 20.

## 4. Cycloidal Free Normal Subgroups of $H(\sqrt{3})$ of Finite Index

Because of the similarity between the underlying fields of $H(\sqrt{2})$ and

$H(\sqrt{3})$, one may expect to have a similar result about the cycloidal free subgroups of $H(\sqrt{3})$. But we show that the situation is quite different:

**Theorem 4.1.** *$H(\sqrt{3})$ has no cycloidal free normal subgroups of finite index.*

*Proof.* Let us suppose that $N$ is a cycloidal free normal subgroup of $H(\sqrt{3})$ with index $\mu$. As $R$ goes to transpositions and $S$ goes to 6-cycles, $\mu$ must be a multiple of 6. Let us put $\mu = 6k$, $k \in \mathbb{N}$. If the genus of $N$ is $g$, by the Riemann-Hurwitz formula,

$$2g - 2 + 1 = 6k(-2 + \frac{1}{2} + \frac{5}{6} + 1) \tag{4.1}$$

and hence

$$g = \frac{1 + 2k}{2} \tag{4.2}$$

is obtained. But this can not be an integer implying the result. □

## 5. Cycloidal Free Normal Subgroups of $H(\lambda_5)$ of Finite Index

Let $N$ be a cycloidal free normal subgroup of $H(\lambda_5)$ with finite index $\mu$. Since $R$ goes to $\mu/2$ 2-cycles and $S$ goes to $\mu/5$ 5-cycles, $\mu$ is divisible by 10. Let $\mu = 10k,\ k \in \mathbb{N}$. By the Riemann-Hurwitz formula

$$2g - 2 + 1 = 10k(-2 + \frac{1}{2} + \frac{4}{5} + 1) \tag{5.1}$$

and therefore

$$g = \frac{1 + 3k}{2}, \tag{5.2}$$

where $g$ is the genus of $N$. Hence $k$ must be odd. Then we have

**Theorem 5.1.** *Let $\mu = 10k$ with $k \in \mathbb{N}$ odd. Then $H(\lambda_5)$ has a cycloidal free normal subgroup $N$ of index $\mu$ given by the signature*

$$\left(\frac{1 + 3k}{2}; \infty\right). \tag{5.3}$$

*Proof.* We have just found the genus $g$ of $N$ to be $g = \frac{1+3k}{2}$. Therefore it only remains to prove the existence of $N$. We define the permutations as follows:

We map every number between 1 and $\mu/2$ to the number $\mu/2$ bigger than itself except the following two cases:

$$\begin{aligned}(2+10n \quad 7+10n) \qquad n &= 0,1,\ldots,k-1,\\ (6+10n \quad 11+10n) \qquad n &= 0,1,\ldots,\frac{k-3}{2},\\ (1+10n \quad 6+10n) \qquad n &= \frac{k+1}{2},\ldots,k-1.\end{aligned}$$

In this way we obtain all the permutations for $R$. Define the permutation for $S$ as (1 2 3 4 5), (6 7 8 9 10), and so on. Then one can find the permutation corresponding to $T=R.S$ as a unique $\mu$-cycle proving that we got a cycloidal normal subgroup. □

**Example 5.1.** Let $\mu=50$. We define

$$\begin{aligned}R \to\ & (1\ \ 26)(2\ \ 7)(3\ \ 28)(4\ \ 29)(5\ \ 30)(6\ \ 11)(8\ \ 33)(9\ \ 34)(10\ \ 35)\\ & (12\ \ 17)(13\ \ 38)(14\ \ 39)(15\ \ 40)(16\ \ 21)(18\ \ 43)(19\ \ 44)(20\ \ 45)\\ & (22\ \ 27)(23\ \ 48)(24\ \ 49)(25\ \ 50)(31\ \ 36)(32\ \ 37)(41\ \ 46)(42\ \ 47) \qquad (5.4)\end{aligned}$$

$$S \to (1\ \ 2\ \ 3\ \ 4\ \ 5)(6\ \ 7\ \ 8\ \ 9\ \ 10)\cdots\cdots(46\ 47\ \ 48\ \ 49\ \ 50). \qquad (5.5)$$

Then

$$\begin{aligned}T \to\ & (1\ \ 27\ \ 23\ \ 49\ \ 25\ \ 46\ \ 42\ \ 48\ \ 24\ \ 50\ \ 21\ \ 17\ \ 13\ \ 39\ \ 15\ \ 36\\ & 32\ \ 38\ \ 14\ \ 40\ 11\ \ 7\ \ 3\ 29\ \ 5\ \ 26\ \ 2\ \ 8\ \ 34\ \ 10\ \ 31\ \ 37\ \ 33\ \ 9\\ & 35\ \ 6\ \ 12\ \ 18\ \ 44\ \ 20\ \ 41\ \ 47\ \ 43\ \ 19\ \ 45\ \ 16\ \ 22\ \ 28\ \ 4\ \ 30). \qquad (5.6)\end{aligned}$$

## 6. Cycloidal Free Subgroups of $H(\lambda_q)$ of Finite Index

We now consider the general case. Let $N$ be as usual. Again $R$ will go to $\mu/2$ transformations and $S$ to $\mu/q$ $q$-cycles. Hence $\mu$ must be divisible by 2 and $q$. Therefore, if $q$ is odd $\mu$ must have the form $\mu=2qk$ with $k\in\mathbb{N}$, whereas when $q$ is even $\mu=qk$ with $k\in\mathbb{N}$.

We are going to determine whether $H(\lambda_q)$ may have a cycloidal free normal subgroup of finite index $\mu$ or not. Let, first, $q$ be odd. By the Riemann-Hurwitz formula

$$g = \frac{1 + k(q-2)}{2}. \tag{6.1}$$

Hence $k$ must be odd. In other words, $\mu = (4n+2)q$ for $n \in \mathbb{N}$.

Secondly, let $q = 2m$, $m \in \mathbb{N}$. Similarly

$$g = \frac{1 + \frac{k}{2}(q-2)}{2} = \frac{1 + k(m-1)}{2}.$$

Hence to have a non-negative integer $g$, we must have $k(m-1)$ odd, i.e. $k$ is odd and $m$ is even. These imply that $q$ is divisible by 4. We therefore have

**Theorem 6.1.** *If $q \equiv 2 \pmod 4$, then $H(\lambda_q)$ can not have any cycloidal free subgroups of finite index.*

For the modular group these subgroups are studied by Millington in detail, [8]. For larger values of $q$, it is difficult to study cycloidal free normal subgroups by means of $q$-gons.

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