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SHAPE PRESERVING APPROXIMATION BY COMPLEX POLYNOMIALS IN THE UNIT DISK

BY

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Abstract

The purpose of this paper is to obtain new results concerning the preservation of some properties in Geometric Function Theory, in approximation of analytic functions by polynomials, with best approximation types of rates. In addition, the approximating polynomials satisfy some interpolation conditions too.

1. Introduction

A central concept in Geometric Function Theory is that of univalence. Many sufficient conditions of geometric kind that imply univalence are important, like : starlikeness, convexity, close-to-convexity, α-convexity, spirallikeness, bounded turn. Also, there are known many other sufficient analytic conditions of univalence.

All these geometric sufficient conditions for univalence are mainly studied for analytic functions, because in this case they can easily be expressed by nice (and simple) differential inequalities. Also, because of the *Riemann Mapping Theorem*, in general it suffices to study these properties on the open unit disk denoted by $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$.

Concerning these properties, it is natural to ask how well can be approximated an analytic function having a given property in Geometric Function Theory, by polynomials having the same property.

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The history of this problem, in our best knowledge, contains three main directions of research, depending on the methods used :

(1) Approximation preserving geometric properties by the partial sums of the Taylor expansion, see e.g., Szegö [27], Alexander [1], Ruscheweyh [18], Ruscheweyh-Wirths [22], Suffridge [25, p. 236], Suffridge [26] and the references cited therein;

(2) Approximation preserving geometric properties by Cesàro means and by convolution polynomials based on other trigonometric kernels, see e.g., Fejér [6], Robertson [17], Pólya-Schoenberg [16], Bustoz [4], Lewis [12], Egerváry [5], Ruscheweyh [19, 20], Ruscheweyh-Sheil-Small [21], Gal [7, 8, 9] and the references cited therein;

(3) Approximation of univalent functions by subordinate polynomials in the unit disk, by using the concept of maximal polynomial range, see e.g., Andrievskii-Ruscheweyh [2], Greiner [10], Greiner-Ruscheweyh [11] and the references cited therein.

In this paper we use other methods in order to obtain new results concerning the preservation of geometric properties by approximating and interpolating polynomials.

2. Main Results

The basic tools are represented by the simultaneous approximation results. Theorems A, B and Theorem 2.1 below summarize these kinds of results. Before to state them, let us first make some useful comments.

Theorem A was proved by Vorob'ev [28] for the so-called domains of type A in the complex plane (including the unit disk) and Theorem B was proved by Andrievskii-Pritsker-Varga [3] for general continua in the complex plane (including the unit disk). For our purpose, we will re-state them here for the particular case of unit disk only. Unfortunately, the constants appearing in these estimates, are claimed in the corresponding papers, as independent of n and z only, without to be mentioned the independence of f too. Because of complicated technical details, it seems to be very difficult to deduce from the proofs of Theorems A and B that possibly these constants are independent of f too. For this reason, in the case of unit disk, by Theorem 2.1, we prefer to present here a new simple proof which clearly shows that the constant is independent of n, z, and f too.

Denote by $A(\mathbb{D}) = \{f : \overline{\mathbb{D}} \to \mathbb{C}; f$ is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}\}$ and for $p \in \mathbb{N}$, $A^p(\mathbb{D})$ denotes the space of p-times continuous differentiable functions on $\overline{\mathbb{D}}$.

The following two results are known.

Theorem A. (Vorob'ev [28]) *Let $p \in \mathbb{N}$. For any $f \in A^p(\mathbb{D})$ and $n \geq p$, there exists a polynomial P_n of degree $\leq n$, such that for all $j = 0, \ldots, p$ we have*

$$|f^{(j)}(z) - P_n^{(j)}(z)| \leq An^{j-p}\omega_1(f^{(p)}; \frac{1}{n}), \forall z \in \partial\mathbb{D},$$

where A is independent of n and z. Here $\omega_1(g;\delta) = sup\{|f(u) - f(v)|; u, v \in \overline{\mathbb{D}}, |u - v| \leq \delta\}$.

Theorem B. (Andrievskii-Pritsker-Varga [3]) *Let us suppose that $p, q, r \in \mathbb{N}$, $f \in A^p(\mathbb{D})$ and consider the distinct points $|z_l| = 1, l = 1, \ldots, q$. Then, for any $n \in \mathbb{N}$, $n \geq qp + r$, there exists a polynomial P_n of degree $\leq n$, such that for all $j = 0, \ldots, p$ we have*

$$|f^{(j)}(z) - P_n^{(j)}(z)| \leq cn^{j-p}\omega_r^*(f^{(p)}; \frac{1}{n}), \forall z \in \partial\mathbb{D},$$

and

$$P_n^{(j)}(z_l) = f^{(j)}(z_l), l = 1, \ldots, q,$$

where c is independent of n and z. Here

$$\omega_r^*(g;\delta) := sup_{z\in\overline{\mathbb{D}}}\{E_{r-1}(g; \overline{\mathbb{D}} \cap B(z;\delta))\},$$

$B(z;\delta) = \{\xi \in \mathbb{C}; |\xi - z| \leq \delta\}$, $E_m(g; M) := inf\{||g - P||_M$; P complex polynomial of degree $\leq m\}$, $||\cdot||_M$ is the uniform norm on the set M.

Our main result is the following.

Theorem 2.1. *Let $p \in \mathbb{N}$. For any $f \in A^p(\mathbb{D})$ and $n \geq p$, there exists a polynomial P_n of degree $\leq n$, such that for all $j = 0, \ldots, p$ we have*

$$||f^{(j)} - P_n^{(j)}|| \leq Cn^{j-p}E_{n-p}(f^{(p)}),$$

where $C > 0$ depends on p but it is independent of n and f. Here $E_n(f^{(p)}) = inf\{||f^{(p)} - P||; P$ is polynomial of degree $\leq n\}$ and $||\cdot||$ represents the uniform norm in $C(\overline{\mathbb{D}})$.

Proof. First we recall some known facts about the de la Vallée Poussin trigonometric sums. According to e.g., Stechkin [24, p. 61], relation (0.3), the de la Vallée Poussin sums attached to a continuous 2π periodic function g is given by

$$\tilde{\sigma}_{n,m}(g)(x) = \frac{1}{m+1}\sum_{j=n-m}^{n} s_j(g)(x),$$

where $0 \leq m \leq n$, $n = 0, 1, \ldots,$ and $s_j(g)(x)$ denotes the jth Fourier partial sum attached to g. We also have the representation (see e.g. Stechkin [24], p. 63)

$$\tilde{\sigma}_{n,m}(g)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x+t)V_{n,m}(t)dt,$$

where

$$V_{n,m}(t) = \frac{1}{m+1}\sum_{k=n-m}^{n} D_k(t),$$

and $D_k(t) = \frac{1}{2} + \sum_{j=1}^{k}\cos(jt)$ represents the Dirichlet kernel of order k.

Now, for $f \in A^p(\mathbb{D})$ and $0 \leq m \leq n$, let us define

$$\sigma_{n,m}(f)(z) = \frac{1}{m+1}\sum_{k=n-m}^{n} T_k(f)(z),$$

where $T_k(f)(z) = \sum_{j=0}^{k}\frac{f^{(j)}(0)}{j!}z^j$ represents the kth Taylor partial sum of f.

It is easy to see that we have the representation (by using the same kinds of reasonings as in , e.g., the proof of Lemma 1, p. 881-882 in Mujica [15])

$$\sigma_{n,m}(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})V_{n,m}(t)dt.$$

In what follows we will consider some properties of $\sigma_{2n,n-p}(f)(z)$, $n \geq p$, by using everywhere the notation $T'_k(f)$ instead of $\{T_k(f)\}'$. First, from the obvious property $T'_k(f)(z) = T_{k-1}(f')(z)$, we easily get

$$\sigma^{(k)}_{2n,n-p}(f)(z) = \sigma_{2n-k,n-p}(f^{(k)})(z),$$

for all $k = 0, \ldots, p$. Also, $\sigma_{2n,n-p}(f)(z)$ is a polynomial of degree $\leq 2n$. Concerning the approximation property, analysing all the proofs in Stechkin [24], it is not difficult to see (by repeating the reasonings there) that we can get the same kind of estimate as that in the trigonometric case, that is relation (0.10) in Stechkin [24, p. 62], which is

$$||f - \sigma_{n,m}(f)|| \leq A \sum_{j=0}^{n} \frac{E_{n-m+j}(f)}{m+j+1},$$

where $||\cdot||$ denotes the uniform norm in $C(\overline{\mathbb{D}})$, $A > 0$ is an absolute constant (independent of f, n and m) and $E_n(f) = inf\{||f - P||;\ P$ is polynomial of degree $\leq n\}$.

This implies that for all $k = 0, \ldots, p$ and $n \geq p$, we have the estimate

$$\begin{aligned}
||f - \sigma_{2n-k,n-p}(f)|| &\leq A \sum_{j=0}^{2n-k} \frac{E_{n+p-k+j}(f)}{n-p+j+1} \\
&\leq AE_{n+p-k}(f) \sum_{j=0}^{2n-k} \frac{1}{n-p+j+1} \\
&\leq AE_{n+p-k}(f) \frac{2n-k+1}{n-p+1} \\
&\leq AE_{n+p-k}(f) \frac{2n+1}{n-p+1} \\
&= AE_{n+p-k}(f)[2 + (2p-1)/(n-p+1)] \\
&\leq A(2p+1)E_{n+p-k}(f).
\end{aligned}$$

Let $P_n(z)$ be the best approximation polynomial of degree $\leq n$, that is $E_n(f) = ||f - P_n||$ (or, any near to the best approximation polynomial of degree $\leq n$, that is $||f - P_n|| \leq CE_n(f)$, with $C > 1$ independent of n and f).

For any $n \geq p$ and $k = 0, \ldots, p$, taking into account the above error estimate, the Bernstein's inequality for complex polynomials and the well-known inequality $E_n(f) \leq C_p n^{-p} E_{n-p}(f^{(p)})$, we obtain (notice that below

C_p stands for a constant depending only on p, which may change from line to line)

$$\begin{aligned}
||f^{(k)} - P_n^{(k)}|| &\le ||f^{(k)} - \sigma_{2n,n-p}^{(k)}(f)|| + ||\sigma_{2n,n-p}^{(k)}(f) - P_n^{(k)}|| \\
&= ||f^{(k)} - \sigma_{2n-k,n-p}(f^{(k)})|| + ||\sigma_{2n,n-p}^{(k)}(f) - P_n^{(k)}|| \\
&\le C_p E_{n+p-k}(f^{(k)}) + ||(\sigma_{2n,n-p}(f) - P_n)^{(k)}|| \\
&\le C_p E_{n+p-k}(f^{(k)}) + (2n)^k ||\sigma_{2n,n-p}(f) - P_n|| \\
&\le C_p E_{n+p-k}(f^{(k)}) + C_p n^k [||\sigma_{2n,n-p}(f) - f|| + ||f - P_n||] \\
&\le C_p E_{n+p-k}(f^{(k)}) + C_p n^k [E_{n+p}(f) + E_n(f)] \\
&\le C_p E_{n+p-k}(f^{(k)}) + C_p n^k E_n(f) \\
&\le C_p E_{n+p-k}(f^{(k)}) + C_p n^k n^{-p} E_{n-p}(f^{(p)}) \\
&\le C_p (n+p-k)^{-p+k} E_n(f^{(p)}) + C_p n^{-p+k} E_{n-p}(f^{(p)}) \\
&\le C_p n^{-p+k} E_{n-p}(f^{(p)}),
\end{aligned}$$

which proves the theorem. □

Remark. Theorem 2.1 raises the natural question if instead of the de la Vallée Poussin sums, we could use the Taylor polynomials attached to f, denoted by $T_n(f)(z) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} z^j$. Indeed, these polynomials reproduce any polynomial of degree $\le n$ and satisfy $T_n^{(k)}(f) = T_{n-k}(f^{(k)})$. However, as linear operators on $A(\mathbb{D})$, the family $T_n, n \in \mathbb{N}$, obviously is not uniformly bounded on $A(\mathbb{D})$. This shortcoming could be solved by supposing a stronger hypothesis on f, that is, $f \in A^p(\mathbb{D}_R)$ with $R > 1$ and $p \in \mathbb{N}$, where $\mathbb{D}_R = \{|z| < R\}$. In this case, taking into account the Cauchy's estimates for the coefficients, for any fixed $1 < r < R$, we have $\frac{|f^{(j)}(0)|}{j!} \le \frac{||f||_r}{r^j}$, where $||f||_r$ denotes the uniform norm in $C(\overline{\mathbb{D}_r})$ (for simplicity, $||\cdot||_1$ is denoted by $||\cdot||$) and

$$||T_n(f)|| \le \frac{r}{r-1} ||f||_r, \forall n \in \mathbb{N},$$

which shows that $T_n : A(\overline{\mathbb{D}_r}) \to A(\mathbb{D}), n \in \mathbb{N}$, is a family of bounded linear operators. Reasoning similar with the proof of Theorem 2.1, let P_n be the polynomial of best approximation of degree $\le n$ of f and q_{n-k} the polynomial of best approximation of degree $\le n-k$ of $f^{(k)}$, both on $\overline{\mathbb{D}_r}$. We get

$$||f^{(k)} - P_n^{(k)}|| \le ||f^{(k)} - T_{n-k}(f^{(k)})|| + ||T_n^{(k)}(f) - P_n^{(k)}||$$

$$
\begin{aligned}
&\leq ||f^{(k)} - q_{n-k}|| + ||q_{n-k} - T_{n-k}(f^{(k)})|| + ||[T_n(f) - P_n]^{(k)}|| \\
&\leq E_{n-k}(f^{(k)}; \overline{\mathbb{D}_r}) + |||T_{n-k}||| \cdot ||f^{(k)} - q_{n-k}||_r + n^k ||T_n(f) - P_n|| \\
&\leq E_{n-k}(f^{(k)}; \overline{\mathbb{D}_r}) + \frac{r}{r-1} E_{n-k}(f^{(k)}; \overline{\mathbb{D}_r}) + n^k ||T_n[f - P_n]|| \\
&\leq C_{r,p,k} n^{-p+k} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}) + |||T_n||| \cdot n^k ||f - P_n||_r \\
&\leq C_{r,p,k} n^{-p+k} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}) + C_{r,p,k} n^{-p+k} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}) \\
&\leq \frac{C_{r,p,k}}{n^{p-k}} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}).
\end{aligned}
$$

Therefore, for any $f \in A^p(\mathbb{D}_R)$, $p \geq 1$, (with $R > 1$), any fixed $1 < r < R$ and any $n > p$, there exists a sequence of polynomials $P_n(f)$, with degree$(P_n(f)) \leq n$, such that for any $k = 0, 1, \ldots, p$, we have

$$||f^{(k)} - P_n^{(k)}(f)|| \leq \frac{C_{r,p}}{n^{p-k}} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}), \quad k = 0, 1, \ldots, p,$$

where $C_{r,p} > 0$ is a constant independent of f and n. This inequality is very similar to that in the statement of Theorem 2.1. Here $E_n(F; \overline{\mathbb{D}_r})$ denotes the best approximation of F on $\overline{\mathbb{D}_r}$ by polynomials of degree $\leq n$.

The first shape preserving result is the following.

Theorem 2.2. *Let us consider a function $f \in A^p(\mathbb{D})$, the fixed integers $h, k, p \in \mathbb{N}$, $0 \leq h \leq k \leq p$, functions $a_j : \overline{\mathbb{D}} \to \mathbb{C}$, continuous on $\overline{\mathbb{D}}$, for all $j = h, \ldots, k$, such that $a_h(z) = 1$, for all $z \in \overline{\mathbb{D}}$, and distinct interpolation points $|z_i| \leq 1, i = 1, \ldots, h$ (if h=0, then by convention we do not consider any interpolation point).*

Define a differential operator L by $L(f)(z) = \sum_{j=h}^{k} a_j(z) f^{(j)}(z)$, $z \in \mathbb{D}$ and suppose that $Re[L(f)(z)] \geq 0$, for all $z \in \mathbb{D}$.

Then, for every $n \in \mathbb{N}$, $n \geq p$, there exists a complex polynomial $P_n(z)$ of degree $\leq n$, such that

$$||f - P_n|| \leq C n^{k-p} E_{n-p}(f^{(p)}),$$

with C independent of n, f and, in addition, $Re[L(P_n)(z)] \geq 0$, for all $z \in \mathbb{D}$ and $P_n(z_i) = f(z_i), i = 1, \ldots, h$ (if $h = 0$ then we don't have interpolative conditions). Here $|| \cdot ||$ denotes the uniform norm on $C(\overline{\mathbb{D}})$.

Proof. We use the above Theorem 2.1, that is, for any $g \in A^p(\mathbb{D})$ and $n \geq p$, there exists a polynomial $p_n(z)$ of degree $\leq n$, such that

$$||g^{(j)} - p_n^{(j)}|| \leq Cn^{j-p}E_{n-p}(g^{(p)}),$$

$j = 0, 1, \ldots, p$, with $C > 0$ independent of n and g.

Define $q_n(z) = p_n(z) + Q(g - p_n)(z)$, where $Q(g - p_n)(z)$ represents the Lagrange interpolation polynomial attached to $g - p_n$ on the points $z_1, \ldots, z_h$. It is immediate that $q_n(z_i) = g(z_i), i = 1, \ldots, h$ and

$$||q_n - g|| \leq ||p_n - g|| + ||Q(g - p_n)|| \leq c_1 n^{-p} E_{n-p}(g^{(p)}),$$

with c_1 independent of n and g.

Since each $a_j(z)$ is continuous on $\overline{\mathbb{D}}$, denoting $A_j = ||a_j||$, $j = h, \ldots, k$, it easily follows $A_h = 1$ and there exists $M > 0$ with $A_j \leq M, j = h + 1, \ldots, k$. Since $c_1 E_{n-p}(f^{(p)}) \sum_{j=h}^{k} A_j n^{j-p} \leq c_1 \cdot max\{1, M\}(k - h + 1)n^{k-p}E_{n-p}(f^{(p)}) =: \eta_n$, taking $g(z) = f(z) + \eta_n[(z - z_1) \cdots (z - z_h)]/(h!)$, (if $h = 0$ then $g(z) = f(z) + \eta_n$), let $P_n(z)$ be the polynomial of degree $\leq n$ satisfying $P_n(z_i) = g(z_i), i = 1, \ldots, h$ and

$$||g^{(j)} - P_n^{(j)}|| \leq c_1 n^{j-p} E_{n-p}(g^{(p)}) = c_1 n^{j-p} E_{n-p}(f^{(p)}), \quad j = 0, 1, \ldots, p.$$

(Here c_1 is independent of n and g, so independent of f too). First it is obvious that $P_n(z_i) = g(z_i) = f(z_i), i = 1, \ldots, h$.

We get

$$||f - P_n|| \leq 2^h \eta_n (h!)^{-1} + c_1 n^{-p} E_{n-p}(f^{(p)}) \leq Cn^{k-p}E_{n-p}(f^{(p)}),$$

with C independent of n and f, which implies the estimate in theorem.

On the other hand, if $z \in \mathbb{D}$, with the convention on the case $h = 0$, it is easy to check

$$L(P_n)(z) = L(f)(z) + \eta_n + \sum_{j=h}^{k} a_j(z)\Big\{P_n(z) - f(z) - [(z - z_1) \cdots (z - z_h)]\eta_n/(h!)\Big\}^{(j)},$$

and we get

$$\begin{aligned} Re[L(P_n)(z)] &= Re[L(f)(z)] + \eta_n \\ &+Re\Big\{\sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) - [(z-z_1)\cdots(z-z_h)]\eta_n/(h!)\}^{(j)}\Big\}. \end{aligned}$$

By

$$\begin{aligned} &\Big|Re\Big\{\sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) - [(z-z_1)\cdots(z-z_h)]\eta_n/(h!)\}^{(j)}\Big\}\Big| \\ &\leq \Big|\sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) - [(z-z_1)\cdots(z-z_h)]\eta_n/(h!)\}^{(j)}\Big| \\ &\leq \max\{1, M\}c_1 \cdot (k-h+1)n^{k-p}E_{n-p}(f^{(p)}) = \eta_n, \end{aligned}$$

we get

$$\eta_n + Re\Big\{\sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) - [(z-z_1)\cdots(z-z_h)]\eta_n/(h!)\}^{(j)}\Big\} \geq 0,$$

and since $Re[L(f)(z)] \geq 0$, we finally obtain $Re[L(P_n)(z)] \geq 0$, for all $z \in \mathbb{D}$. □

Remarks. (1) The statement of Theorem 2.2 obviously remains valid if we replace the real part "Re" of the corresponding quantities, with the imaginary part "Im".

(2) If in the statement of Theorem 2.2 we have $Re[L(f)(z)] > 0$, for all $z \in \mathbb{D}$, then from the proof it easily follows $Re[L(P_n)(z)] > 0$, for all $z \in \mathbb{D}$, $n \geq p$.

Another consequence of Theorem 2.1 is the following

Corollary 2.3. *Let us consider a function $f \in A^p(\mathbb{D})$, the fixed integers $h, k, p \in \mathbb{N}$, $0 \leq h \leq k \leq p$, the functions $a_j : \overline{\mathbb{D}} \to \mathbb{C}$, continuous on $\overline{\mathbb{D}}$, for all $j = h, \ldots, k$, such that $a_h(z) = 1$, for all $z \in \overline{\mathbb{D}}$, and the fixed point $|z_0| \leq 1$.*

Define the complex differential operator $L(f)(z) = \sum_{j=h}^{k} a_j(z)f^{(j)}(z)$, $z \in \mathbb{D}$ and suppose that $Re[L(f)(z)] \geq 0$, for all $z \in \mathbb{D}$.

Then, for every $n \in \mathbb{N}$, $n \geq p$, there exists a complex polynomial $P_n(z)$ of degree $\leq n$, such that

$$||f - P_n|| \leq Cn^{k-p} E_{n-p}(f^{(p)}),$$

with C independent of n, f and, in addition, $Re[L(P_n)(z)] \geq 0$, for all $z \in \mathbb{D}$ and $P_n^{(i)}(z_0) = f^{(i)}(z_0), i = 0, \ldots, h$.

Proof. As in the proof of Theorem 2.2, for any $g \in A^p(\mathbb{D})$ and $n \geq p$, there exists a polynomial $p_n(z)$ of degree $\leq n$, such that

$$||g^{(j)} - p_n^{(j)}|| \leq cn^{j-p} E_{n-p}(g^{(p)}), \qquad j = 0, 1, \ldots, p.$$

Define now $q_n(z) = p_n(z) + T_h(z)$, where $T_h(z)$ denotes the Taylor polynomial of degree h attached to the point z_0 and to $g - p_n$, i.e. $T_h(z) = \sum_{j=0}^{h} \frac{(z-z_0)^j}{j!} [g - p_n]^{(j)}(z_0)$.

We easily get $q_n^{(j)}(z_0) = g^{(j)}(z_0), j = 0, \ldots, h$ and $||q_n - g|| \leq c_1 \sum_{j=0}^{h} ||g^{(j)} - p_n^{(j)}|| \leq c_1 n^{k-p} E_n(g^{(p)})$, with $c_1 > 0$ independent of n and f, since $h \leq k$.

Defining $\eta_n = c_1 max\{1, M\}(k-h+1)n^{k-p} E_{n-p}(f^{(p)})$ (where M is given by the proof of Theorem 2.2) and taking $g(z) = f(z) + \eta_n (z - z_0)^h/(h!)$, let $P_n(z)$ be the polynomial of degree $\leq n$ satisfying $P_n^{(i)}(z_0) = g^{(i)}(z_0), i = 0, \ldots, h$ and

$$\begin{aligned} ||g^{(j)} - P_n^{(j)}|| &\leq c_1 n^{j-p} E_{n-p}(g^{(p)}) \\ &= c_1 n^{j-p} E_{n-p}(f^{(p)}) \leq Cn^{k-p} E_{n-p}(f^{(p)}), \quad j = 0, 1, \ldots, k. \end{aligned}$$

The rest of the proof follows the lines in the proof of Theorem 2.2. □

In what follows we present some applications of Corollary 2.3.

Theorem 2.4.

(i) *Let $p \in \mathbb{N}$, $f \in A^p(\mathbb{D})$ be normalized in $\mathbb{D}$, i.e. $f(0) = f'(0) - 1 = 0$, and satisfying $Re[f'(z)] > 0$, for all $z \in \mathbb{D}$.*

For any $n \geq p$*, there exists a polynomial of degree* $\leq n$ *such that* $P_n(0) = f(0), P'_n(0) = f'(0)$*,* $Re[P'_n(z)] > 0$*, for all* $z \in \mathbb{D}$ *and*

$$||f - P_n|| \leq C \frac{1}{n^{p-1}} E_{n-p}(f^{(p)}),$$

with C *independent of* n*,* f*;*

(ii) *Let* $f \in A^p(\mathbb{D})$*,* $p \in \mathbb{N}, p \geq 2$*, be normalized in* $\mathbb{D}$*. For any* $n \in \mathbb{N}$*,* $n \geq p$*, there exists a polynomial of degree* $\leq n$ *such that* $P_n(0) = f(0), P'_n(0) = f'(0)$*,*

$$||f - P_n|| \leq C \frac{1}{n^{p-2}} E_{n-p}(f^{(p)}),$$

with C *independent of* n*,* f*, that in addition, has the following properties (the choice of* $P_n(z)$ *depends on the property):*

(a) *If* $Re[f'(z) + \frac{1}{\gamma} z f''(z)] > 0$*, for all* $z \in \mathbb{D}$*, where* $-1 < \gamma \leq \gamma_0 = 1.869...$*, then* $Re[P'_n(z) + \frac{1}{\gamma} z P''_n(z)] > 0$*,* $\forall z \in \mathbb{D}$*.*
(b) *If* $Re[f'(z) + \frac{1}{2} z f''(z)] > 0$*, for all* $z \in \mathbb{D}$*, then* $Re[P'_n(z) + \frac{1}{2} z P''_n(z)] > 0$*, for all* $z \in \mathbb{D}$*;*

(iii) *If* $g \in A^p(\mathbb{D})$*,* $p \in \mathbb{N}, p \geq 2$*, satisfies* $g(0) = a$*, with* $Re[a] > 0$ *and* $Re[g(z) + z g'(z) + z^2 g''(z)] > 0$*, for all* $z \in \mathbb{D}$*, then for all* $n \in \mathbb{N}$*,* $n \geq p$*, there exists a polynomial* P_n *of degree* $\leq n$*, such that* $P_n(0) = g(0)$*,* $Re[P_n(z) + z P'_n(z) + z^2 P''_n(z)] > 0$*, for all* $z \in \mathbb{D}$ *and*

$$||g - P_n|| \leq C \frac{1}{n^{p-2}} E_{n-p}(g^{(p)}),$$

with C *independent of* n*,* f*.*

(iv) *If* $g \in A^p(\mathbb{D})$*,* $p \in \mathbb{N}$*, satisfies* $g(0) = a$*, with* $Re[a] > 0$ *and* $Re[g(z) + z B(z) g'(z)] > 0$*, where* $B(z)$ *is analytic in* $\mathbb{D}$ *and* $Re[B(z)] > 0$*, for all* $z \in \mathbb{D}$*, then for all* $n \in \mathbb{N}$*,* $n \geq p$*, there exists a polynomial* P_n *of degree* $\leq n$*, such that* $P_n(0) = g(0)$*,* $Re[P_n(z) + z B(z) P'_n(z)] > 0$*, for all* $z \in \mathbb{D}$ *and*

$$||g - P_n|| \leq C \frac{1}{n^{p-1}} E_{n-p}(g^{(p)}),$$

with C *independent of* n*,* f*.*

Proof.

(i) Let $L(f)(z) = f'(z)$. Apply Remark 2 of Theorem 2.2 and Corollary 2.3 for $z_0 = 0$, $h = k = 1$.

(ii) (a), (b) Let $L(f)(z) = f'(z) + \frac{1}{\gamma} z f''(z)$ (or $L(f)(z) = f'(z) + \frac{z}{2} f''(z)$, respectively). Apply Remark 2 of Theorem 2.2 and Corollary 2.3 for $z_0 = 0$, $h = 1$, $k = 2$.

(iii) Apply Remark 2 of Theorem 2.2 and Corollary 2.3 for $z_0 = 0$, $h = 0$ and $k = 2$.

(iv) Apply Remark 2 of Theorem 2.2 and Corollary 2.3 for $z_0 = 0$, $h = 0$ and $k = 1$. □

Remarks. (1) It is well-known (see e.g., Mocanu-Bulboaca-Salagean [14, p. 78]) that if f is normalized and satisfies the condition $Re[f'(z)] > 0$, for all $z \in \mathbb{D}$, then f is univalent and of bounded turn in $\mathbb{D}$ (i.e. $|arg[f'(z)]| < \frac{\pi}{2}$, for all $z \in \mathbb{D}$). As a consequence, the approximation polynomials $P_n, n \geq p$, are univalent and of bounded turn on $\mathbb{D}$.

(2) By Singh-Singh [23], Mocanu [13] and Mocanu-Bulboaca-Salagean [14, p. 358], respectively, the fact that f is normalized together with any from the two conditions (a) and (b) in Theorem 2.4, (ii), implies the starlikeness of f (and as a consequence the starlikeness of P_n too) in $\mathbb{D}$ (for the above sufficient conditions of starlikeness (a) and (b), see also Mocanu-Bulboaca-Salagean [14, p. 363].

(3) The conditions on g in Theorem 2.4, (iii), imply $Re[g(z)] > 0$ for all $z \in \mathbb{D}$ (see e.g. Mocanu-Bulboaca-Salagean [14, Problem 9.6.5, (ii), p. 221]).

(4) The conditions in Theorem 2.4, (iv), imply $Re[g(z)] > 0$, for all $z \in \mathbb{D}$ (see e.g., Mocanu-Bulboaca-Salagean [14, p. 192]).

Other results of the same kind are given by the following.

Theorem 2.5.

(i) *If $f \in A(\mathbb{D})$, satisfies $Re[f(z)] > 0$, for all $z \in \mathbb{D}$, then for each $n \in \mathbb{N}$, there exists P_n-complex polynomial of degree $\leq n$, such that $||P_n - f|| \leq 2E_n(f)$, and , in addition $Re[P_n(z)] > 0$, for all $z \in \mathbb{D}$. Here $E_n(f)$ denotes the best approximation of f by polynomials of degree $\leq n$.*

(ii) *Let* $f \in A^1(\mathbb{D})$ *be such that* $f'(0) = 1$ *and there exists* $\gamma \in (-\pi/2, \pi/2)$, *with* $Re[e^{i\gamma} f'(z)] > 0$, *for all* $z \in \mathbb{D}$. *Then, for any* $n \geq 1$, *there exists a polynomial* $Q_n(f)(z)$, *of degree* $\leq n$, *such that* $Re[e^{i\gamma} Q_n'(z)] > 0$, *for all* $z \in \mathbb{D}$, *and* $||Q_n(f) - f|| \leq cE_{n-1}(f')$, *where* c *is independent of* n *and* f.

Proof. (i) Let P_n^* be the polynomial of degree $\leq n$ that satisfies $||f - P_n^*|| = E_n(f) > 0$. Then it is easy to check that $P_n(z) = P_n^*(z) + E_n(f)$ satisfies the required conditions, since $|Re[P_n^*(z) - f(z)]| \leq ||f - P_n^*|| = E_n(f)$ and $Re[P_n(z) - f(z)] = Re[P_n^*(z) - f(z)] + E_n(f) > 0$.

If $E_n(f) = 0$ and $Re[f] > 0$, then $f = P_n$-polynomial of degree $\leq n$ and the result is obvious.

(ii) First suppose $E_{n-1}(f') > 0$. By Theorem 2.1, there exists $P_n(z)$ with the properties $||P_n - f|| \leq c\frac{1}{n} E_{n-1}(f') =: \alpha_n$, and $||P_n' - f'|| \leq cE_{n-1}(f') := \beta_n$.

Denote $Q_n(z) = P_n(z) + z\frac{2\beta_n}{\cos(\gamma)}$. We get $||Q_n - f|| \leq ||P_n - f|| + 2\beta_n \leq \alpha_n + 2\beta_n$, which proves the approximation error.

Also,

$$\begin{aligned} Re[e^{i\gamma}(Q_n'(z) - f'(z))] &= Re[e^{i\gamma}(P_n'(z) - f'(z))] + 2\beta_n Re\left[\frac{e^{i\gamma}}{\cos(\gamma)}\right] \\ &= Re[e^{i\gamma}(P_n'(z) - f'(z))] + 2\beta_n > 0, \end{aligned}$$

since $|Re[e^{i\gamma}(P_n'(z) - f'(z))]| \leq |e^{i\gamma}[P_n'(z) - f'(z)]| \leq ||P_n' - f'|| \leq \beta_n < 2\beta_n$. The case $E_{n-1}(f') = 0$ implies $f = P_n$-polynomial of degree $\leq n$ and we choose $Q_n(f) = P_n$, which proves the theorem. □

Remark. It is well-known that $Re[e^{i\gamma} f'(z)] > 0$, for all $z \in \mathbb{D}$, is the Noshiro, Warschawski, Wolff's sufficient condition of univalence for f (see also e.g. Mocanu-Bulboaca-Salagean [14, p. 78]).

All the above results show that there exists polynomials with good approximation properties, that preserve some known subclasses of univalent functions. Also, we note that the approximation polynomials constructed in Gal [7, 8, 9], in general do not preserve these subclasses and even if they preserve some subclasses (the cases of subclasses in Theorem 2.4, (i) and Theorem 2.5, (i)), the error estimates presented here are much better than those in Gal [7, 8, 9], (which are expressed in terms of moduli of smoothness of order one or two).

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