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BEST PROXIMITY PAIRS IN UNIFORMLY CONVEX SPACES

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Abstract

In this paper we prove existence theorems of best proximity pairs in uniformly convex spaces, using a fixed point theorem for Kakutani factorizable multi-functions.

1. Introduction

Let X and Y be two non-empty subsets of a normed linear space E and let $T : X \to 2^Y$ be a multi-function. The pair $(x, T(x))$ is called a *best proximity pair* if $d(x, T(x)) = d(X, Y)$. The existence of best proximity pairs is closely related to the existence of equilibrium pairs for free abstract economies and the existence of fixed points for multi-functions. Several authors have studied these problems ([5], [6], [2] and [3]).

In [3] the authors established the following existence theorem of best proximity pairs.

Theorem 1.1. *For each $i \in I = \{1, \dots, m\}$, let X and Y_i be non-empty compact and convex subsets of a normed linear space E, and let $T_i : X \to 2^{Y_i}$ be a upper semi-continuous multi-function on X^0 such that $T_i(x) \subset Y_i$ is a non-empty closed and convex subset for each $x \in X$. Assume that $X^0 \neq \emptyset$ and $T_i(x) \cap Y_i^0 \neq \emptyset$, for each $x \in X^0$. Then there exists a system of best proximity pairs $\{x_i\} \times T_i(x_i) \subset X \times Y_i$, i.e., for each $i \in I, d(x_i, T_i(x_i)) = d(X, Y_i)$.*

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Here X^0 and Y_i^0 are certain sets which we shall describe later.

The main purpose of this paper is to prove similar results to Theorem 1.1 for uniformly convex Banach spaces or finite dimensional spaces, only assuming compactness for one of the involved sets. The interest in these type of results are based on the very limited condition of being a compact set in infinite dimensional spaces. In the case in which E is a finite dimensional normed space, no additional condition over the space E is required. As in [2], one theorem of existence of equilibrium pairs for a free 1-person game can be obtained in uniformly convex spaces or finite dimensional normed spaces. We observe that it is usual in the literature, even for finite dimensional spaces, to require compactness of all the considered sets.

We shall use in this paper, in part, the technique employed by Kim and Lee in [2].

We shall also give an example which shows that the authors in [3] have forgotten an essential condition in the statement of Theorem 1, when $m > 1$.

2. Preliminaries

In what follows we shall enumerate some classical notions and results regarding the multi-valued mappings. Although many of these are available in a more general framework, we shall mention them only in the form we need in the present paper.

Let $(E, \|\cdot\|)$ be a Banach space. Let $X, Y \subset E$. We call *multi-valued mapping* (or *multi-function*) defined on X to every application $F : X \to 2^Y$. We call F *upper semi-continuous* at $x \in X$ (in brief u.s.c.) if for all open U subset of Y, with $F(x) \subset U$, there exists an open ball $B(x, s) \subset X$, of center at x and radius $s > 0$, such that $F(B(x, s)) \subset U$. We call F *u.s.c. on* X if it is u.s.c. in each point of X. The multi-function F is said to be a *Kakutani multi-function* if the following conditions are satisfied: i) F is u.s.c.; ii) either $F(x)$ is a singleton for each $x \in X$ or Y is a convex set and for each $x \in X$, $F(x)$ is a non-empty compact and convex set. F is said to be a *Kakutani factorizable multi-function* if it can be expressed as a composition of finitely many Kakutani multi-functions.

Given $X \subset E$ and $Y_i \subset E, i \in I = \{1, \dots, m\}, m \in \mathbb{N}$, we denote

$$X^0 = \{x \in X : \text{ for each } i \in I, \exists y_i \in Y_i \text{ such that } d(x, y_i) = d(X, Y_i)\}$$

and

$$Y_i^0 = \{y \in Y_i : \exists x \in X \text{ such that } d(x, y) = d(X, Y_i)\}.$$

We also consider $P_X : E \to 2^X$ the *metric projection* defined by $P_X(z) = \{x \in X : d(z, x) = d(z, X)\}$.

Now, we introduce some classical concepts of the Functional Analysis which will be necessary for our purposes. (See [1],[4])

Let E be a Banach space. If $(x_n) \subset E$ is a sequence which weakly converges to $x \in E$ then $\|x\| \leq \underline{\lim}\|x_n\|$. If $X \subset E$ is a convex set, then it is strongly closed iff it is weakly closed. If E is reflexive then every bounded subset of E is weakly sequentially compact. E is said to be *uniformly convex* (or *uniformly rotund*) if for $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|x\| \leq 1, \|y\| \leq 1$, and $\|x - y\| \geq \epsilon$ imply $\|x + y\| \leq 2(1 - \delta)$. It is well known that every uniformly convex Banach space is reflexive and satisfies the property **H**, i.e, if

(a) $(x_n) \subset E$ is a sequence which weakly converges to $x \in E$, and
(b) $\|x_n\| \to \|x\|$,

then $\|x_n - x\| \to 0$, as $n \to \infty$.

The following form of Lassonde's fixed point theorem is due to Srinivasan and Veeramani.

Lemma 2.1. *Let X be a non-empty compact and convex subset of a locally convex Hausdorff topological vector space, then any Kakutani factorizable multi-function $F : X \to 2^X$ has a fixed point, i.e., there exists $x \in X$ such that $x \in F(x)$.*

3. Main Results

We begin with an auxiliary Lemma.

Lemma 3.1. *Let E be a reflexive Banach space satisfying the property* **H**. *Let $X \subset E$ be a non-empty closed and convex set, and let P_X be the* *metric projection. Then* $P_X(z)$ *is a non-empty compact convex set for all* $z \in E$*, and* P_X *is a u.s.c. multi-function on* E.

Proof. Let $z \in E$. Since X is reflexive, it is well known that $P_X(z)$ is a non-empty convex set. Let $(x_n) \subset P_X(z)$ be a sequence. Then (x_n) is strongly bounded, therefore there is a subsequence x_{n_k} weakly converging to a point $x \in X$. Thus $(x_{n_k} - z)$ weakly converges to $x - z$, it follows immediately that $x \in P_X(z)$. Now the property **H** implies (x_{n_k}) strongly converges to x. In consequence $P_X(z)$ is a compact set.

We show that it is u.s.c.. Let $z_0 \in E$ and let U be an open set which contains to $P_X(z_0)$. Since $P_X(z_0)$ is a compact set, there is $r > 0$ such that $P_X(z_0) + r := \{y \in E : d(y, P_X(z_0)) < r\} \subset U$. Then, it will be sufficient to prove that there is $s > 0$ such that $P_X(B(z_0, s)) \subset P_X(z_0) + r$. Suppose that it is not true, so there are two sequences $(z_n) \subset E$ and $(y_n) \subset P_X(z_n)$ such that $\|z_n - z_0\| < \frac{1}{n}$ and $d(y_n, P_X(z_0)) \geq r$. The sequence (y_n) is strongly bounded, in consequence we can find a subsequence (y_{n_k}) weakly converging to a point $y \in X$. Now, $(z_{n_k} - y_{n_k})$ weakly converges to $z_0 - y$ and $\|z_0 - y\| \leq \underline{\lim}\|z_{n_k} - y_{n_k}\| = d(z_0, X)$. It follows that $y \in P_X(z_0)$ and $\|z_{n_k} - y_{n_k}\| \to \|z_0 - y\|$. The property **H** implies $\|y_{n_k} - y\| \to 0$, so $d(y_{n_k}, P_X(z_0)) \to 0$, a contradiction. This completes the proof. □

Next, we establish a theorem of existence of best proximity pairs.

Theorem 3.2. *Let* $I = \{1, \ldots, m\}$ *and let* E *be a reflexive Banach space satisfying the property* **H**. *Let* X *and* $Y_i, i \in I$*, be non-empty closed convex subsets of* E. *Let* $T_i : X \to 2^{Y_i}, i \in I$*, be u.s.c. multi-functions on* X^0 *such that* $T_i(x)$ *is a non-empty closed and convex subset of* Y_i *for each* $x \in X$*, and* $T_i(x) \cap Y_i^0 \neq \emptyset$*, for each* $x \in X^0, i \in I$. *If*

(a) $|I| > 1$, *X is a compact set, and* $P_X(Y_i^0) \subset X^0, i \in I$, *or*
(b) $|I| = 1$, *and either of the sets X and Y_1 is a compact set,*

then there is a system of best proximity pairs $\{(x_i, T_i(x_i)) : i \in I\}$*, i.e., for each* $i \in I, d(x_i, T_i(x_i)) = d(X, Y_i)$.

Proof. (a) First, we show that $Y_i^0 \neq \emptyset$. In fact, since X is a compact set there is $x \in X$ such that $d(x, Y_i) = d(X, Y_i)$. We can choose a sequence $(y_n) \subset Y$ which satisfies $\lim d(x, y_n) = d(x, Y_i)$. So, (y_n) is strongly bounded, therefore there is a subsequence y_{n_k} weakly converging to a point $y \in Y_i$ and $\|y - x\| \leq \lim \|y_{n_k} - x\| = d(X, Y_i)$. Thus $y \in Y_i^0$.

The convexity of the sets X^0 and Y_i^0 follows immediately from the convexity of the sets X and Y_i. We shall prove that X^0 and Y_i^0 are compact sets.

Let (x_n) be a sequence in X^0, and let $y_i^n \in Y_i, i \in I$, be such that $d(x_n, y_i^n) = d(X, Y_i)$. The compactness of X implies that there exists a subsequence (x_{n_k}) and $x \in X$ such that $\|x_{n_k} - x\| \to 0$. As x_{n_k} is strongly bounded, then for each $i \in I$ the sequence $(y_i^{n_k})$ is strongly bounded. Since E is reflexive, there is a subsequence, which for simplicity we denote again by $(y_i^{n_k})$, and $y_i \in Y_i$, verifying $(y_i^{n_k})$ weakly converges to y_i for all $i \in I$. In consequence, the sequence $(x_{n_k} - y_i^{n_k})$ weakly converges to $x - y_i$, for all $i \in I$. In addition, we have $\|x - y_i\| \leq \underline{\lim}\|x_{n_k} - y_i^{n_k}\| = d(X, Y_i)$, so $x \in X^0$. Therefore X^0 is a compact set.

If $i \in I$, in order to prove that Y_i^0 is compact, consider a sequence $(y_i^n) \subset Y_i^0$ and let $x_n \in X$ be such that $d(x_n, y_i^n) = d(X, Y_i)$. As X is a compact set, (x_n) has a strongly convergent subsequence, say $(x_{n_k}) \to x \in X$. So, (y_{n_k}) is strongly bounded. Therefore, it has a subsequence which we denote again for simplicity by (y_{n_k}) such that (y_{n_k}) weakly converges to $y \in Y_i$. Since $\|x - y\| \leq \underline{\lim}\|x_{n_k} - y_{n_k}\| = d(X, Y_i)$, we get $\|x - y\| = \underline{\lim}\|x_{n_k} - y_{n_k}\|$. It follows that there is a subsequence $x_{n'_k} - y_{n'_k}$ such that $\|x_{n'_k} - y_{n'_k}\| \to \|x - y\|$. Now the property **H** implies $x_{n'_k} - y_{n'_k}$ strongly converges to $x - y$, so $y_{n'_k}$ strongly converges to $y \in Y_i^0$.

(b) If either X or Y_1 is a compact set, as in the part a), we can see that $X^0 \neq \emptyset \neq Y_1^0$, and both are convex and compact sets. In this case it is easy to show that $P_X(Y_1^0) \subset X^0$.

Now, using the Lemma 3.1, the remainder of the proof follows the same patterns that the proof of [2], Theorem 1. In fact, we define the functions

$$P'_X(y_1, \ldots, y_m) := \prod_{i \in I} P_X(y_i), \quad (y_1, \ldots, y_m) \in \prod_{i \in I} Y_i^0,$$

and

$$T'(x_1, \ldots, x_m) := \prod_{i \in I} T'_i(x_i), \quad (x_1, \ldots, x_m) \in \prod_{i \in I} X^0,$$

where $T_i'(x) := T_i(x) \cap Y_i^0, x \in X^0$. Since in both cases, (a) and (b), $P_X(Y_i^0) \subset X^0, i \in I$, holds, we can apply the Lemma 2.1 to the multifunction $P_X' \circ T'$. □

Corollary 3.3. *Let $I = \{1, \ldots, m\}$ and let E be a uniformly convex Banach space or a finite dimensional normed space. Let X and $Y_i, i \in I$, be non-empty closed convex subsets of E. Let $F_i : X \to 2^{Y_i}$ be a u.s.c. multifunction on X^0 such that $F_i(x)$ is a non-empty closed and convex subset of Y_i for each $x \in X$, and $F_i(x) \cap Y_i^0 \neq \emptyset$ for each $x \in X^0, i \in I$. If*

(a) *$|I| > 1$, X is a compact set, and $P_X(Y_i^0) \subset X^0, i \in I$, or*
(b) *$|I| = 1$, and either of the sets X and Y_1 is a compact set,*

then there is a system of best proximity pairs $\{(x_i, F_i(x_i)) : i \in I\}$, i.e., for each $i \in I, d(x_i, F_i(x_i)) = d(X, Y_i)$.

Proof. For E a uniformly convex Banach space, the Corollary follows immediately from Theorem 3.2 and the fact that every uniformly convex Banach space is reflexive and satisfies the property **H**. If dim $E < \infty$, we observe that the conditions of being reflexive and verifying the property **H** are automatically satisfied for E. □

Remark 3.4. First we observe that we do not require compactness of all sets X and $Y_i, i \in I$, as in [3], Theorem 1, but we assume certain properties about the normed space E.

The assumption of either X or Y_1 being a compact set in Corollary 3.3, (b) can not be relaxed as the following example shows.

Example. Let $E = \mathbb{R}^2$ be with the Euclidean norm and let $X = \{(x, 0) : x \in \mathbb{R}\}$, and $Y = \{(x, 1) : x \in \mathbb{R}\}$ be. Let $F : X \to Y$ defined by $F(x, 0) = (x + 1, 1)$. Here, $X^0 = X$ and $Y^0 = Y$. Clearly a pair $(\overline{x}, F(\overline{x})) \in X \times Y$ such that $\|\overline{x} - F(\overline{x})\| = d(X, Y) = 1$ does not exist.

The next example shows that the condition of either of the sets being compact is not necessary for the existence of best proximity pairs.

Example. Let $E = \mathbb{R}^2$ be with the Euclidean norm and let $Y = \{(x, y) : 0 \leq x \leq 2, 1 \leq y\}$ and $X = \{(x, y) : 0 \leq x \leq 2, y \leq 0\}$. If $F = P_X$, clearly there are best proximity pairs.

Next we give other Theorem over existence of best proximity pairs closely related to Theorem 2, of Kim and Lee [3].

Theorem 3.5. *Let $I = \{1, \ldots, m\}$ and let E be a reflexive Banach space satisfying the property* **H**. *Let X and $Y_i, i \in I$, be non-empty closed convex subsets of E. Assume*

$$\bigcap_{i \in I} P_X(y_i) \neq \emptyset \quad \textit{for all} \quad (y_1, \ldots, y_m) \in \prod_{i \in I} Y_i^0. \tag{$*$}$$

Let $F_i : X \to 2^{Y_i}, i \in I$, be multi-functions on X^0 such that $F_i(x)$ is a non-empty closed and convex subset of Y_i for each $x \in X$, and $F_i(x) \cap Y_i^0 \neq \emptyset$, for each $x \in X^0, i \in I$. If $F_i, i \in I$, are u.s.c. on X^0 and X is a compact set, then there is $\overline{x} \in X$ satisfying the system of best proximity pairs, i.e., for each $i \in I, \{\overline{x}\} \times F_i(\overline{x}) \subset X \times Y_i$ such that $d(\overline{x}, F_i(\overline{x})) = d(X, Y_i)$.

Proof. As it was proved in [2], if $(y_1, \ldots, y_m) \in \prod_{i \in I} Y_i^0$ then $\bigcap_{i \in I} P_X(y_i) \subset X^0$, so from $(*)$ we get $X^0 \neq \emptyset$. The remainder of the proof follows analogously to the proof of Theorem 3.2 and Theorem 2,[2], by defining the multi-functions $P'_X(y_1, \ldots, y_m) := \bigcap_{i \in I} P_X(y_i), (y_1, \ldots, y_m) \in \prod_{i \in I} Y_i^0$, and $T'(x) := \prod_{i \in I} T'_i(x), x \in X^0$, where $T'_i(x) := F_i(x) \cap Y_i^0, x \in X^0$. □

Remark 3.6. We observe that the hypothesis $X^0 \neq \emptyset$ in [3], Theorem 2, can be deduced of the hypothesis $\bigcap_{i \in I} P_X(y_i) \neq \emptyset$ for each $(y_1, \ldots, y_m) \in \prod_{i \in I} Y_i^0$.

Now, we give an example which shows that the Theorem 1 in [3] is not true for $m > 1$.

Example 3.7. Let $E = \mathbb{R}^2$ be with the Euclidean norm. Let $X = \{(x, 0) : 0 \leq x \leq 1\}, Y_1 = \{(x, y) : 0 \leq x \leq 1, 1 \leq y \leq 2\}$ and $Y_2 = \{(1, -1)\}$. Then $d(X, Y_1) = d(X, Y_2) = 1, Y_1^0 = \{(x, 1) : 0 \leq x \leq 1\}, Y_2^0 = Y_2$ and $X^0 = \{(1, 0)\}$. We consider the continuous functions $F_i : X \to Y_i, i = 1, 2$, defined by $F_1((x, 0)) = (0, 2 - x)$ and $F_2((x, 0)) = (1, -1)$. Clearly, $F_i((x, 0)) \cap Y_i^0 \neq \emptyset$, for $(x, 0) \in X^0, i = 1, 2$. Although $X^0 \neq \emptyset$, $P_X(Y_1^0) \nsubseteq X^0$. Further, we observe that there is not a pair $((x, 0), F_1((x, 0)))$ such that $d((x, 0), F_1((x, 0))) = 1$.

Remark 3.8. We observe that the Theorem 1 in [3], with the additional condition $P_X(Y_i^0) \subset X^0, i \in I$, is true and it can be worked analogously as in [2]. The previous example shows that if $|I| > 1$, the hypothesis $P_X(Y_i^0) \subset X^0$, for all $i \in I$, is stronger than the condition $X^0 \neq \emptyset$. Justly, we note that there is a gap in the proof ([2], p. 438), when it is proved that $P_X(Y_i^0) \subset X^0$ always occurs, whenever $X^0 \neq \emptyset$.

As we have mentioned in the Introduction, we can obtain one existence theorem of equilibrium pairs for free 1-person games in the setting of uniformly convex Banach spaces. In fact, given $\Gamma := (X, Y, A, P)$ a free 1-person game as in [2] with the same hypothesis as [2], Theorem 3, except that we assume only compactness for the set Y, and using the Theorem 3.2 (b), we get the existence of an equilibrium pair, with the same proof that in [2], Theorem 3.

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