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ON KAN EXTENSION OF HOMOLOGY AND ADAMS COCOMPLETION

BY

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Abstract

Under a set of conditions, it is shown that the Kan extension of an additive homology theory over smaller admissible subcategory is again a homology theory. Furthermore, using a Serre class of abelian groups, it is shown that a homology theory over the category of simply connected based topological spaces and continuous maps arising through Kan extension process from an additive homology theory over a smaller subcategory always admits global Adams cocompletion.

1. Introduction

Let $\mathscr{T}$ be the category of based topological spaces and base point preserving maps. A subcategory $\mathscr{J}$ of $\mathscr{T}$ is said to be *admissible* if it is nonempty, full, closed under the formation of mapping cones and contains (based) homotopy types [12]. It is evident that an admissible subcategory of $\mathscr{T}$ contains singletons and is closed under suspensions. We denote by $\tilde{\mathscr{J}}$ the homotopy category of $\mathscr{J}$.

A *homology theory* h on an admissible category $\mathscr{J}$ is a sequence of functors $h_n : \mathscr{J} \to \mathcal{A}$ where $\mathcal{A}$ is the category of abelian groups, together with natural transformations $\sigma_n : h_n \to h_{n+1}\Sigma$ (Σ denoting the suspension) satisfying the homotopy, suspension and exactness axioms:

(i) If $f_0 \simeq f_1$ then $h_n(f_0) = h_n(f_1)$.

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(ii) $\sigma_n : h_n \cong h_{n+1}\Sigma$.

(iii) If $f : X \to Y$ is in $\mathscr{J}$ and C_f is the mapping cone of f and $P_f : Y \to C_f$ is the canonical embedding then

$$h_n(X) \xrightarrow{h_n(f)} h_n(Y) \xrightarrow{h_n(P_f)} h_n(C_f)$$

is exact.

Let $\mathscr{J}_0$ and $\mathscr{J}_1$ be two admissible subcategories of $\mathscr{T}$ with $\mathscr{J}_0 \subset \mathscr{J}_1$ and let h be a homology theory defined on $\mathscr{J}_0$. The left Kan extension of h over $\mathscr{J}_1$, say h', is a functor on $\mathscr{J}_1$ having values in the category of abelian groups. In this note we give a set of conditions under which the functor h' is a homology functor. It may be recalled that Deleanu and Hilton have considered this question for cohomology theories [1, 2, 9] and Piccinini [12] has considered the same question for homology theories on stable categories. It will be evident from the conditions imposed and the method of proof of the main result that the description given here is largely dualization of the results obtained in [2].

2. The Extension Procedure

We assume that the categories $\mathscr{J}_0$ and $\mathscr{J}_1$ have the following three properties:

(i) $\tilde{\mathscr{J}}_0$ has weak local push-outs relative to $\tilde{\mathscr{J}}_1$; that is, given a commutative diagram in $\tilde{\mathscr{J}}_1$

$$\begin{array}{ccc} Y & \xrightarrow{u_0} & Y_0 \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ Y_1 & \xrightarrow[f_1]{} & X \end{array}$$

with Y_0, Y_1 and Y in $\mathscr{J}_0$, there exists a diagram

$$\begin{array}{ccccc} Y & \xrightarrow{u_0} & Y_0 & & \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle v_0} & \searrow{\scriptstyle f_0} & \\ Y_1 & \xrightarrow[v_1]{} & Z & & \\ & {\scriptstyle f_1}\searrow & & \searrow{\scriptstyle g} & \\ & & & & X \end{array}$$

in $\tilde{\mathscr{J}_1}$ with Z in $\mathscr{J}_0$.

(ii) The suspension map $\Sigma : \tilde{\mathscr{J}_1} \to \tilde{\mathscr{J}_1}$ is locally left $\tilde{\mathscr{J}_0}$-adjunctable, which means that condition (a) and (b) below are satisfied.

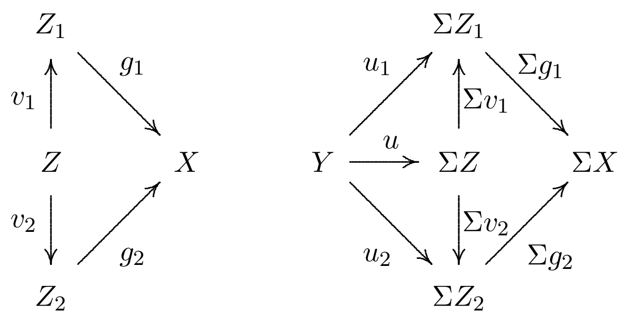
(a) Given $f : Y \to \Sigma X$ in $\tilde{\mathscr{J}_1}$ with Y in $\mathscr{J}_0$, there exists an object Z in $\mathscr{J}_0$ and $g : Z \to X$ in $\tilde{\mathscr{J}_1}$ and $u : Y \to \Sigma Z$ in $\tilde{\mathscr{J}_0}$ such that $f = (\Sigma g)u$.

$$\begin{array}{ccccc} Y & \xrightarrow{f} & \Sigma X & \qquad & X \\ {\scriptstyle u}\downarrow & \nearrow{\scriptstyle \Sigma g} & & & \uparrow{\scriptstyle g} \\ \Sigma Z & & & & Z \end{array}$$

(b) Given diagrams

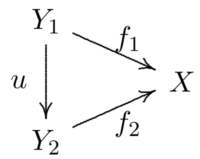
$$\begin{array}{ccccc} & & \Sigma Z_1 & & & \qquad & Z_1 \\ & {\scriptstyle u_1}\nearrow & & \searrow{\scriptstyle \Sigma g_1} & & & \downarrow{\scriptstyle g_1} \\ Y & & & & \Sigma X & & X \\ & {\scriptstyle u_2}\searrow & & \nearrow{\scriptstyle \Sigma g_2} & & & \uparrow{\scriptstyle g_2} \\ & & \Sigma Z_2 & & & & Z_2 \end{array}$$

with Z_1, Z_2 and Y in $\mathscr{J}_0$ and $(\Sigma g_1)u_1 = (\Sigma g_2)u_2$ in $\tilde{\mathscr{J}_1}$, there exist $v_1 : Z \to Z_1$, $v_2 : Z \to Z_2$ and $u : Y \to \Sigma Z$ in $\tilde{\mathscr{J}_0}$ with $g_1 v_1 = g_2 v_2$, $(\Sigma v_1)u = u_1$ and $(\Sigma v_2)u = u_2$. Thus we have the commutative diagrams



(iii) $\mathscr{J}_0$ is closed under finite sums i.e., if Y_1 and Y_2 are in $\mathscr{J}_0$, then so is $Y_1 \vee Y_2$.

We now describe the extension procedure. Let X be an object of $\mathscr{J}_1$. Form the category $\tilde{\mathscr{J}}_{01}(X)$ of all $\tilde{\mathscr{J}}_0$-objects over X: An object in this category is a morphism $f : Y \to X$ in $\tilde{\mathscr{J}}_1$ with Y in $\mathscr{J}_0$; a morphism $u : f_1 \to f_2$ in this category is a morphism $u : Y_1 \to Y_2$ in $\tilde{\mathscr{J}}_0$ such that the diagram



is commutative in $\tilde{\mathscr{J}}_1$. It is easy to check that $\tilde{\mathscr{J}}_{01}(X)$ is a category. The *left Kan extension* h'_n of the homology functor h_n is defined as follows:

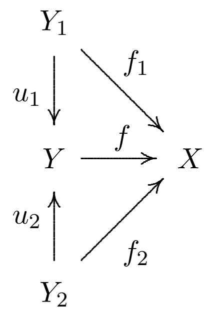
$$h'_n(X) = \underset{f}{\overset{\lim}{\longrightarrow}}(h_n(Y), h_n(u)).$$

It is obvious that h'_n is an extension of h_n and defines a covariant functor on $\tilde{\mathscr{J}}_1$ with values in the category of abelian groups.

An alternative description of the groups $h'_n(X)$ can now be given. Let

$$P_n(X) = \{(\alpha, f) f : Y \to X \text{ in } \tilde{\mathscr{J}}_1 \text{ with } Y \text{ in } \mathscr{J}_0, \quad \alpha \in h_n(Y)\}.$$

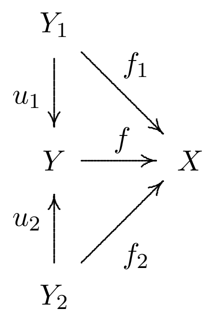
In $P_n(X)$ define a relation $\sim$ by the rule: $(\alpha_1, f_1) \sim (\alpha_2, f_2)$ if and only if there is a commutative diagram

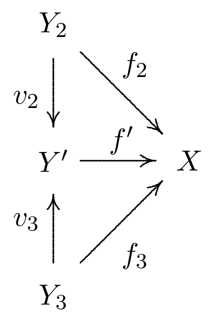


in $\tilde{\mathscr{J}}_1$ with Y in $\mathscr{J}_0$, u_1, u_2 in $\tilde{\mathscr{J}}_0$, $fu_1 = f_1$, $fu_2 = f_2$ and $h_n(u_1)\alpha_1 = h_n(u_2)\alpha_2$.

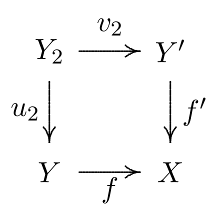
Proposition 2.1. *Under the assumptions* (i), (ii) *and* (iii), $\sim$ *is an equivalence relation on* $P_n(X)$.

Proof. Only transitivity is in question. Suppose that $(\alpha_1, f_1) \sim (\alpha_2, f_2)$ and $(\alpha_2, f_2) \sim (\alpha_3, f_3)$. We then have two commutative diagrams in $\tilde{\mathscr{J}}_1$ as follows:

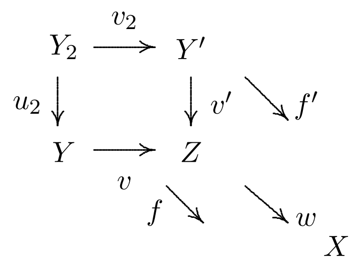




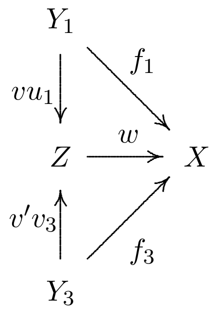
with Y and Y' in $\mathscr{J}_0$. Moreover, $h_n(u_1)\alpha_1 = h_n(u_2)\alpha_2$ and $h_n(v_2)\alpha_2 = h_n(v_3)\alpha_3$. By condition (i), the commutative diagram



can be embedded in a diagram



in $\tilde{\mathscr{J}}_1$ with Z in $\mathscr{J}_0$. Now consider the diagram



which is clearly commutative. Moreover $h_n(vu_1)\alpha_1 = h_n(v)h_n(u_1)\alpha_1 = h_n(v)h_n(u_2)\alpha_2 = h_n(vu_2)\alpha_2 = h_n(v'v_2)\alpha_2 = h_n(v')h_n(v_2)\alpha_2 = h_n(v')h_n(v_3)\alpha_3 = h_n(v'v_3)\alpha_3$. Hence $(\alpha_1, f_1) \sim (\alpha_3, f_3)$. This completes the proof of Proposition 2.1.

We denote the equivalence class of (α, f) by $[\alpha, f]$ and denote the set of such equivalence classes by $E_n(X)$. Now define 'addition' on $E_n(X)$ as follows: Given $[\alpha_1, f_1]$ and $[\alpha_2, f_2]$ in $E_n(X)$ with $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$, let $i_1 : Y_1 \to Y_1 \vee Y_2$ and $i_2 : Y_2 \to Y_1 \vee Y_2$ denote the usual injections. Define

$$[\alpha_1, f_1] + [\alpha_2, f_2] = [h_n(i_1)\alpha_1 + h_n(i_2)\alpha_2, f_1 \vee f_2]$$

where $f_1 \vee f_2 : Y_1 \vee Y_2 \to X$ is defined as usual. This 'addition' is easily checked to be well-defined, associative and commutative. The additive identity of the group is easily seen to be the element $[0, i]$ where $i : * \to X$ is the map that takes the singleton $*$ to the base point of X. We note that $[\alpha, f] = [0, i]$ if and only if we have a commutative diagram

$$\begin{array}{ccc} Y & & \\ {\scriptstyle u}\downarrow & \searrow^{f} & \\ Y_0 & \xrightarrow[f_0]{} & X \end{array}$$

in $\tilde{\mathscr{J}}_1$ with $Y_0 \in \mathscr{J}_0$ and $h_n(u)\alpha = 0$. The additive inverse of an element $[\alpha, f]$ is easily seen to be $[-\alpha, f]$. $E_n(X)$ is thus a commutative group.

For a given $f : Y \to X$ in $\tilde{\mathscr{J}}_1$ with Y in $\mathscr{J}_0$, define $i_f : h_n(Y) \to E_n(X)$ by the rule $i_f(\alpha) = [\alpha, f]$. $E_n(X)$ together with the maps $\{i_f\}$ has the required universal property; thus, $E_n(X) = h'_n(X)$. Note also that if $g : X \to X'$ is in $\tilde{\mathscr{J}}_1$, then $h'_n(g) = g_* : h'_n(X) \to h'_n(X')$ is defined by the rule $g_*[\alpha, f] = [\alpha, gf]$.

3. The Suspension Axiom

We now show that h' satisfies the suspension axiom. First identify the suspension map $\sigma : h_n(Y) \to h_{n+1}(\Sigma Y)$ in $\mathscr{J}_0$ as follows: $\sigma[\alpha, f] = [\sigma\alpha, \Sigma f]$. We then define the suspension isomorphism $\sigma' : h'_n(X) \to h'_{n+1}(\Sigma X)$ for any X in $\mathscr{J}_1$ by the rule $\sigma'[\alpha, f] = [\sigma\alpha, \Sigma f]$.

The map σ' is onto: If we take $[\beta, g] \in h'_{n+1}(\Sigma X)$, then we have a map $g : Y \to \Sigma X$ for some Y in $\mathscr{J}_0$ and $\beta \in h_{n+1}(Y)$. By assumption (ii) we can factorize this map $g = (\Sigma u)k : Y \xrightarrow{k} \Sigma Z \xrightarrow{\Sigma u} \Sigma X$ with Z in $\mathscr{J}_0$. Then the commutative diagram

$$\begin{array}{ccc} Y & & \\ {\scriptstyle k}\downarrow & \searrow^{g} & \\ \Sigma Z & \xrightarrow{\Sigma u} & \Sigma X \\ {\scriptstyle 1_{\Sigma Z}}\downarrow & \nearrow_{\Sigma u} & \\ \Sigma Z & & \end{array}$$

shows that $[\beta, g] = [h_{n+1}(k)\beta, \Sigma u] = [\sigma\sigma^{-1}(h_{n+1}(k)\beta), \Sigma u] = \sigma'[\sigma^{-1}(h_{n+1}(k)\beta), u]$. Thus σ' is onto.

To show that σ' is one-one, assume that $\sigma'[\alpha, f] = 0,\ f : Y \to X$ and $\alpha \in h_n(Y)$. We then have a commutative diagram

$$\begin{array}{ccc} \Sigma Y & & \\ {\scriptstyle u_1}\downarrow & \searrow^{\Sigma f} & \\ Y_1 & \xrightarrow[g_1]{} & \Sigma X \end{array}$$

with Y_1 in $\tilde{\mathscr{J}}_0$ and $h_{n+1}(u_1)(\sigma\alpha) = 0$. We can now factorize $g_1 = (\Sigma g)u_2 : Y_1 \xrightarrow{u_2} \Sigma Z \xrightarrow{\Sigma g} \Sigma X$. We thus have a diagram

$$\begin{array}{ccc} \Sigma Y & & \\ {\scriptstyle u_1}\downarrow & \searrow^{\Sigma f} & \\ Y_1 & \xrightarrow[g_1]{} & \Sigma X \\ {\scriptstyle u_2}\downarrow & \nearrow_{\Sigma g} & \\ \Sigma Z & & \end{array}$$

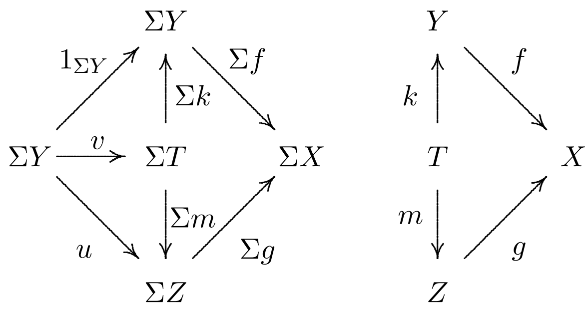
Let $u = u_2 u_1$; then we have a commutative diagram

$$\begin{array}{ccc} \Sigma Y & & \\ {\scriptstyle u}\downarrow & \searrow^{\Sigma f} & \\ \Sigma Z & \xrightarrow[\Sigma g]{} & \Sigma X \end{array}$$

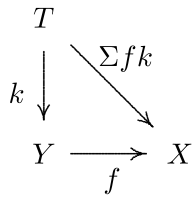
Moreover, $h_{n+1}(u)(\sigma\alpha) = h_{n+1}(u_2)h_{n+1}(u_1)(\sigma\alpha) = 0$. Consider the commutative diagram

$$\begin{array}{ccccc} & & \Sigma Y & & \\ & {\scriptstyle 1_{\Sigma Y}}\nearrow & & \searrow{\scriptstyle \Sigma f} & \\ \Sigma Y & & & & \Sigma X \\ & {\scriptstyle u}\searrow & & \nearrow{\scriptstyle \Sigma g} & \\ & & \Sigma Z & & \end{array}$$

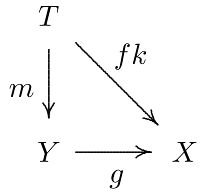
By assumptions (ii)(b), we have commutative diagrams



with T in $\mathscr{J}_0$. Consider the element $h_{n+1}(v)(\sigma\alpha) \in h_{n+1}(\Sigma T)$. Since $\sigma : h_n(T) \xrightarrow{\approx} h_{n+1}(\Sigma T)$ is an isomorphism, it follows that $h_{n+1}(v)(\sigma\alpha) = \sigma\beta$ for some $\beta \in h_n(T)$. Moreover, $1_{\Sigma Y} = (\Sigma k)v$, so that $\sigma\alpha = h_{n+1}(1_{\Sigma Y})(\sigma\alpha) = h_{n+1}(\Sigma k)h_{n+1}(v)(\sigma\alpha) = h_{n+1}(\Sigma k)h_{n+1}(\sigma\beta) = \sigma h_n(k)(\beta)$, showing that $\alpha = h_n(k)(\beta)$. Consideration of the diagram



shows that $[\beta, fk] = [\alpha, f]$. On the other hand we have $h_{n+1}(\Sigma m)(\sigma\beta) = h_{n+1}(\Sigma m)h_{n+1}(v)(\sigma\alpha) = h_{n+1}(u)(\sigma\alpha) = \sigma(h_n(u)(\alpha)) = 0$. Thus, $h_n(m)(\beta) = 0$. Considering the diagram



we arrive at the conclusion that $[\beta, fk] = 0$, so that $[\alpha, f] = 0$. Thus, σ' is also one-one and this completes the proof of the suspension axiom.

4. The Exactness Axiom

We shall show now, using the suspension axiom (proved above), that the functor h'_n satisfies the exactness axiom, i.e., h'_n carries every cokernel sequence $A \xrightarrow{g} X \xrightarrow{P_g} C_g$ in $\tilde{\mathscr{J}}_1$ into an exact sequence. Let $[\alpha, f] \in \ker h'_n(P_g)$ with $f : Y \to X$. This implies that $[\alpha, P_g f] = 0$ so that we have a map $u : Y \to Y_0$ in $\tilde{\mathscr{J}}_0$ and a map $f_0 : Y_0 \to C_g$ in $\tilde{\mathscr{J}}_1$ such that $h_n(u) = 0$. Consider the diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{g} & X & \xrightarrow{P_g} & C_g & \longrightarrow & \Sigma A & \xrightarrow{\Sigma g} & \Sigma X \\ & & f \uparrow & & f_0 \uparrow & & f_1 \uparrow & & \Sigma f \uparrow \\ & & Y & \xrightarrow[u]{} & Y_0 & \xrightarrow[P_u]{} & C_u & \xrightarrow[Q_u]{} & \Sigma Y & \xrightarrow[\Sigma u]{} & \Sigma Y_0 \end{array}$$

where f_1 and Σf are induced maps between the corresponding terms of the Puppe sequences. Since $\sigma : h_n(Y) \overset{\approx}{\to} h_{n+1}(\Sigma Y)$ is an isomorphism, it follows that $h_{n+1}(\Sigma u)(\sigma\alpha) = 0$. But the homology functor on the bottom Puppe sequence is exact; so there is an element $\beta \in h_{n+1}(C_u)$ such that $h_{n+1}(Q_u)\beta = \sigma\alpha$. From the third sequence, we therefore deduce that $[\beta, (\Sigma g) f_1] = [\sigma\alpha, \Sigma f]$. If $\gamma = [\beta, f_1] \in h'_{n+1}(\Sigma A)$, then $h'_{n+1}(\Sigma g)(\gamma) = h'_{n+1}(\Sigma g)[\beta, f_1] = [\beta, (\Sigma g) f_1] = [\sigma\alpha, \Sigma f] = \sigma'[\alpha, f]$. Since we have an isomorphism $\sigma' : h'_n(A) \overset{\approx}{\to} h'_{n+1}(A)$ we take $\delta^{-1} = (\sigma')^{-1}(\gamma)$, so that $h'_n(g)(\delta) = h'_n(g)(\sigma')^{-1}(\gamma) = (\sigma')^{-1} h'_{n+1}(\Sigma g)(\gamma) = (\sigma')^{-1}[\sigma\alpha, \Sigma f] = (\sigma')^{-1}\sigma'[\alpha, f] = [\alpha, f]$. This shows that $\ker h'_n(P_g) \subset \operatorname{Image} h'_n(g)$. To show that $\operatorname{Image} h'_n(g) \subset \ker h'_n(P_g)$, it is enough to show that for any element $[\alpha, f] \in h'_n(A)$, $h'_n(P_g)\ h'_n(g)[\alpha, f] = 0$, i.e., $[\alpha, P_g g f] = 0$. In the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{g} & X & \xrightarrow{P_g} & C_g \\ f \uparrow & & gf \uparrow & & \uparrow k \\ Y & \xrightarrow[1_Y]{} & Y & \xrightarrow[P_{1_Y}]{} & C_{1_Y} \end{array}$$

it is clear that $[\alpha, P_g g f] = [h_n(P(1_Y)) h_n(1_Y)\alpha, k]$. But the bottom Puppe sequence is carried by h_n into an exact sequence so that $h_n(P(1_Y)) h_n(1_Y)\alpha = 0$. Thus we have the desired result.

5. Examples

We present some examples where the assumptions made on the categories $\mathscr{J}_0$ and $\mathscr{J}_1$ are valid.

Example 5.1. In the stable categories, the suspension functor is an isomorphism, so that there is no difficulty in proving that the three axioms hold in such categories (see [9, 14]).

Example 5.2. Let $\mathscr{J}_0$ to be the set of all based topological spaces having the homotopy type of finite CW-complexes and $\mathscr{J}_1$ to be the set of all spaces having the homotopy type of CW-complexes. It is easy to prove that the three axioms are true in this case.

Example 5.3. Let $\mathscr{J}_1$ be the category of 1-connected based topological spaces having the homotopy type of a CW-complex and $\mathscr{J}_0$ be the subcategory of spaces whose homotopy groups are all finitely generated and P-local, where P denotes a fixed set of primes. Then any cohomolgy theory h on $\mathscr{J}_0$ extends to a cohomolgy theory h_1 on $\mathscr{J}_1$ through the Kan extension process [10].

Example 5.4. Let $\mathcal{C}$ be a Serre class of abelian groups and $\mathscr{T}_0(\mathcal{C})$ be the subcategory $\mathscr{T}_S$ (the category of 1-connected spaces) consisting of spaces whose homotopy groups belong to $\mathcal{C}$ in the sense of Serre [15]. Then any cohomology theory h on $\mathscr{T}_0(\mathcal{C})$ extends to a cohomology theory h_1 on $\mathscr{T}_S$ through Kan extension process ([9], Theorem 4.1).

Note 5.5. Example 5.3 is a sort of variant of Example 5.4.

Example 5.6. Let $\mathcal{CW}_{CF}$ and $\mathcal{CW}_C$ be the categories of connected finite based CW-complexes and connected based CW-complexes respectively. Let h be a homology theory on $\mathcal{CW}_{CF}$. Then the Kan extension h_1 of h to $\mathcal{CW}_C$ is also a homology theory; indeed, it is naturally equivalent to a homology theory defined by a spectrum [13].

Example 5.7. Let $\mathscr{T}$ be a triangulated category and let h, k be two homology theories on $\mathscr{T}$. Let X in $\mathscr{T}$ admit the Adams h-completion X_h, and form the triangle $X\xrightarrow{e}X_h\xrightarrow{i}\mathcal{C}_e\xrightarrow{j}\Sigma X$ in $\mathscr{T}$. Let $\mathscr{T}_0$ be the full subcategory of $\mathscr{T}$ whose objects are those Y such that $h(Y)=0$. It is plain that $\mathscr{T}_0$ is a triangulated subcategory of $\mathscr{T}$. Let k^0 be the restriction of k to $\mathscr{T}_0$. Then the Kan extension k^1 of k^0 is given by $k^1_n(X) = k_{n+1}(\mathcal{C}_e)$ ([3], Theorem 4.1).

Example 5.8. Let $\boldsymbol{S}$ be the stable CW-category and let $\boldsymbol{S}_0$ be the full subcategory consisting of spaces in the Serre class $\mathcal{C}_P$ of P-torsion groups. Let k be a homology theory on $\boldsymbol{S}$. The Kan extension k^1 to $\boldsymbol{S}$ of the restriction of k to $\boldsymbol{S_0}$, for X in $\boldsymbol{S}$, is given by $k^1_n(X) = k_{n+1}(X; \mathbb{Z}_{P^\infty})$ ([3], Theorem 4.4).

6. Adams Cocompletion for the Homology Theory h'

Next we show that a homology theory over the category of simply connected based topological spaces and continuous maps arising through Kan extension process from an additive homology theory over a smaller subcategory always admits global Adams cocompletion. We do it in the context of Serre class of abelian groups. We recall the following.

Let $\mathcal{C}$ be a category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect to S and $F : \mathcal{C} \to \mathcal{C}[S^{-1}]$ the canonical functor. Let $\boldsymbol{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$, $\mathcal{C}[S^{-1}](Y, -) : \mathcal{C} \to \boldsymbol{S}$ defines a covariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, that is, $\mathcal{C}[S^{-1}](Y, -) \cong \mathcal{C}(Y_S, -)$, then Y_S is called the (*generalized*) *Adams cocompletion* of Y with respect to the set of morphisms S or simply the S-*cocompletion* of Y. We shall often refer to Y_S simply as the *cocompletion* of Y [5].

Let $\mathscr{T}$, $\mathscr{J}$ and $\tilde{\mathscr{J}}$ be as in Section 1. Let h be a generalized homology (cohomology) theory defined on $\tilde{\mathscr{J}}$. Let S be the set of morphisms of $\tilde{\mathscr{J}}$ which are carried into isomorphisms by h. If every object of $\tilde{\mathscr{J}}$ admits a cocompletion with respect to S, then we say that the homology theory h admits global Adams cocompletion. Deleanu [6] has shown that any additive theory h on the homotopy category of based CW-complexes and based continuous maps admits global Adams cocompletion.

In this note, we show that every homology theory on an admissible category arising from an additive homology theory on a smaller admissible category through Kan extension process (see Deleanu and Hilton [1, 2], Hilton [11]) always admits global Adams cocompletion. More precisely, let $\mathscr{J}_0$ and $\mathscr{J}_1$ be admissible complete categories with $\mathscr{J}_0 \subset \mathscr{J}_1$ and $\tilde{\mathscr{J}}_1$ be small $\mathscr{U}$-category, where $\mathscr{U}$ is a fixed Grothendieck universe. Let h be an additive homology theory on $\mathscr{J}_0$ such that its Kan extension h' over $\tilde{\mathscr{J}}_1$ is also a homology theory; then we show that h' admits global Adams cocompletion. The proof of this result depends mainly on the particularly nice description of the homology group $h'_n(X)$ as described in Section 2 and additivity of the functor h. We shall use the following theorem for showing the global existence of the Adams cocompletion of the homology theory h'; the result is essentially Theorem 1 in [7] (also see, Theorem [12]).

Theorem 6.1. *Let $\mathcal{C}$ be a complete small $\mathscr{U}$-category ($\mathscr{U}$ is a fixed Grothendieck universe) and S a set of morphisms of $\mathcal{C}$ that admits a calculus of right fractions. Suppose that the following compatibility condition with product is satisfied:*

(P) *If each $s_i : X_i \to Y_i$, $i \in I$, is an element of S where the index set I is an element of $\mathscr{U}$, then*

$$\prod_{i\in I} s_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$

is an element of S.

Then every object X of $\mathcal{C}$ has an Adams cocompletion X_S with respect to the set of morphisms S.

We now apply this result to the category $\tilde{\mathscr{J}}_1$. Let S be the set of morphisms of $\tilde{\mathscr{J}}_1$ which are carried into $\mathcal{C}$-isomorphisms in all dimensions by the homology functor h' where $\mathcal{C}$ is a Serre class of abelian groups which is moreover an acyclic ideal of abelian groups.

It is well known that $\tilde{\mathscr{J}}_1$ is complete. We prove the following propositions.

Proposition 6.2. *S is saturated.*

Proof. This is evident from Proposition 1.1 ([5], p. 63).

Proposition 6.3. *S admits a calculus of right fractions.*

Proof. Clearly S is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.3* [5]. For this, it is enough to prove that every diagram of the form

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\gamma]{} & B \end{array}$$

in $\tilde{\mathscr{J}}_1$ with $\gamma \in S$, can be imbedded in a weak pull-back diagram

$$\begin{array}{ccc} D & \xrightarrow{\delta} & A \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\gamma]{} & B \end{array}$$

with $\delta \in S$. The essential idea, as has been explained in [8], is to factorize these maps in terms of fibrations and then taking the pull-back of these fibrations. Suppose $\alpha = [f]$ and $\gamma = [s]$. We replace the maps f and s by fibrations to get the following diagram

$$\begin{array}{ccccc} & & & & A \\ & & & & {\scriptstyle \overline{r}}\uparrow\downarrow{\scriptstyle r} \\ & & D & \xrightarrow{p} & P_f \\ & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f'} \\ C & \underset{t}{\overset{\bar{t}}{\rightleftarrows}} & P_s & \xrightarrow[s']{} & B \end{array}$$

where f' and s' are fibrations; r and t are mod-$\mathcal{C}$ homotopy equivalences; $\overline{r}$ and $\bar{t}$ are mod-$\mathcal{C}$ homotopy inverses of r and t; P_f and P_s are mapping tracks of f and s; D is the usual pull-back of f' and s' and p, q are the respective projections. So we have $f = f'r$ and $s = s't$. Let $\delta = [\overline{r}p]$, $\beta = [\bar{t}q]$. Thus $\alpha\delta = [f][\overline{r}p] = [f\overline{r}p] = [f'r\overline{r}p] = [f'p] = [s'q] = [s't\bar{t}q] = [s\bar{t}q] = [s][\bar{t}q] = \gamma\beta$. Moreover, if $\alpha\mu = r\lambda$, let $u : U \to A$, $v : U \to C$ be in the classes μ, λ respectively so that $fu \simeq sv$ or $f'ru \simeq sv$. Let $F : U \times I \to B$ be a homotopy with $F_0 = f'ru$ and $F_1 = sv$. Consider the following diagram

$$\begin{array}{ccc} & & P_f \\ & {\scriptstyle G_t}\nearrow\!\!\!\nearrow {\scriptstyle ru} & \downarrow f' \\ U & \xrightarrow[F_t]{} & B \end{array}$$

Since f' is a fibration there exists a homotopy $G : U \times I \to P_f$ such that (a) $f'G_t = F_t$, (b) $G_0 = ru$. Thus $f'G_1 = F_1 = sv = s'tv$. Consider the following diagram

$$\begin{array}{cccc} U & & & \\ & \searrow^{k} & \searrow^{G_1} & \\ & {\scriptstyle tv}\searrow & D \xrightarrow{p} & P_f \\ & & q\downarrow & \downarrow f' \\ & & P_s \xrightarrow[s']{} & B \end{array}$$

By the pull-back property of D, there exists a map $k : U \to D$ such that $pk = G_1 \simeq ru$ and $qk = tv$. Thus if $\rho = [k] : U \to D$, then $\delta\rho = [\overline{r}p][k] = [\overline{r}pk] = [\overline{r}ru] = [u] = \mu$ and $\beta\rho = [\overline{t}q][k] = [\overline{t}qk] = [\overline{t}tv] = [v] = \lambda$.

It now remains to be shown that $\delta \in S$. We assume that the map $\alpha : A \to B$ is a fibration with fiber F; then F is also the fiber of the map $\beta : B \to D$ and from the commutative diagram

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ D & \xrightarrow{\delta} & A \\ \beta\downarrow & & \downarrow\alpha \\ C & \xrightarrow[\gamma]{} & B \end{array}$$

we have the following commutative diagram

$$\begin{array}{ccccccccccc} \cdots \to & \pi_{m+1}(C) & \to & \pi_m(F) & \to & \pi_m(D) & \to & \pi_m(C) & \to & \pi_{m-1}(F) & \to \cdots \\ & \gamma_*\downarrow & & \| & & \downarrow\delta_* & & \downarrow\gamma_* & & \| & \\ \cdots \to & \pi_{m+1}(B) & \to & \pi_m(F) & \to & \pi_m(A) & \to & \pi_m(B) & \to & \pi_{m-1}(F) & \to \cdots \end{array}$$

The mod-$\mathcal{C}$ Five lemma [15] implies that δ_* is a $\mathcal{C}$-isomorphism for all $m \geq 0$. Thus $\delta \in S$. $\square$

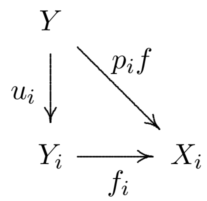
Proposition 6.4. *Let* $\{s_i : X_i \to Y_i,\ i \in I\}$ *be a subset of* S. *Then*

$$\prod_{i\in I} s_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$

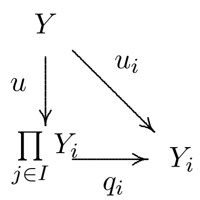
is an element of S, *where the index set* I *is an element of* $\mathscr{U}$.

Proof. First we prove that the homology theory h' satisfies the wedge axiom. For each $i \in I$, let p_i denote the homology class of the projection map $\prod_{j\in I} X_j \to X_i$. Thus we have group homomorphisms $h'(p_i) : h'(\prod_{j\in I} X_j) \to h'(X_i)$. We need to prove that the group homomorphism $\{h'(p_i)\} : h'(\prod_{j\in I} X_j) \to \prod_{i\in I} h'(X_i)$ is a $\mathcal{C}$-isomorphism.

We first show that $\{h'(p_i)\}$ is a $\mathcal{C}$-monomorphism, i.e., Ker $\{h'(p_i)\} \in \mathcal{C}$ (see [15]; p. 505). Let $[\alpha, f]$ be class in $h'(\prod_{j\in I} X_j)$ such that $\{h'(p_i)\}([\alpha, f]) = 0$, i.e., $[\alpha, fp_i] = 0$ for each $i \in I$, where $f : Y \to \prod_{j\in I} X_j$ is in $\tilde{\mathscr{J}_1}$ with $Y \in \mathscr{J}_0$ and $\alpha \in h(Y)$. Hence for each $i \in I$, there exists a space Y_i in $\mathscr{J}_0$ and maps $u_i : Y \to Y_i$ in $\tilde{\mathscr{J}_0}$ and $f_i : Y_i \to X_i$ such that the following diagram



commutes, i.e., $f_i u_i = p_i f$ and $h_0(u_i)(\alpha) = 0$. Let $q_i : \prod_{j\in I} Y_j \to Y_i$ denote the homotopy class of the projection map. By the universal property of the product, there exists a map $u : Y \to \prod_{j\in I} Y_j$ making the diagram



commutative, i.e., $q_i u = u_i$. Hence for each $i \in I$, $h(q_i)h(u)(\alpha) = h(u_i)(\alpha) = 0$. Since h is an additive homology theory, we note that $\{h(q_i)\} : h(\prod_{i\in I} Y_i) \to$

$\prod\limits_{i\in I} h(Y_i)$ is an isomorphism and hence $h(u)(\alpha) = 0$. In the following diagram

$$\begin{array}{ccc}
\prod\limits_{j\in I} Y_j & & \\
& \searrow^{q_i} & \\
{\scriptstyle v}\,\Big\downarrow & & Y_i \\
& & \searrow^{f_i} \\
\prod\limits_{j\in I} X_j & \xrightarrow[p_i]{} & X_i
\end{array}$$

by the definition of the product, there exists a map $v : \prod\limits_{j\in I} Y_j \to \prod\limits_{j\in I} X_j$ in $\tilde{\mathscr{J}}_1$ such that $p_i v = f_i q_i$. Thus $p_i v u = f_i q_i u = f_i u_i = p_i f$ i.e., the following diagram

$$\begin{array}{ccc}
Y & & \\
\Big\downarrow {\scriptstyle u} & \searrow^{u_i} & \\
f \downarrow \ \prod\limits_{j\in I} Y_j & & Y_i \\
\Big\downarrow {\scriptstyle v} & & \searrow^{f} \\
\prod\limits_{j\in I} X_j & \xrightarrow[p_i]{} & X_i
\end{array}$$

is commutative and by the universal property of the product we have $vu = f$. Thus we have the following commutative diagram

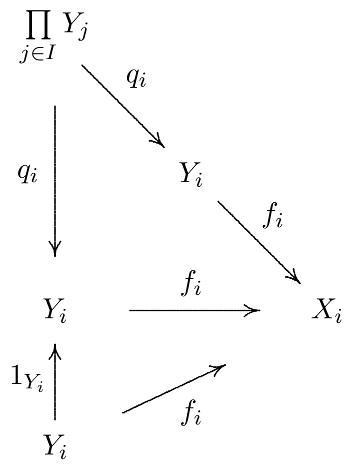
$$\begin{array}{ccc}
Y & & \\
{\scriptstyle u}\,\Big\downarrow & \searrow^{f} & \\
\prod\limits_{j\in I} Y_j & \xrightarrow[v]{} & \prod\limits_{j\in I} X_j
\end{array}$$

with $u \in \tilde{\mathscr{J}}_0$ and $h(u)(\alpha) = 0$. Hence $[\alpha, f] = 0$. Thus $\mathrm{Ker}\{h'(p_i)\}$ is a trivial group and hence by Theorem 9.6.1 [15], $\mathrm{Ker}\{h'(p_i)\} \in \mathcal{C}$.

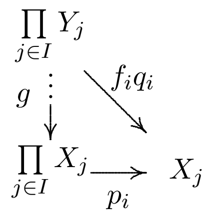
Next we show that $\{h'(p_i)\}$ is a $\mathcal{C}$-epimorphism, i.e., $\mathrm{Coker}\{h'(p_i)\} \in \mathcal{C}$. Consider an arbitrary element $\{[\alpha_i, f_i]\}_{i\in I}$ in $\prod\limits_{i\in I} h'(X_i)$ where for each $i \in I$, the class $[\alpha_i, f_i]$ is represented by $f_i : Y_i \to X_i$ with $Y_i \in \mathscr{J}_0$ and

$\alpha_i \in h(Y_i)$. Since $\{h(q_i)\} : h(\prod_{i\in I} Y_i) \to \prod_{i\in I} h(Y_i)$ is an isomorphism, the element $\{\alpha_i\} \in \prod_{i\in I} h(Y_i)$ corresponds to some element $\alpha \in h(\prod_{i\in I} Y_i)$ such that $\{h(q_i)\}(\alpha) = \{\alpha_i\}$. Thus for each $i \in I$, $h(q_i)(\alpha) = \alpha_i$.

In the following diagram



the two triangles commute and for each $i \in I$, $h(1_{Y_i})(\alpha_i) = \alpha_i = h(q_i)(\alpha)$. Hence $[\alpha_i, f_i] = [\alpha, f_i q_i]$, for each $i \in I$. By the universal property of q_i in the following diagram



there exists a unique map $g : \prod_{j\in I} Y_j \to \prod_{j\in I} X_j$ such that $p_i g = f_i q_i$. Consider the class $[\alpha, g] \in h'(\prod_{j\in I} X_j)$. For each $i \in I$, we have $h'(p_i)([\alpha, g]) = [\alpha, p_i g] = [\alpha, f_i q_i] = [\alpha_i, f_i]$. Hence $\{h'(p_i)\}([\alpha, g]) = \{[\alpha_i, f_i]\}_{i\in I}$. So $\operatorname{Im}\{h'(p_i)\} = \prod_{i\in I} h'(X_i)$; thus $\operatorname{Coker}\{h'(p_i)\}$ is a trivial group and it is in $\mathcal{C}$ by Theorem 9.6.1 [15].

Now we prove that h' satisfies the compatibility axiom with products. Consider the commutative diagram

$$
\begin{array}{ccc}
h'\Big(\prod_{i\in I} X_i\Big) & \xrightarrow{\{h'(p_i)\}} & \prod_{i\in I} h'(X_i) \\
{\scriptstyle h'\left(\prod_{i\in I} s_i\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle \prod_{i\in I} h'(s_i)} \\
h'\Big(\prod_{i\in I} Y_i\Big) & \xrightarrow[\{h'(p'_i)\}]{} & \prod_{i\in I} h'(Y_i)
\end{array}
$$

Since $h'(s_i)$ is a $\mathcal{C}$-isomorphism, so is $h'(\prod_{i\in I} s_i)$. By the wedge axiom, as proved above, it follows that $\{h'(p_i)\}$ and $\{h'(p'_i)\}$ are $\mathcal{C}$-isomorphisms. Thus it follows that $h'(\prod_{i\in I} s_i)$ is a $\mathcal{C}$-isomorphism. This completes the proof of Proposition 6.4.

Hence from Propositions 6.2, 6.3 and 6.4, it follows that all the conditions of Theorem 6.1 are satisfied and so we obtain the following theorem.

Theorem 6.5. *The homology theory h' admits global Adams cocompletion.*

Acknowledgments

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