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MIXTURE LEMMA

BY

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Abstract

The Mixture Lemma plays a central role in the study of the singularity of solutions of the Boltzmann equation. We offer a detailed proof of the lemma. The proof depends on the proper switching of the differentiations with respect to the space variables to those of the microscopic velocities, and depends on the precise regularity properties of the collision operator.

1. Introduction

Consider the Boltzmann equation

$$F_t + \xi \cdot \nabla_x F = Q(F,F),\ \ F = F(x,t,\xi) \geq 0, \qquad x, \xi \in \mathbb{R}^3,\ \ t \in \mathbb{R}_+,$$

for the hard sphere model

$$\begin{aligned} Q(F,G) \ \equiv\ & \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+} [F(\xi^{'})G(\xi^{'}_*) + F(\xi^{'}_*)G(\xi^{'}) - F(\xi)G(\xi_*) \\ & -F(\xi_*)G(\xi)]|(\xi-\xi_*)\cdot\Omega|d\Omega d\xi_*. \end{aligned}$$

In this paper we offer a detailed proof of the Mixture Lemma of [5], [6].

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The Lemma is to study the singularity of the solutions for the Boltzmann equation and is essential for the construction of the Green's function for the linearized Boltzmann equation. The Green's function approach, e.g. [4], [7], [8], [9], [10], offers more quantitative informations for the solutions of the Boltzmann equation and is useful for the studies of interesting physical phenomena. There is the celebrated Velocity Averaging Lemma, [3], which is often used for the study of the combined effect of transport and collision. It serves a different purpose of gaining space compactness for macroscopic quantities, e.g. [2]. The Boltzmann equation for the hard sphere models is semilinear hyperbolic and therefore the propagation of the singularities of the solution can be studied on the level of linearized Boltzmann equation. Consider the linearization about the normalized Maxwellian

$$F = M + \sqrt{M}g,$$

$$M \equiv (2\pi)^{-3/2} e^{-|\xi|^2/2}.$$

and the resulting linearized Boltzmann equation

$$\partial_t g + \xi \cdot \nabla_x g = Lg. \tag{1.1}$$

For the hard sphere model under consideration, the linearized collision operator

$$Lg \equiv 2\frac{1}{\sqrt{M}} Q(M, \sqrt{M}g)$$

has explicit form consisting of a multiplicative operator $\nu(\xi)$ and an integral operator K, [1]:

$$Lg(\xi) = -\nu(\xi)g(\xi) + Kg(\xi),$$

$$\begin{aligned}
Kg(\xi) &\equiv \int_{\mathbb{R}^3} K(\xi,\xi_*)g(\xi_*)d\xi_*,\\
K(\xi,\xi_*) &\equiv \frac{2}{\sqrt{2\pi}|\xi-\xi_*|}\exp\Big\{-\frac{(|\xi|^2-|\xi_*|^2)^2}{8|\xi-\xi_*|^2}-\frac{|\xi-\xi_*|^2}{8}\Big\}\\
&\quad -\frac{|\xi-\xi_*|}{2}\exp\Big\{-\frac{(|\xi|^2+|\xi_*|^2)}{4}\Big\},\\
\nu(\xi) &\equiv \frac{1}{\sqrt{2\pi}}\Big[2e^{-\frac{|\xi|^2}{2}}+\Big(|\xi|+\frac{1}{|\xi|}\Big)\int_0^{|\xi|} e^{-\frac{u^2}{2}}du\Big].
\end{aligned}$$

Rewrite the linearized Boltzmann equation as follows:

$$\partial_t g + \xi \cdot \nabla_x g + \nu(\xi) g = Kg,$$

and view this as the coupling of the integral operator K with the damped transport equation

$$\begin{aligned} &\partial_t h + \xi \cdot \nabla_x h + \nu(\xi) h = 0, \\ &h(x, 0, \xi) = h_0(x, \xi). \end{aligned} \tag{1.2}$$

The damped transport equation has the solution operator $\mathbb{S}^t$:

$$\mathbb{S}^t h_0(x, \xi) = e^{-\nu(\xi)t} h_0(x - \xi t, \xi). \tag{1.3}$$

The above coupling leads to the Mixture operators $\mathbb{M}_k^t,\ k = 1, 2, \ldots,$ when the Picard-type iterations are used in extracting parts of a Boltzmann solution with varying degree of regularity, [5] and [6]:

$$\mathbb{M}_k^t g_0 \equiv \int_0^t \int_0^{s_1} \cdots \int_0^{s_{2k-1}} \mathbb{S}^{t-s_1} K \mathbb{S}^{s_1 - s_2} K \cdots \mathbb{S}^{s_{2k-1} - s_{2k}} K \mathbb{S}^{s_{2k}} g_0 ds_{2k} \cdots ds_1.$$

The function $\nu(\xi)$ behaves like $1 + |\xi|$. Let ν_0 be a positive lower bound of $\nu(\xi)$:

$$\nu(\xi) > \nu_0,\ \xi \in \mathbb{R}^3.$$

The main theorem as stated in [6] is the following:

Theorem 1.1. *For each given $k \geq 0$, there exists a positive constant C_k such that, for any*

$$\beta \in \mathbb{N}_0^3,\ |\beta| = \beta^1 + \beta^2 + \beta^3 = k,$$

$$\|D_x^\beta \mathbb{M}_k^t g_0\|_{L_x^2(L_\xi^2)} \leq C_k e^{\frac{-\nu_0 t}{2}} \sum_{\gamma \in \mathbb{N}_0^3, \gamma^i \leq \beta^i} \|D_\xi^\gamma g_0\|_{L_x^2(L_\xi^2)}.$$

The details of its proof for $k = 1$ has been carried out in [5, 6] using the combination of Fourier transform and characteristic method. In this paper, we make use of switching the differentiations along the characteristic curves to prove the Mixture Lemma using characteristic method only. Shih-Hsien

Yu has recently told us that he has also carried out the proof for the case of $k = 1$ using characteristic method only.

There is a delicate regularity property for the kernel $K(\xi, \xi_*)$ that we need in order for the switching of differentiations method to work. This is done in Section 2, Lemma 2.2. The proof of the Mixture Lemma is done in Section 3 and Section 4. For easier reading, in Section 2 and Section 3, we present the proofs for the case of one space dimension, and in Section 4 indicate the generalization to the case of three space dimensions.

2. Regularity of Collision Kernels

In this and next sections we consider the case of one space dimention $x \in \mathbb{R}^1$; the microscopic velocity is still of three dimensions $\xi \in \mathbb{R}^3$. The linearized Boltzmann equation becomes

$$\partial_t g + \xi^1 \partial_x g = Lg. \tag{2.1}$$

The Mixture Lemma takes the following form:

Theorem 2.1. *For each given $k \geq 0$, there exists a positive constant C_k such that*

$$\|\partial_x^k \mathbb{M}_k^t g_0\|_{L^2_x(L^2_\xi)} \leq C_k e^{-\frac{\nu_0 t}{2}} \sum_{l=0}^{k} \|\partial_{\xi^1}^l g_0\|_{L^2_x(L^2_\xi)}.$$

In preparation for its proof, we now study the regularity property of the functions $\nu(\xi)$ and $K(\xi, \xi_*)$.

Lemma 2.1. *For any $l \geq 1$, l-th derivatives of $\nu(\xi)$ is bounded, i.e., for some constant C_l,*

$$\left|\partial_{\xi^i}^l \nu(\xi)\right| \leq C_l, \quad i = 1, 2, \quad \textit{and } 3. \tag{2.2}$$

Proof. It suffices to consider the term $\frac{1}{|\xi|}\int_0^{|\xi|} e^{-\frac{u^2}{2}} du$ for $0 < |\xi| < 1$. We have by Taylor expansion

$$\frac{1}{|\xi|}\int_0^{|\xi|} e^{-\frac{u^2}{2}} du = \frac{1}{|\xi|}\int_0^{|\xi|} \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{u^2}{2})^n du$$

$$= \frac{1}{|\xi|} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{|\xi|} (-\frac{u^2}{2})^n du$$
$$= \frac{1}{|\xi|} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} \frac{|\xi|^{2n+1}}{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} \frac{|\xi|^{2n}}{2n+1}$$

which is absolutely convergent as the function of $|\xi|^2$ and therefore is analytic in (ξ^1, ξ^2, ξ^3). This completes the proof.

Unlike $\nu(\xi)$, the differentiations of the kernel $K(\xi, \xi_*)$ are not all smooth, even integrable. However, it is the smoothness of certain combinations of the differentiations of $K(\xi, \xi_*)$ that are needed. To express this, we have the following notations:

(1) $\mathcal{A} \equiv \Big\{\alpha \in \mathbb{N}_0^{6k} \Big| \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2k})\ \alpha_{2j} = 0, \quad \alpha_{2j-1}^1 = 0, 1,\ \alpha_{2j-1}^2 = \alpha_{2j-1}^3 = 0, \text{ for } j = 1, \ldots, k\Big\}$

(2) $H(\xi, \xi_*) \equiv |K(\varepsilon\xi, \varepsilon\xi_*)| + |\partial_{\xi_*^1} K(\varepsilon\xi, \varepsilon\xi_*)|$, for any fixed $\varepsilon < 1$.

(3) using the change of variables $\mathcal{T} : \Xi = (\xi_1, \xi_2, \ldots, \xi_{2k}) \to V = (V_1, V_2, \ldots, V_{2k})$ where $V_1 \equiv \xi - \xi_1$, $V_i \equiv \xi_{i-1} - \xi_i$ for $i = 2, 3, \ldots, 2k$, to rewrite the kernel:

(4) $K_i \equiv K(\xi - \sum_{j=1}^{i-1} V_j, \xi - \sum_{j=1}^{i} V_j)$, $H_i \equiv H(\xi - \sum_{j=1}^{i-1} V_j, \xi - \sum_{j=1}^{i} V_j)$.

Lemma 2.2. *For $1 < l < 2k$ and $\alpha \in \mathcal{A}$, we have, for some constant C_α,*

$$|D_V^\alpha K_l| \le C_\alpha H_l.$$

Proof. We focus only on the most interesting term in K_l:

$$F_l \equiv \frac{1}{|V_l|} e^{G_l} \equiv \frac{1}{|V_l|} \exp\Big\{ - \frac{(|\xi - \sum_{j=1}^{l-1} V_j|^2 - |\xi - \sum_{j=1}^{l} V_j|^2)^2}{|V_l|^2} \Big\}.$$

Note that we can find m such that $\alpha_i = 0$ for $i > m$ and $\alpha_m^1 = 1$ for the given α in $\mathcal{A}$, $\alpha \neq 0$. And it is easy to see that $D_V^\alpha K_l = 0$ in case of $m > l$, so we consider the following two cases.

Case A $(m < l)$ Note that

$$\frac{\partial G_l}{\partial V_n^1} = \frac{-2(|\xi - \sum_{j=1}^{l-1} V_j|^2 - |\xi - \sum_{j=1}^{l} V_j|^2)(-2V_l^1)}{|V_l|^2} = \frac{\partial G_l}{\partial V_1^1}, \text{ for } n < l, \tag{2.3}$$

$$\frac{\partial^2 G_l}{\partial V_{n_1}^1 \partial V_{n_2}^1} = \left(\frac{V_l^1}{|V_l|}\right)^2 = \frac{\partial^2 G_l}{\partial [V_1^1]^2}, \text{ for } n_1 \text{ and } n_2 < l \text{ and } l > 2, \tag{2.4}$$

we obtain multi index α' by replacing α_m with 0. Direct calculation then gives us

$$\begin{aligned} D_V^\alpha F_l &= D_V^{\alpha'}\big(F_l \cdot \frac{\partial G_l}{\partial V_1^1}\big) \\ &= \sum_{\beta < \alpha'} \begin{pmatrix} \alpha' \\ \beta \end{pmatrix} \big(D_V^\beta F_l\big)\big(D_V^{\alpha' - \beta} \frac{\partial G_l}{\partial V_1^1}\big) \\ &\leq C_m F_l \sum_{n=1}^{[|\alpha|/2]} \Big(\frac{\partial G_l}{\partial V_1^1}\Big)^{|\alpha| - 2n} \Big(\frac{\partial^2 G_l}{\partial [V_1^1]^2}\Big)^n \\ &\leq C_m F_l \Big(\frac{|V_l^1|}{|V_l|}\Big)^{|\alpha|} \sum_{n=1}^{[|\alpha|/2]} \Big(\frac{4||\xi - \sum_{j=1}^{l-1} V_j|^2 - |\xi - \sum_{j=1}^{l} V_j|^2|}{|V_l|}\Big)^{|\alpha| - 2n}. \end{aligned}$$

Here we have used the fact that the third derivatives of G_l vanishes, (2.3) and (2.4). It is easy to see that, for any fixed $\varepsilon < 1$,, there is a constant C_m such that $|D_V^\alpha F_l|$ is bounded by

$$|D_V^\alpha F_l| \leq C_m \frac{1}{|V_l|} \exp\Big\{ -\varepsilon^2 \frac{(|\xi - \sum_{j=1}^{l-1} V_j|^2 - |\xi - \sum_{j=1}^{l} V_j|^2)^2}{|V_l|^2} \Big\}.$$

Case B $(m = l)$ In this case, we observe that

$$\partial_{V_l^1} K_l = -\partial_{\xi_*} K(\xi - \sum_{j=1}^{l-1} V_j, \xi - \sum_{j=1}^{l} V_j),$$

and use the same argument as in Case A to complete the proof.

3. One Space Dimension

Equipped with the above lemmas, we are now ready to prove the Mixture

Lemma.

The Proof of Theorem 2.1.

With the explicit form of the solution operator (1.3) for the damped transport equation, the Mixture operator $\mathbb{M}_k^t g_0$ is

$$\mathbb{M}_k^t g_0 = \int_{\mathbb{T}} \int_{\mathbb{R}^{6k}} e^{-\nu(\xi)(t-s_1)} \Big[\prod_{i=1}^{2k} e^{-\nu(\xi_i)(s_i-s_{i+1})} K(\xi_{i-1}, \xi_i) \Big] \times g_0(x - \sum_{i=0}^{2k} \xi_i^1 (s_i - s_{i+1}), \xi_{2k}) d\Xi dS,$$

where we have set $\xi_0 \equiv \xi,\ , s_0 \equiv t,\ , s_{2k+1} \equiv 0$ and used the notations

$$S \equiv (s_1, s_2, \ldots, s_{2k}), \quad \mathbb{T} \equiv [0,t] \times [0, s_1] \times \cdots \times [0, s_{2k-1}].$$

We now change variables $\mathcal{T}$ as introduced in previous section, put ξ as V_0. The key observation is that one should do integrate by parts with respect to V instead of with respect to ξ. This way the differentiations can be evenly distributed to the differentiation with respect to the components of V:

$$\begin{aligned}
&\partial_x^k \mathbb{M}_k^t g_0 \\
&= \int_{\mathbb{T}} \int_{\mathbb{R}^{6k}} e^{-\nu(\xi)(t-s_1)} \Big[\prod_{i=1}^{2k} e^{-\nu(\xi_i)(s_i-s_{i+1})} K(\xi_{i-1}, \xi_i) \Big] \\
&\qquad \times \partial_x^k g_0(x - \sum_{i=0}^{2k} \xi_i^1 (s_i - s_{i+1}), \xi_{2k}) d\Xi dS \\
&= \int_{\mathbb{T}} \int_{\mathbb{R}^{6k}} e^{-\nu(\xi)(t-s_1)} \Big[\prod_{i=1}^{2k} e^{-\nu(\xi - \sum_{j=1}^{i} V_j)(s_i-s_{i+1})} K_i \Big] \frac{1}{s_1 \cdot s_3 \cdots s_{2k-1}} \\
&\qquad \times \sum_{|\alpha|=0, \alpha\in\mathcal{A}}^{k} D^\alpha \partial_{\xi_{2k}^1}^{k-|\alpha|} g_0(x - \sum_{i=0}^{2k} V_i s_i, \xi - \sum_{i=1}^{2k} V_i) dV dS \\
&= \sum_{|\alpha|=0, \alpha\in\mathcal{A}}^{k} (-1)^{|\alpha|} \int_{\mathbb{T}} \int_{\mathbb{R}^{6k}} D^\alpha \Big\{ e^{-\nu(\xi)(t-s_1)} \Big[\prod_{i=1}^{2k} e^{-\nu(\xi - \sum_{j=1}^{i} V_j)(s_i-s_{i+1})} K_i \Big] \Big\} \\
&\qquad \times \frac{1}{s_1 \cdot s_3 \cdots s_{2k-1}} \partial_{\xi_{2k}^1}^{k-|\alpha|} g_0(x - \sum_{i=0}^{2k} V_i s_i, \xi - \sum_{i=1}^{2k} V_i) dV dS. \qquad (3.1)
\end{aligned}$$

With (2.2) and Lemma 2.2, we can easily see that

$$\begin{aligned}
&D^{\alpha}\Big\{e^{-\nu(\xi)(t-s_1)}\Big[\prod_{i=1}^{2k} e^{-\nu(\xi-\sum_{j=1}^{i} V_j)(s_i-s_{i+1})}K_i\Big]\Big\}\\
&\leq\ C_k e^{-\nu_0 t}\Big[\prod_{i=1}^{2k} H_i\Big]\times\Big[\prod_{i=1}^{2k}(s_i-s_{i+1})^k\Big],
\end{aligned}$$

where C_k is a generic constant depending only on k. Thus from the Hölder inequality,

$$\begin{aligned}
&|\partial_x^k \mathbb{M}_k^t g_0|\\
\leq\ & C_k e^{-\nu_0 t}\sum_{\alpha\in\mathcal{A},|\alpha|=0}^{k}\Big(\int_{\mathbb{T}}\int_{\mathbb{R}^{6k}}\Big[\prod_{i=1}^{2k}H_i\Big]\frac{\prod_{i=0}^{2k}(s_i-s_{i+1})^{2k}}{s_1\cdot s_3\cdots s_{2k-1}}dV\,dS\Big)^{\frac{1}{2}}\\
&\times\ \Big(\int_{\mathbb{T}}\int_{\mathbb{R}^{6k}}\frac{1}{s_1\cdot s_3\cdots s_{2k-1}}\Big[\prod_{i=1}^{2k}H_i\Big]|\partial_{\xi^1}^{k-|\alpha|}g_0(x-\sum_{i=0}^{2k}V_is_i,\xi-\sum_{i=1}^{2k}V_i)|^2\\
&dV\,dS\Big)^{\frac{1}{2}}\\
\leq\ & C_k e^{-\frac{\nu_0 t}{2}}\sum_{|\alpha|=0}^{k}\Big(\int_{\mathbb{T}}\int_{\mathbb{R}^{6k}}\frac{\Big[\prod_{i=1}^{2k}H_i\Big]}{s_1\cdot s_3\cdots s_{2k-1}}|\partial_{\xi^1}^{k-|\alpha|}g_0(x-\sum_{i=0}^{2k}V_is_i,\xi-\sum_{i=1}^{2k}V_i)|^2\\
&dV\,dS\Big)^{\frac{1}{2}}, \qquad (3.2)
\end{aligned}$$

Here we have used the integrability of kernel K and $\partial_\xi K$ and the fact that the time integral is of the polynomial order. We now integrate the square of (3.2) over $\mathbb{R}\times\mathbb{R}^3$ to get

$$\|\partial_x^k\mathbb{M}_k^t g_0\|_{L^2_x(L^2_\xi)}\leq C_k e^{\frac{-\nu_0 t}{2}}\sum_{m=0}^{k}\|\partial_{\xi^1}^{k-m}g_0\|_{L^2_x(L^2_\xi)}.$$

This completes the proof of Theorem 2.1. □

Remark 3.1. This result can be generalized to the function space $L^p_x(L^p_\xi)$ for $1\leq p\leq\infty$.

For the case of $1\leq p<\infty$, we apply Hölder inequality to (3.1) with

Hölder conjugate $(\frac{p}{p-1}, p)$ to yield

$$|\partial_x^k \mathbb{M}_k^t g_0| \le C_k e^{-\nu_0 t} \sum_{\alpha\in\mathcal{A},|\alpha|=0}^{k} \Big(\int_{\mathbb{T}}\int_{\mathbb{R}^{6k}} \Big[\prod_{i=1}^{2k} H_i\Big] \frac{\prod_{i=0}^{2k}(s_i - s_{i+1})^{2k}}{s_1\cdot s_3\cdots s_{2k-1}} dV dS\Big)^{1-\frac{1}{p}}$$
$$\times\Big(\int_{\mathbb{T}}\int_{\mathbb{R}^{6k}} \frac{1}{s_1\cdot s_3\cdots s_{2k-1}}\Big[\prod_{i=1}^{2k} H_i\Big]|\partial_{\xi^1}^{k-|\alpha|} g_0(x-\sum_{i=0}^{2k} V_i s_i, \xi - \sum_{i=1}^{2k} V_i)|^p dV dS\Big)^{\frac{1}{p}}. \tag{3.3}$$

We can then apply the same argument as in the above proof with the exponent p to establish the same estimate as in Theorem 2.1 for the spaces $L_x^p(L_\xi^p)$ for $1 \le p < \infty$.

For the case of $p = \infty$, it is easy to see that

$$\begin{aligned}|\partial_x^k \mathbb{M}_k^t g_0| \ \le\ & C_k e^{-\nu_0 t} \sum_{\alpha\in\mathcal{A},|\alpha|=0}^{k} \Big(\int_{\mathbb{T}}\int_{\mathbb{R}^{6k}} \Big[\prod_{i=1}^{2k} H_i\Big] \frac{\prod_{i=0}^{2k}(s_i - s_{i+1})^{k}}{s_1\cdot s_3\cdots s_{2k-1}} dV dS\Big)\\ & \times \|\partial_{\xi^1}^{k-|\alpha|} g_0\|_{L_x^\infty(L_\xi^\infty)}.\end{aligned}$$

4. Three Space Dimensions

For the three dimensional space case, the distribution of space differentiations to that of microscopic differentiations is also done evenly and we have the corresponding notations: For any $\beta \in \mathbb{N}_0^3$ with $|\beta| = k$, one can decompose β into $\beta = \sum_{i=1}^k \beta_i$ so that $\beta_i \in \mathbb{N}^3$ and $|\beta_i| = 1$. Define the set

$$\mathcal{A}^\beta \ \equiv\ \Big\{\alpha \in \mathbb{N}_0^{6k} : \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2k}),\ \alpha_{2j} = 0,\ \alpha_{2j-1} = \beta_j \text{ or } 0, \quad \text{for } j = 1, \ldots, k\Big\}$$

and for $\alpha \in \mathcal{A}^\beta$, denote $\widetilde{\alpha} \equiv \sum_{i=1}^k \alpha_{2i-1} \in \mathbb{N}_0^3$.

A direct computation yields

$$\begin{aligned}&D_x^\beta g_0(x - \sum_{i=0}^{2k} \xi_i^1 (s_i - s_{i+1}), \xi_{2k})\\ &\quad = \frac{1}{s_1\cdot s_3\cdots s_{2k-1}} \sum_{|\alpha|=0,\alpha\in\mathcal{A}^\beta}^{k} D_V^\alpha D_{\xi_{2k}}^{\beta-\widetilde{\alpha}} g_0(x - \sum_{i=0}^{2k} V_i s_i, \xi - \sum_{i=1}^{2k} V_i).\end{aligned}$$

Then we can apply the same argument as for the one space dimensional case to establish

$$\|D_x^{\beta}\mathbb{M}_k^t g_0\|_{L_x^2(L_\xi^2)} \leq C_k e^{\frac{-\nu_0 t}{2}} \sum_{\gamma\in\mathbb{N}_0^3,\gamma^i\leq\beta^i} \|D_\xi^{\gamma} g_0\|_{L_x^2(L_\xi^2)}.$$

As for the one space dimensional case, the estimate also hold for L^p spaces.

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