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ON CERTAIN SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS

BY

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Abstract

In this paper, we introduce new classes, $MB_k(\alpha, \lambda, q, s, \rho)$ and $MT_k(\alpha, \lambda, q, s, \rho)$ of meromorphic functions defined by using a meromorphic analougue of the Choi-Saigo-Srivastava operator for the generalized hypergeometric function and investigate a number of inclusion relationships of these classes. We also derive some interesting properties of these classes, which also includes radius problem for the class $MB_k(\alpha, \lambda, q, s, \rho)$.

1. Introduction

Let M denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

which are analytic in the punctured unit disk

$$E^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = E \setminus \{0\}.$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \tag{1.1}$$

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where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [16]. We note that $P_k(0) = P_k$, see [15], $P_2(\rho) = P(\rho)$, the class of analytic functions with positive real part greater than ρ and $P_2(0) = P$, the class of functions with positive real part. From (1.1) we can write $p \in P_k(\rho)$ as

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z), \tag{1.2}$$

where $p_i(z) \in P(\rho)$, $i = 1, 2$ and $z \in E$.

For complex parameters

$$\alpha_1, \ldots, \alpha_q \text{ and } \beta_1, \ldots, \beta_s \quad (\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}; \quad j = 1, \ldots, s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; z)$ [13, 18] as follows:

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k z^k}{(\beta_1)_k \cdots (\beta_q)_k k!},$$
$$(q \leq s+1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1, 2, \ldots\}; z \in E),$$

where $(v)_k$ is the Pochhammer symbol (or the shifted factorial) defined in (terms of the Gamma function) by

$$(v)_k := \frac{\Gamma(v+\kappa)}{\Gamma(v)} = \begin{cases} 1 & \text{if } k = 0 \text{ and } v \in \mathbb{C} \setminus \{0\}, \\ v(v+1) \cdots (v+\kappa-1) & \text{if } k \in \mathbb{N} \text{ and } v \in \mathbb{C}. \end{cases}$$

Corresponding to a function $\mathscr{F}\,(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$\mathscr{F}\,(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-1}\ {}_q\mathscr{F}_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \tag{1.3}$$

Liu and Srivastava [8] considered a linear operator $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : M \to M$ defined by the following Hadamard product (or convolution):

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = \mathscr{F}\,(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). \tag{1.4}$$

We note that the linear operator $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)$ was motivated essentially by Dziok and Srivastava [2]. Some interesting developments with the generalized hypergeometric function were considered recently by Dzoik and Srivastava [3, 4] and Liu and Srivastava [6, 7].

Corresponding to the function $\mathscr{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ defined by (1.3), we introduce a function $\mathscr{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ given by

$$\begin{aligned}&\mathscr{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*\mathscr{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1\ldots,\beta_s;z)\\ &\quad=\frac{1}{z(1-z)^\lambda}\quad(\lambda>0).\end{aligned}\tag{1.5}$$

Analogous to $H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$ defined by (1.4), we now define the linear operator $H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$ on M as follows:

$$\begin{aligned}&H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=\mathscr{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z)\\ &\quad(\alpha_i,\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ i=1,\ldots,q;\ j=1,\ldots,s;\ \lambda>0;z\in E^*;\ f\in M).\end{aligned}\tag{1.6}$$

For convenience, we write

$$H_{\lambda,q,s}(\alpha_1):=H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s).$$

It is easily verified from the definition (1.5) and (1.6) that

$$z(H_{\lambda,q,s}(\alpha_1+1)f(z))'=\alpha_1H_{\lambda,q,s}(\alpha_1)f(z)-(\alpha_1+1)H_{\lambda,q,s}(\alpha_1+1)f(z),\tag{1.7}$$

and

$$z(H_{\lambda,q,s}(\alpha_1)f(z))'=\lambda H_{\lambda+1,q,s}(\alpha_1)f(z)-(\lambda+1)H_{\lambda,q,s}(\alpha_1)f(z).\tag{1.8}$$

We note that the operator $H_{\lambda,q,s}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator [1] for analytic functions, which includes the integral operator studied by Liu [5] and Noor et al. [10, 11].

Next by using the operator $H_{\lambda,q,s}(\alpha_1)$, we introduce some new classes of meromorphic functions.

Definition 1.1. Let $f\in M$. Then $f\in MB_k(\alpha,\lambda,q,s,\rho)$, if and only if

$$-(1-\alpha)z^2(H_{\lambda,q,s}(\alpha_1)f(z))'-\alpha z^2(H_{\lambda+1,q,s}(\alpha_1)f(z))'\in P_k(\rho),\quad z\in E,$$

where $\alpha>0$, $k\geq 2$ and $0\leq\rho<1$.

Definition 1.2. Let $f\in M$. Then $f\in MT_k(\alpha,\lambda,q,s,\rho)$, if and only if

$$(1-\alpha)z(H_{\lambda,q,s}(\alpha_1)f(z))+\alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))\in P_k(\rho),\quad z\in E,$$

where $\alpha > 0$, $k \geq 2$ and $0 \leq \rho < 1$.

2. Preliminary Results

Lemma 2.1.([17]). *If $p(z)$ is analytic in E with $p(0) = 1$, and if λ_1 is a complex number satisfying* $\text{Re}(\lambda_1) \geq 0$ $(\lambda_1 \neq 0)$, *then*

$$\text{Re}\{p(z) + \lambda_1 z p'(z)\} > \beta \quad (0 \leq \beta < 1).$$

Implies

$$\text{Re } p(z) > \beta + (1-\beta)(2\gamma - 1),$$

where γ is given by

$$\gamma = \gamma(\text{Re}\lambda_1) = \int_0^1 (1 + t^{\text{Re}\lambda_1})^{-1} dt,$$

which is an increasing function of Re(λ_1) and $\frac{1}{2} \leq \gamma < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.2. ([19]). *If $p(z)$ is analytic in E, $p(0) = 1$ and* $\text{Re } p(z) > \frac{1}{2}$, *$z \in E$, then for any function F analytic in E, the function $p * F$ takes values in the convex hull of the image of E under F.*

Lemma 2.3. (cf., e.g., Pashkouleva [14]). *Let $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in P(\rho)$. Then*

$$\text{Re } p(z) \geq 2\rho - 1 + \frac{2(1-\rho)}{1 + |z|}.$$

3. Main Results

Theorem 3.1. *Let $f \in MB_k(\alpha, \lambda, q, s, \rho)$. Then*

$$-z^2(H_{\lambda,q,s}(\alpha_1 f(z))' \in P_k(\rho_1),$$

where ρ_1 is given by

$$\rho_1 = \rho + (1-\rho)(2\gamma - 1), \tag{3.1}$$

and

$$\gamma = \int_0^1 \left(1 + t^{\mathrm{Re}(\frac{\alpha}{\lambda})}\right)^{-1} dt.$$

Proof. Let

$$-z^2(H_{\lambda,q,s}(\alpha_1)f(z))' = p(z) = \Big(\frac{k}{4} + \frac{1}{2}\Big)p_1(z) - \Big(\frac{k}{4} - \frac{1}{2}\Big)p_2(z). \qquad (3.2)$$

Then $p(z)$ is analytic in E with $p(0) = 1$. Applying the identity (1.8) in (3.2) and differentiating the resulting equation with respect to z, we have

$$-(1-\alpha)z^2(H_{\lambda,q,s}(\alpha_1)f(z))' - \alpha z^2(H_{\lambda+1,q,s}(\alpha_1)f(z))' = \Big\{p(z) + \frac{\alpha}{\lambda}zp'(z)\Big\}.$$

Since $f \in MB_k(\alpha,\lambda,q,s,\rho)$, so $\Big\{p(z) + \frac{\alpha}{\lambda}zp'(z)\Big\} \in P_k(\rho)$ for $z \in E$. This implies that

$$\mathrm{Re}\Big\{p_i(z) + \frac{\alpha}{\lambda}zp_i'(z)\Big\} > \rho, \quad i = 1,2.$$

Using Lemma 2.1, we see that $\mathrm{Re}\{p_i(z)\} > \rho_1$, where ρ_1 is given by (3.1). Consequently $p \in P_k(\rho_1)$ for $z \in E$, and the proof is complete. □

Theorem 3.2. *Let $f \in MB_k(0,\lambda,q,s,\rho)$ for $z \in E$. Then $f \in MB_k(\alpha,\lambda,q,s,\rho)$ for $|z| < R(\alpha,\lambda)$, where*

$$R(\alpha,\lambda) = \frac{\lambda}{\alpha + \sqrt{a^2 + \lambda^2}}\,. \qquad (3.3)$$

Proof. Set

$$-z^2(H_{\lambda,q,s}(\alpha_1)f(z))' = (1-\rho)h(z) + \rho, \quad h \in P_k.$$

Now proceeding as in Theorem 3.1, we have

$$\begin{aligned} &-(1-\alpha)z^2(H_{\lambda,q,s}(\alpha_1)f(z))' - \alpha z^2(H_{\lambda+1,q,s}(\alpha_1)f(z))' - \rho \\ &= (1-\rho)\Big\{h(z) + \frac{\alpha}{\lambda}zh'(z)\Big\} \\ &= (1-\rho)\Big[\Big(\frac{k}{4}+\frac{1}{2}\Big)\Big\{h_1(z) + \frac{\alpha}{\lambda}zh_1'(z)\Big\} - \Big(\frac{k}{4}-\frac{1}{2}\Big)\Big\{h_2(z) + \frac{\alpha}{\lambda}zh_2'(z)\Big\}\Big], \end{aligned} \qquad (3.4)$$

where we have used (1.2) and $h_1, h_2 \in P$, $z \in E$. Using the following well known estimate [9]

$$|zh_i'(z)| \leq \frac{2r}{1-r^2}\text{Re}\{h_i(z)\}, \quad (|z| = r < 1), \quad i = 1, 2,$$

we have

$$\begin{aligned}\text{Re}\left\{h_i(z) + \frac{\alpha}{\lambda}zh_i'(z)\right\} &\geq \text{Re}\left\{h_i(z) - \frac{\alpha}{\lambda}|zh_i'(z)|\right\} \\ &\geq \text{Re}h_i(z)\left\{1 - \frac{2\alpha r}{\lambda(1-r^2)}\right\}.\end{aligned}$$

The right hand side of this inequality is positive if $r < R(\alpha, \lambda)$, where $R(\alpha, \lambda)$ is given by (3.3). Consequently it follows from (3.4) that $f \in MB_k(\alpha, \lambda, q, s, \rho)$ for $|z| < R(\alpha, \lambda)$. Sharpness of this result follows by taking $h_i(z) = \dfrac{1+z}{1-z}$ in (3.4), $i = 1, 2$. □

Theorem 3.3. *Let $f \in MB_k(0, \lambda, q, s, \rho)$ and let*

$$\mathcal{F}_\delta(f)(z) = \frac{\delta}{z^{\delta+1}}\int_0^z t^\delta f(t)dt \quad (\delta > 0, \ z \in E^*). \tag{3.5}$$

Then

$$-z^2(H_{\lambda,q,s}(\alpha_1)\mathcal{F}(f)(z))' \in P_k(\rho_2),$$

where ρ_2 is given by

$$\rho_2 = \rho + (1-\rho)(2\gamma_1 - 1), \tag{3.6}$$

and

$$\gamma_1 = \int_0^1 \left(1 + t^{\text{Re}(\frac{1}{\delta})}\right)^{-1} dt.$$

Proof. Setting

$$-z^2(H_{\lambda,q,s}(\alpha_1)\mathcal{F}(f)(z))' = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z). \tag{3.7}$$

Then $p(z)$ is analytic in E with $p(0) = 1$. Using the following operator identity:

$$z(H_{\lambda,q,s}(\alpha_1)\mathcal{F}(f)(z))' = \delta(H_{\lambda,q,s}(\alpha_1)f(z)) - (\delta+1)(H_{\lambda,q,s}(\alpha_1)\mathcal{F}(f)(z)) \tag{3.8}$$

in (3.7), and differentiating the resulting equation with respect to z, we find that

$$-z^2(H_{\lambda,q,s}(\alpha_1)f(z))' = \left\{p(z) + \frac{1}{\delta}zp'(z)\right\} \in P_k(\rho) \quad \text{for} \quad z \in E.$$

Using Lemma 2.1, we see that $-z^2(H_{\lambda,q,s}(\alpha_1)\mathcal{F}(f)(z))' \in P_k(\rho_2)$, for $z \in E$, where ρ_2 is given by (3.6), and the proof is complete. □

Theorem 3.4. *Let $\varphi(z) \in M$ satisfy the inequality:*

$$\mathrm{Re}(z\varphi(z)) > \frac{1}{2} \quad (z \in E). \tag{3.9}$$

Let $f \in MT_k(\alpha,\lambda,q,s,\rho)$. Then $\varphi * f \in MT_k(\alpha,\lambda,q,s,\rho)$.

Proof. Let $G = \varphi * f$. Then

$$\begin{aligned}
&(1-\alpha)z(H_{\lambda,q,s}(\alpha_1)G(z)) + \alpha z(H_{\lambda+1,q,s}(\alpha_1)G(z)) \\
&\quad = (1-\alpha)z(H_{\lambda,q,s}(\alpha_1)(\varphi * f)(z)) + \alpha z(H_{\lambda+1,q,s}(\alpha_1)(\varphi * f)(z)) \\
&\quad = z\varphi(z) * h(z), \qquad h \in P_k(\rho). \\
&\quad = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(1-\rho)(z\varphi(z) * h_1(z)) + \rho\right\} \\
&\qquad -\left(\frac{k}{4}-\frac{1}{2}\right)\left\{(1-\rho)(z\varphi(z) * h_2(z)) + \rho\right\}, \quad h_1, h_2 \in P.
\end{aligned}$$

Since $\mathrm{Re}(z\varphi(z)) > \frac{1}{2}$, $(z \in E)$, and so using Lemma 2.2, we can conclude that $G = \varphi * f \in MT_k(\alpha,\lambda,q,s,\rho)$. □

Theorem 3.5. *Let $\varphi(z) \in M$ satisfy the inequality* (3.9), *and $f \in MB_k(0,\lambda,q,s,\rho)$. Then $\varphi * f \in MB_k(0,\lambda,q,s,\rho)$.*

Proof. We have

$$-z^2(H_{\lambda,q,s}(\alpha_1(\varphi * f)(z))' = -z^2(H_{\lambda,q,s}(\alpha_1)f(z))' * z\varphi(z) \quad (z \in E).$$

Now the remaining part of Theorem 3.5 follows by employing the techniques that we used in proving Theorem 3.4 above. □

Theorem 3.6. *For* $0\le\alpha_2<\alpha_1$, $MT_k(\alpha_1,\lambda,q,s,\rho)\subset MT_k(\alpha_2,\lambda,q,s,\rho)$.

Proof. For $\alpha_2=0$, the proof is immediate. Let $\alpha_2>0$ and let $f\in MT_k(\alpha_1,\lambda,q,s,\rho)$. Then

$$\begin{aligned}
&(1-\alpha_2)z(H_{\lambda,q,s}(\alpha_1)f(z))+\alpha_2 z(H_{\lambda+1,q,s}(\alpha_1)f(z))\\
&=\frac{\alpha_2}{\alpha_1}\Big[\Big(\frac{\alpha_1}{\alpha_2}-1\Big)z(H_{\lambda,q,s}(\alpha_1)f(z))+(1-\alpha_1)(H_{\lambda,q,s}(\alpha_1)f(z))\\
&\quad+\alpha_1(H_{\lambda+1,q,s}(\alpha_1)f(z))\Big]\\
&=\Big(1-\frac{\alpha_2}{\alpha_1}\Big)H_1(z)+\frac{\alpha_2}{\alpha_1}H_2(z),\quad H_1,H_2\in P_k(\rho).
\end{aligned}$$

Since $P_k(\rho)$ is a convex set, see [12], we conclude that $f\in MT_k(\alpha_2,\lambda,q,s,\rho)$ for $z\in E$. Now by using Theorem 3.1 and the lines of proof of Theorem 3.6 we have the following Theorem. □

Theorem 3.7. *For* $0\le\alpha_2<\alpha_1$, $MB_k(\alpha_1,\lambda,q,s,\rho)\subset MB_k(\alpha_2,\lambda,q,s,\rho)$.

Theorem 3.8. *Let* $f\in MT_k(\alpha,\lambda,q,s,\rho_3)$ *and* $g\in MT_k(\alpha,\lambda,q,s,\rho_4)$ *and let* $F=f*g$. *Then* $F\in MT_k(\alpha,\lambda,q,s,\rho_5)$, *where*

$$\rho_5=1-4(1-\rho_3)(1-\rho_4)\left[1-\frac{\lambda}{\alpha}\int_0^1\frac{u^{(\frac{\lambda}{(1-\alpha)})-1}}{1+u}du\right].\tag{3.10}$$

This result is sharp.

Proof. Since $f\in MT_k(\alpha,\lambda,q,s,\rho_3)$, it follows that

$$S(z)=(1-\alpha)z(H_{\lambda,q,s}(\alpha_1)f(z))+\alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))\in P_k(\rho_3),$$

and so using identity (1.8) in the above equation, we have

$$H_{\lambda,q,s}(\alpha_1)f(z)=\frac{\lambda}{(\alpha)}z^{-1-\frac{\lambda}{(\alpha)}}\int_0^z t^{\frac{\lambda}{(\alpha)}-1}S(t)dt.\tag{3.11}$$

$$H_{\lambda,q,s}(\alpha_1)g(z)=\frac{\lambda}{(\alpha)}z^{-1-\frac{\lambda}{(\alpha)}}\int_0^z t^{\frac{\lambda}{(\alpha)}-1}S(t)dt.\tag{3.12}$$

where $S^*(z) \in P_k(\rho_4)$.

Using (3.11) and (3.12), we have

$$H_{\lambda,q,s}(\alpha_1)F(z) = \frac{\lambda}{(\alpha)} z^{-1-\frac{\lambda}{(\alpha)}} \int_0^z t^{\frac{\lambda}{(\alpha)}-1} Q(t)dt, \tag{3.13}$$

where

$$\begin{aligned} Q(z) &= \Big(\frac{k}{4}+\frac{1}{2}\Big)q_1(z) - \Big(\frac{k}{4}+\frac{1}{2}\Big)q_2(z), \\ &= \frac{\lambda}{(\alpha)} z^{-\frac{\lambda}{(\alpha)}} \int_0^z t^{\frac{\lambda}{(\alpha)}-1}(S * S^*)(t)dt. \end{aligned} \tag{3.14}$$

Now

$$\begin{aligned} S(z) &= \Big(\frac{k}{4}+\frac{1}{2}\Big)s_1(z) - \Big(\frac{k}{4}-\frac{1}{2}\Big)s_2(z), \\ S^*(z) &= \Big(\frac{k}{4}+\frac{1}{3}\Big)s_1^*(z) - \Big(\frac{k}{4}-\frac{1}{2}\Big)s_2^*(z), \end{aligned} \tag{3.15}$$

where $s_i \in P(\rho_3)$ and $s_i^* \in P(\rho_4)$, $i = 1, 2$.

Since

$$P_i^*(z) = \frac{s_i^*(z) - \rho_4}{2(1-\rho_4)} + \frac{1}{2} \in P\Big(\frac{1}{2}\Big), \quad i = 1, 2,$$

we obtain that $(s_i * p_i^*)(z) \in P(\rho_3)$, by using the Herglots formula. Thus

$$(s_i * s_i^*) \in P(\rho_5)$$

with

$$\rho_5 = 1 - 2(1-\rho_3)(1-\rho_4). \tag{3.16}$$

Using (3.13), (3.14), (3.15), (3.16) and Lemma 2.3, we have

$$\begin{aligned} \text{Re } q_i(z) &= \frac{\lambda}{(\alpha)} \int_0^1 u^{\frac{\lambda}{(\alpha)}-1} \text{Re}\{(s_i * s_i^*)(uz)\}du \\ &\geq \frac{\lambda}{(\alpha)} \int_0^1 u^{\frac{\lambda}{(\alpha)}-1}\Big(2\rho_5 - 1 + \frac{2(1-\rho_5)}{1+u|z|}\Big)du \\ &\geq \frac{\lambda}{(\alpha)} \int_0^1 u^{\frac{\lambda}{(\alpha)}-1}\Big(2\rho_5 - 1 + \frac{2(1-\rho_5)}{1+u}\Big)du \end{aligned}$$

$$= 1-4(1-\rho_3)(1-\rho_4)\left[1-\frac{\lambda}{(\alpha)}\int_0^1 \frac{u^{\frac{\lambda}{(\alpha)}-1}}{1+u}du\right].$$

From this we conclude that $F \in MT_k(\alpha, \lambda, q, s, \rho_5)$, where ρ_5 is given by (3.10). We discuss the sharpness as follows:

We take

$$\begin{aligned} S(z) &= \Big(\frac{k}{4}+\frac{1}{2}\Big)\frac{1+(1-2\rho_3)z}{1-z} - \Big(\frac{k}{4}-\frac{1}{2}\Big)\frac{1-(1-2\rho_3)z}{1+z}, \\ S^*(z) &= \Big(\frac{k}{4}+\frac{1}{2}\Big)\frac{1+(1-2\rho_4)z}{1-z} - \Big(\frac{k}{4}-\frac{1}{2}\Big)\frac{1-(1-2\rho_4)z}{1+z}, \end{aligned}$$

Since

$$\left(\frac{1+(1-2\rho_3)z}{1-z}\right)*\left(\frac{1+(1-2\rho_4)z}{1-z}\right) = 1-4(1-\rho_3)(1-\rho_4)+\frac{4(1-\rho_3)(1-\rho_4)}{1-z},$$

it follows from (3.14) that

$$\begin{aligned} q_i(z) &= \frac{\lambda}{(\alpha)}\int_0^1 u^{\frac{\lambda}{(\alpha)}-1}\left\{1-4(1-\rho_3)(1-\rho_4)+\frac{4(1-\rho_3)(1-\rho_4)}{1-z}\right\}du \\ &\longrightarrow 1-4(1-\rho_3)(1-\rho_4)\left\{1-\frac{\lambda}{(\alpha)}\int_0^1 \frac{u^{\frac{\lambda}{(\alpha)}-1}}{1+u}du\right\} \quad \text{as} \quad z\to -1. \end{aligned}$$

This completes the proof. □

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