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NON-OSCILLATION OF SOLUTIONS OF SECOND AND THIRD ORDER DIFFERENCE EQUATIONS

BY

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Abstract

In this paper, sufficient conditions are obtained for non-oscillation of all solutions of a class of linear homogeneous second order difference equations. These results are used to get sufficient conditions for non-oscillation of all solutions of a class of linear homogeneous third order difference equations. An attempt is made to obtain necessary conditions for non-oscillation of solutions of second and third order difference equations. Many unsolved problems are stated.

1. Introduction

Linear difference equations of second order has been the subject of study for last several years. A good deal of attention is paid to the oscillation/non-oscillation of solutions of such equations. Although difference equations of the form

$$y_{n+2} + p_n y_{n+1} + q_n y_n = 0 \tag{1.1}$$

are viewed as the discrete analogue of differential equations

$$y'' + p(t)y' + q(t)y = 0, \tag{1.2}$$

all the properties of solutions of (1.2) cannot be carried over to (1.1). If $p(t)$

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and $q(t)$ are continuous, then (1.2) can always be put in the self-adjoint form

$$(r_1(t)y')' + q_1(t)y = 0.$$

However, (1.1) can be put in the self-adjoint form (see p.252, [5])

$$\Delta(p^*_{n-1}\Delta y_{n-1}) + q^*_n y_n = 0,$$

if $q_n > 0$, where Δ denotes the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$. We may note that the Fibonacci equation

$$y_{n+2} - y_{n+1} - y_n = 0 \tag{1.3}$$

cannot be put in the self-adjoint form. Linear difference equations of third order is being studied vigorously in recent years. In this paper, sufficient conditions are obtained for non-oscillation of all solutions of a class of linear second order difference equations and these results are used to obtain sufficient conditions for non-oscillation of linear third order difference equations.

By a solution of (1.1), we mean a sequence $\{y_n\}, n \geq 0$, of real numbers which satisfies the recurrence relation (1.1). A solution $\{y_n\}$ of (1.1) is said to be nontrivial if, for every integer $N > 0$, there exists an integer $n_0 > N$ such that $y_{n_0} \neq 0$. By a solution of (1.1) we always mean a nontrivial solution. A solution $\{y_n\}$ of (1.1) is said to be non-oscillatory if there exists an integer $N_0 > 0$ such that $y_n > 0$ or < 0 for $n \geq N_0$; otherwise, $\{y_n\}$ is called oscillatory. In that case, for every integer $N > 0$, we can find an integer $m > N$ such that $y_{m-1}y_m \leq 0$. Equation (1.1) is said to be non-oscillatory if all its solutions are non-oscillatory. It is said to be weakly non-oscillatory if it admits a non-oscillatory solution. It is said to be oscillatory if all its solutions are oscillatory. Such a definition is needed in view of the fact that the Fibonacci equation (1.3) admits both oscillatory and non-oscillatory solutions;$\{(\frac{1+\sqrt{5}}{2})^n\}$ is the positive solution and $\{(\frac{1-\sqrt{5}}{2})^n\}$ is the oscillatory solution of (1.3). Such a situation does not arise in case of linear second order ordinary differential equations or self-adjoint linear second order difference equations in view of Sturm's separation theorem. These definitions hold good for linear third order difference equations. However, self-adjoint linear third order difference equations are equations with constant coefficients (see [7]) and Sturm's separation like theorem is not available for linear third order difference equations. There are third order difference equations all of whose solutions are non-oscillatory or oscillatory or where both oscillatory and non-oscillatory solutions coexist. Moreover, a solution $\{y_n\}$ of (1.1) is said to have a simple zero at $n_0 \geq 0$ if $y_{n_0} = 0$. It is said to have a sign-changing zero at $n_0 \geq 1$ if $y_{n_0-1}y_{n_0} < 0$. The solution $\{y_n\}$ of (1.1) is said to have a generalized zero at n_0 if $n_0 \geq 0$ is a simple zero or $n_0 \geq 1$ is a sign-changing zero.

There are some results concerning necessary and sufficient conditions for non-oscillation of (1.2) in [14]. It seems that there are not many results available in the literature concerning necessary and sufficient conditions for oscillation/non-oscillation of (1.1). Oscillation/nonoscillation of solutions of self-adjoint linear second order homogeneous difference equations is briefly discussed in [1, p.320]. The equations considered in this work are more general than self-adjoint equations. Moreover, simple observation of the equation can predict the nonoscillation of all solutions of the equation by our results. But the result concerning nonoscillation of solutions in [1] requires computation. In [2, 4], sufficient conditions for non-oscillation of a class of self-adjoint second order linear difference equations are obtained. However, neither the methods nor the results in these papers can be used to obtain sufficient conditions for non-oscillation of third order linear difference equations. In [5], some sufficient conditions for oscillation and necessary conditions for nonoscillation of self-adjoint second order linear homogeneous difference equations are given. However, our conditions are different from those conditions. In [15], linear third order difference equations with constant coefficients of the form

$$y_{n+3} + ry_{n+2} + qy_{n+1} + py_n = 0, n \geq 0, \tag{1.4}$$

is studied in great detail and some of the results concerning oscillation of (1.4) are generalized to corresponding third order difference equations with variable coefficients in [9, 10]. However, no result in [15] concerning nonoscillation of (1.4) could yet be generalized to variable coefficients. In the following we state some of these results which will be used in Section 3 to verify that the examples cited satisfy them although the nonoscillation results obtained in this paper for equations with variable coefficients are not the generalization of these results.

Proposition 1.1. *Suppose that* $p < 0$, $q = \frac{r^2}{3}$, $r \neq 0$ *and*

$$p - \frac{qr}{3} + \frac{2r^3}{27} = 0. \tag{1.5}$$

Then (1.4)*is nonoscillatory.*

Proposition 1.2. *Let* $p < 0$, $q < \frac{r^2}{3}$, $r < 0$ *and*

$$0 < p - \frac{qr}{3} + \frac{2r^3}{27} \leq \frac{2}{3\sqrt{3}}(\frac{r^2}{3} - q)^{3/2}$$

Then (1.4) *is nonoscillatory.*

Proposition 1.3. *If* $p < 0$, $r < 0$, $\frac{p}{r} \leq q < \frac{r^2}{3}$ *and*

$$\frac{2}{3\sqrt{3}}(\frac{r^2}{3} - q)^{3/2} \geq \frac{qr}{3} - p - \frac{2r^3}{27} > 0,$$

then (1.4) *is nonoscillatory.*

Proposition 1.4. *If* $p < 0$, $0 < q < \frac{r^2}{3}$, $r < 0$ *and* (1.5) *holds, then* (1.4) *is nonoscillatory.*

Due to non-availability of results providing sufficient conditions for nonoscillation of all solutions of

$$y_{n+3} + r_n y_{n+2} + q_n y_{n+1} + p_n y_n = 0,$$

several authors obtained sufficient conditions for the existence of a nonoscillatory solution of the equation. Usually it is done through fixed point theorems. Then the equation is called nonoscillatory. In this paper, sufficient conditions are obtained for nonoscillation of all solutions of a class of linear third order difference equations.

2. Nonoscillation of Second Order Difference Equations

In this section, we obtain sufficient conditions as well as necessary conditions for nonoscillation of a class of difference equations of second order. These results are used to obtain sufficient conditions for nonoscillation of third order difference equations in Section 3. We begin with a result from [11]. For completeness, the proof of the theorem is given.

Theorem 2.1. *If $q_n < 0$ and p_n does not change sign for large n, then each of the equations*

$$y_{n+2} + p_n y_{n+1} + q_n y_n = 0, \ \ n \geq 0, \tag{2.1}$$

and

$$x_{n+2} - p_n x_{n+1} + q_n x_n = 0, \ \ n \geq 0, \tag{2.2}$$

admits both oscillatory and nonoscillatory solutions.

Proof. Setting $y_n = (-1)^n x_n, n \geq 0$, we obtain

$$y_{n+2} + p_n y_{n+1} + q_n y_n = (-1)^n (x_{n+2} - p_n x_{n+1} + q_n x_n).$$

Hence $\{y_n\}$ is a solution of (2.1) if and only if $\{x_n\}$ is a solution of (2.2). Moreover, if (2.1) has a nonoscillatory solution, then (2.2) has an oscillatory solution and vice versa. Thus, to complete the proof of the theorem, it is enough to show that each of (2.1) and (2.2) admits a nonoscillatory solution.

We consider three cases, viz, $p_n \geq 0$, or $p_n \equiv 0$ or $p_n \leq 0$ for $n \geq n_0 > 0$. Let $p_n \geq 0$ for $n \geq n_0$. For each integer $k \geq 2$, let $\{y_n^{(k)}\}$ be a solution of (2.1) with $y_k^{(k)} > 0$ and $y_{k+1}^{(k)} > 0$. Then $y_n^{(k)} > 0$ for $n \in \{1, 2, \ldots, k-1\}$ because $-q_n y_n = y_{n+2} + p_n y_{n+1}$. Let $\{\{z_n^{(1)}\}, \{z_n^{(2)}\}\}$ be a basis of the solution space of (2.1). It is possible to obtain sequences $\{c_{1k}\}$ and $\{c_{2k}\}$ such that

$$y_n^{(k)} = c_{1k} z_n^{(1)} + c_{2k} z_n^{(2)}$$

with $c_{1k}^2 + c_{2k}^2 = 1$. As $\{c_{i,k}\}$, $i = 1, 2$, is a bounded sequence, there exists a subsequence $\{c_{i,k_j}\}$ such that $c_{i,k_j} \to c_i$ for $j \to \infty$. Setting $y_n = c_1 z_n^{(1)} + c_2 z_n^{(2)}$, we obtain $\{y_n\}$ is a solution of (2.1), $y_n^{(k_j)} \to y_n$ as $j \to \infty$ and $y_n > 0$ for $n \geq 1$. Thus (2.1) has a positive solution. Writing (2.2) as

$$x_{n+2} = p_n x_{n+1} - q_n x_n,$$

we observe that it admits a positive solution $\{x_n\}$ with $x_1 = 0$ and $x_2 > 0$.

If $p_n \le 0$, then one may proceed as above by considering (2.2) first and then (2.1). If $p_n \equiv 0$, then (2.1) and (2.2) reduce to an equation of the form

$$u_{n+2} + q_n u_n = 0$$

which admits a positive solution $\{u_n\}$ with $u_1 > 0$ and $u_2 > 0$. Thus the theorem is proved. □

Remark. Similar results are there in [8]. The Fibonacci Eq. (1.3) illustrates Theorem 2.1.

Theorem 2.2. *If $a_n + 1 > 0$, then all solutions of*

$$\Delta^2 y_n - a_n \Delta y_n = 0, n \ge 0,$$

that is,

$$y_{n+2} - (a_n + 2)y_{n+1} + (a_n + 1)y_n = 0 \tag{2.3}$$

are non-oscillatory.

Proof. The Eq.(2.3) can be put in the self-adjoint form. In view of Theorem 6.5 ([5], p.261), it is enough to show that (2.3) admits a nonoscillatory solution. Let $\{y_n\}$ be a solution of (2.3) with $y_{n_0} = 0$ and $y_{n_0+1} > 0$ for some integer $n_0 \ge 0$. Writing (2.3) as

$$y_{n+2} - y_{n+1} = (a_n + 1)(y_{n+1} - y_n), \tag{2.4}$$

we observe that

$$y_{n_0+2} - y_{n_0+1} = (a_{n_0} + 1)(y_{n_0+1} - y_{n_0}) = (a_{n_0} + 1)y_{n_0+1} > 0$$

implies that $y_{n_0+2} > y_{n_0+1} > 0$. Similarly,

$$y_{n_0+3} - y_{n_0+2} = (a_{n_0+1} + 1)(y_{n_0+2} - y_{n_0+1}) > 0$$

implies that $y_{n_0+3} > y_{n_0+2} > 0$. Proceeding as above we obtain $y_n > 0$ for $n \ge n_0 + 1$. Thus the proof of the theorem is complete. □

Remark. If $a_n + 1 \equiv 0$ for $n \ge 0$, then (2.3) reduces to a first order equation. If $a_n + 1 < 0$, for large n, then (2.3) admits both oscillatory and nonoscillatory solutions due to Theorem 2.1. If $a_n + 1 \geq 0$ but not $\equiv 0$, $n \geq 0$, then (2.3) cannot be put in the self-adjoint form and hence Theorem 6.5 in [5] cannot be applied. We have the following theorem in this case:

Theorem 2.3. *If $a_n + 1 \geq 0$, then all solutions of (2.3) are nonoscillatory and these solutions are monotonic increasing or monotonic decreasing.*

Proof. We may observe that, $\{-y_n\}$ is a solution of (2.3) if $\{y_n\}$ is its solution. If possible, let $\{y_n\}$ be an oscillatory solution of (2.3). Hence, for every integer $N > 0$, we can find an integer $n_0 \geq N$ such that $y_{n_0-1}y_{n_0} \leq 0$. Let $y_{n_0} = 0$. Then $y_{n_0+1} \neq 0$ since $\{y_n\}$ is a nontrivial solution. We may take, without any loss of generality, that $y_{n_0+1} > 0$. Writing (2.3) as (2.4) we obtain

$$y_{n_0+2} - y_{n_0+1} = (a_{n_0} + 1)y_{n_0+1} \geq 0$$

which implies that $y_{n_0+2} \geq y_{n_0+1} > 0$. Similarly,

$$y_{n_0+3} - y_{n_0+2} = (a_{n_0+1} + 1)(y_{n_0+2} - y_{n_0+1}) \geq 0$$

implies that $y_{n_0+3} \geq y_{n_0+2} > 0$. Proceeding as above we obtain $y_n > 0$ for $n \geq n_0 + 1$, a contradiction to the assumption that $\{y_n\}$ is oscillatory. Let $y_{n_0} \neq 0$. Without any loss of generality, we may assume that $y_{n_0} > 0$. Then $y_{n_0-1} \leq 0$. From (2.4) we obtain

$$y_{n_0+1} - y_{n_0} = (a_{n_0-1} + 1)(y_{n_0} - y_{n_0-1}) \geq 0$$

which implies that $y_{n_0+1} \geq y_{n_0} > 0$. Similarly,

$$y_{n_0+2} - y_{n_0+1} = (a_{n_0} + 1)(y_{n_0+1} - y_{n_0}) \geq 0$$

in view of the above inequality. Hence $y_{n_0+2} \geq y_{n_0+1} > 0$. Proceeding as above we get $y_n > 0$ for $n \geq n_0$, a contradiction. Hence all solutions of (2.3) are nonoscillatory. Let $\{y_n\}$ be a nonoscillatory solution of (2.3). For some $m \geq 0$, $y_{m+1} - y_m \geq 0$ or < 0. From (2.4) it follows that $\{y_n\}$ is monotonically increasing for $n \geq m$ in the former case and is monotonically decreasing in the latter case. Thus the theorem is proved. □

Remark. If $a_n + 1 \equiv 0$, then (2.3) is reduced to the first order equation $y_{n+2} - y_{n+1} = 0$. Thus $y_{n_0} > 0$ implies that $y_n > 0$ for $n \geq n_0 + 1$ and $y_{n_0} < 0$ implies that $y_n < 0$ for $n \geq n_0 + 1$.

Example 1. All solutions of $\triangle^2 y_n - \triangle y_n = 0$ are nonoscillatory by Theorem 2.2. Indeed, $\{\{1^n\}, \{2^n\}\}$ is a basis of the solution space of the equation. On expansion, the above equation takes the form $y_{n+2} - 3y_{n+1} + 2y_n = 0$. As per the notations in [2, 3], $c_{n+1} = 1, b_{n+1} = 3$ and $c_n = 2$. Hence the assumption $b_n b_{n+1} \geq 4c_n^2$ is not satisfied. Further, the condition $b_n \geq max\{c_{n-1}, 4c_n\}$ fails to hold. Thus Theorem 6 and its Corollary 1 in [2] cannot be applied to this example. Similarly, the nonoscillation results in [3] cannot be applied to this problem.

Example 2. All solutions of $\triangle^2 y_n - (-1)^n \triangle y_n = 0$ are nonoscillatory by Theorem 2.3. In particular, $y_n = 1$ is a positive solution of the equation. Theorem 2.2 cannot be applied to this example. Results in [2, 3] cannot be applied to this example also because the equation cannot be put in the self-adjoint form.

Remark. Although Theorem 2.3 is stronger than Theorem 2.2, both are retained because the method of proof is different. Moreover, Theorem 2.2 demonstrates that the results in [2, 3] cannot be applied to certain situations to which it can be applied even if the equation can be put in the self-adjoint form.

Corollary 2.4. *If $a_n \geq 0, n \geq 0$ and $\alpha > 0$ is a ratio of odd integers, then all solutions of*

$$\triangle^2 y_n - a_n(\triangle y_n)^\alpha = 0, n \geq 0,$$

that is ,

$$y_{n+2} - 2y_{n+1} + y_n - a_n(y_{n+1} - y_n)^\alpha = 0$$

are nonoscillatory.

One may complete the proof of the corollary by proceeding as in the proof of Theorem 2.3 after writing the given equation in the form

$$y_{n+2} - y_{n+1} = a_n(y_{n+1} - y_n)^\alpha + (y_{n+1} - y_n)$$

Theorem 2.5. *If $b_n \geq 0$ and $a_n - b_n + 1 \geq 0$, then all solutions of*

$$\triangle^2 y_n - a_n \triangle y_n - b_n y_n = 0, n \geq 0,$$

that is,

$$y_{n+2} - (2 + a_n)y_{n+1} + (a_n - b_n + 1)y_n = 0 \tag{2.5}$$

are nonoscillatory.

The proof is similar to that of Theorem 2.3 if (2.5) is written as

$$y_{n+2} - y_{n+1} = (a_n - b_n + 1)(y_{n+1} - y_n) + b_n y_{n+1}$$

Example 3. All solutions of

$$\triangle^2 y_n + \frac{1}{2}\triangle y_n - \frac{1}{3} y_n = 0$$

are nonoscillatory. Indeed, $\{\{(\frac{9+\sqrt{57}}{12})^n\}, \{(\frac{9-\sqrt{57}}{12})^n\}\}$ is a basis of the solution space of the equation.

Theorem 2.6. *If $b_n \leq 0$ and $a_n + b_n - 1 \geq 0, n \geq 0$, then all solutions of* (2.5) *are nonoscillatory.*

Proof. We may note that $b_n \leq 0$ and $a_n + b_n - 1 \geq 0$ imply that $a_n \geq 1$. If possible, let $\{y_n\}$ be an oscillatory solution of (2.5). Hence, for every integer $N > 0$, there exists an integer $n_0 > N$ such that $y_{n_0-1} y_{n_0} \leq 0$. Let $y_{n_0} = 0$. Then $y_{n_0+1} \neq 0$. Without any loss of generality, we may assume that $y_{n_0+1} > 0$. Putting (2.5) in the form

$$y_{n+2} - 2y_{n+1} = a_n(y_{n+1} - y_n) - (1 - b_n)y_n \tag{2.6}$$

we get $y_{n_0+2} - 2y_{n_0+1} = a_{n_0} y_{n_0+1} > 0$ which implies that $y_{n_0+2} > 2y_{n_0+1} > 0$ and $y_{n_0+2} - y_{n_0+1} > y_{n_0+1}$. Similarly

$$\begin{aligned} y_{n_0+3} - 2y_{n_0+2} &= a_{n_0+1}(y_{n_0+2} - y_{n_0+1}) - (1 - b_{n_0+1})y_{n_0+1} \\ &> a_{n_0+1} y_{n_0+1} - (1 - b_{n_0+1})y_{n_0+1} \\ &= (a_{n_0+1} + b_{n_0+1} - 1)y_{n_0+1} \\ &\geq 0 \end{aligned}$$

implies that $y_{n_0+3} > 2y_{n_0+2} > 0$ and $y_{n_0+3} - y_{n_0+2} > y_{n_0+2}$. Proceeding as above we obtain $y_n > 0$ for $n \geq n_0 + 1$, a contradiction.

Let $y_{n_0} \neq 0$. We may take, without any loss of generality, $y_{n_0} > 0$. Hence $y_{n_0-1} \leq 0$. From (2.6) we get

$$\begin{aligned} y_{n_0+1} - 2y_{n_0} &= a_{n_0-1}(y_{n_0} - y_{n_0-1}) - (1 - b_{n_0-1})y_{n_0-1} \\ &> 0 \end{aligned}$$

Hence $y_{n_0+1} > 2y_{n_0} > 0$ and $y_{n_0+1} - y_{n_0} > y_{n_0}$. Similarly,

$$\begin{aligned} y_{n_0+2} - 2y_{n_0+1} &= a_{n_0}(y_{n_0+1} - y_{n_0}) - (1 - b_{n_0})y_{n_0} \\ &> a_{n_0}y_{n_0} - (1 - b_{n_0})y_{n_0} \\ &= (a_{n_o} + b_{n_0} - 1) \\ &\geq 0 \end{aligned}$$

implies that $y_{n_0+2} > 2y_{n_0+1} > 0$ and $y_{n_0+2} - y_{n_0+1} > y_{n_0+1}$. Proceeding as above we obtain $y_n > 0$ for $n \geq n_0$, a contradiction. Thus the proof of the theorem is complete. □

Example 4. All solutions of

$$\triangle^2 y_n - 3\triangle y_n + 2y_n = 0, n \geq 0,$$

are nonoscillatory. The set $\{\{3^n\}, \{2^n\}\}$ is a basis of the solution space of the equation.

Remark. It may be easily seen that all solutions of (2.5) are oscillatory if $a_n \leq -2$ and $a_n - b_n + 1 \geq 0$. Taking into account the assumptions in Theorems 2.1, 2.5 and 2.6, we notice that there is no result in the following cases: (i) $-1 < a_n < 1$, $b_n < 0$, (ii) $-2 < a_n < -1$ and (iii) $a_n + b_n < 1$, $a_n > 1$.

In applications, we often come across equations of the form (2.1). In the following, we state two results in terms of coefficient sequences $\{p_n\}$ and $\{q_n\}$ which follow from Theorems 2.5 and 2.6 respectively. Comparing Eqs.(2.1) and (2.5), we obtain $p_n = -(a_n + 2)$ and $q_n = a_n - b_n + 1$. Hence $a_n = -(p_n + 2)$, $b_n = -(p_n + q_n + 1)$ and $a_n + b_n - 1 = -(2p_n + q_n + 4)$.

Theorem 2.7. *If $p_n + q_n + 1 \leq 0$ and $q_n \geq 0$, then all solutions of* (2.1) *are nonoscillatory.*

Theorem 2.8. *If $p_n + q_n + 1 \geq 0$ and $2p_n + q_n + 4 \leq 0$, then all solutions of* (2.1) *are nonoscillatory.*

Remark. We may note that $p_n + q_n + 1 \geq 0$ and $2p_n + q_n + 4 \leq 0$ imply that $p_n + 3 \leq 0$ and hence $q_n \geq 2$.

Theorem 2.9. *Let p_n and q_n do not change sign for large n. If all solutions of* (2.1) *are nonoscillatory, then $p_n < 0$ and $q_n \geq 0$ for large n.*

Proof. If all solutions of (2.1) are nonoscillatory, then $q_n \geq 0$ for large n in view of Theorem 2.1. If $q_n \equiv 0$, then (2.1) reduces to a first order equation $y_{n+2} + p_n y_{n+1} = 0$. Hence $0 < \frac{y_{n+2}}{y_{n+1}} = -p_n$ implies that $p_n < 0$ for large n. If $\{y_n\}$ is a nonoscillatory solution of (2.1), then we may take, without any loss of generality, that $y_n > 0$ for $n \geq n_0 > 0$. Hence $0 < y_{n+1} = -p_n y_{n+1} - q_n y_n$ for $n \geq n_0$ implies that $0 \leq q_n y_n < -p_n y_n$ and $p_n y_{n+1} < -q_n y_n \leq 0$. From either inequality, it follows that $p_n < 0$ for large n. Thus the theorem is proved. □

Remark. We observe that $p_n + q_n + 1 \leq 0$ and $q_n \geq 0$ imply that $p_n \leq -(1 + q_n) < 0$. It indicates a gap $-(1 + q_n) < p_n < 0$ between Theorem 2.7 concerning sufficient conditions and Theorem 2.9 concerning necessary conditions for nonoscillation of (2.1). It seems that all solutions of (2.1) are oscillatory for $q_n \geq 0$ and $-(1 + q_n) < p_n < 0$. For example, all solutions of $y_{n+2} - y_{n+1} + y_n = 0$ are oscillatory because the roots of its characteristic equation $\lambda^2 - \lambda + 1 = 0$ are given by $\lambda_1 = \frac{(1+i\sqrt{3})}{2}$ and $\lambda_2 = \frac{(1-i\sqrt{3})}{2}$.

Theorem 2.10. *Let $\{a_n + 1\}$ and $\{a_n + 2\}$ do not change sign. All solutions of* (2.3) *are nonoscillatory if and only if $a_n + 1 \geq 0$.*

Proof. If $a_n + 1 \geq 0$, then all solutions of (2.3) are nonoscillatory by Theorem 2.3. On the other hand, if all solutions of (2.3) are nonoscillatory , then $a_n + 1 \geq 0$ and $-(a_n + 2) < 0$, that is, $a_n + 1 \geq 0$ and $a_n + 2 > 0$. Hence $a_n + 1 \geq 0$. This completes the proof of the theorem. □

3. Nonoscillation of Third Order Difference Equations

Nonoscillation of solutions of difference equations of third order is dealt with in this section. Although many results concerning oscillation are known [9, 10, 12, 13], very few results on nonoscillation [12, 13] are available in the literature.

Theorem 3.1. *If $b_n \geq 0$ and $a_n - b_n + 1 \geq 0$, then all solutions of*

$$\triangle^3 y_n - a_n \triangle^2 y_n - b_n \triangle y_n = 0,$$

that is,

$$y_{n+3} - (3 + a_n) y_{n+2} + (3 + 2a_n - b_n) y_{n+1} - (1 + a_n - b_n) y_n = 0 \quad (3.1)$$

are nonoscillatory.

Proof. Let $\{y_n\}$ be a solution of (3.1). Setting $z_n = \triangle y_n$ in (3.1), we get

$$\triangle^2 z_n - a_n \triangle z_n - b_n z_n = 0$$

Hence $\{z_n\}$ is a solution of (2.5). From Theorem 2.5 it follows that $z_n > 0$ or < 0 for $n \geq n_0$, that is, $\triangle y_n > 0$ or < 0 for $n \geq n_0$. Hence $\{y_n\}$ is nonoscillatory. Since $\{y_n\}$ is an arbitrary solution of (3.1), then all solutions of (3.1) are nonoscillatory. Thus the theorem is proved. □

Theorem 3.2. *If $a_n \geq 0$, then all solutions of*

$$\triangle^3 y_n - a_n (\triangle^2 y_n)^\alpha = 0 \quad (3.2)$$

are nonoscillatory, where $\alpha > 0$ is a ratio of odd integers.

The proof follows from Corollary 2.4 if we put $z_n = \triangle y_n$ in (3.2) and proceed as in the proof of Theorem 3.1.

Theorem 3.3. *If $b_n \leq 0$ and $a_n + b_n - 1 \geq 0$, then all solutions of* (3.1) *are nonoscillatory.*

It follows from Theorem 2.6.

Example 5. Consider

$$\triangle^3 y_n - 3\triangle^2 y_n - 2\triangle y_n = 0,$$

that is,

$$y_{n+3} - 6y_{n+2} + 7y_{n+1} - 2y_n = 0 \tag{3.3}$$

From Theorem 3.1 it follows that all solutions of (3.3) are nonoscillatory. The set $\{\{1^n\}, \{(\frac{5+\sqrt{17}}{2})^n\}, \{(\frac{5-\sqrt{17}}{2})^n\}\}$ of solutions of (3.3) is a basis of the solution space of (3.3). Moreover, from Proposition 1.3 it follows that all solutions of (3.3) are nonoscillatory.

Example 6. All solutions of

$$\triangle^3 y_n - 3\triangle^2 y_n + 2\triangle y_n = 0,$$

that is,

$$y_{n+3} - 6y_{n+2} + 11y_{n+1} - 6y_n = 0$$

are nonoscillatory because $\{\{1^n\}, \{2^n\}, \{3^n\}\}$ is a basis of the solution space of the equation. This fact also follows from Theorem 3.3 and Proposition 1.4.

Remark. We may note that the results in Section 2 cannot be applied to the following third order difference equations:

$$\triangle^3 y_n - a_n\triangle^2 y_n - b_n\triangle y_n - c_n y_n = 0,$$

that is,

$$y_{n+3} - (3+a_n)y_{n+2} + (3+2a_n-b_n)y_{n+1} - (1+a_n+c_n-b_n)y_n = 0, \tag{3.4}$$

where $c_n \neq 0$.

Remark. The equation

$$y_{n+3} + r_n y_{n+2} + q_n y_{n+1} + p_n y_n = 0 \tag{3.5}$$

can be put in the form (3.4) if and only if $a_n = -(3+r_n), b_n = -(3+2r_n+q_n)$ and $c_n = -(1+r_n+q_n+p_n)$. Hence $c_n = 0$ if and only if $p_n+q_n+r_n+1 = 0$.

Corollary 3.4. *If* $3 + 2r_n + q_n \leq 0$, $r_n + q_n \geq 0$ *and* $p_n + q_n + r_n + 1 = 0$, *then all solutions of* (3.5) *are nonoscillatory.*

This follows from Theorem 3.1 and the above remark.

Corollary 3.5. *If* $3+2r_n+q_n \geq 0$, $7+3r_n+q_n \leq 0$ *and* $p_n+q_n+r_n+1 = 0$, *then all solutions of* (3.5) *are nonoscillatory.*

This follows from Theorem 3.3 and the above remark.

Example 7. Consider

$$y_{n+3} - \frac{13}{12}y_{n+2} + \frac{3}{8}y_{n+1} - \frac{1}{24}y_n = 0 \tag{3.6}$$

This can be put in the form

$$\triangle^3 y_n + \frac{23}{12}\triangle^2 y_n + \frac{29}{24}\triangle y_n + \frac{1}{4}y_n = 0$$

As $\{\{\frac{1}{2^n}\}, \{\frac{1}{3^n}\}, \{\frac{1}{4^n}\}\}$ forms a basis of the solution space of (3.6), then all solutions of (3.6) are nonoscillatory. All the conditions of Proposition 1.3 are satisfied. Hence all solutions of (3.6) are nonoscillatory. We may note that Propositions 1.1, 1.2 and 1.4 cannot be applied to this example.

Let $\{u_n\}$ be a positive solution of (3.4). The assumption $y_n = u_n x_n$ transforms (3.4) into an equation of the form

$$\triangle^3 x_n - A_n\triangle^2 x_n - B_n\triangle x_n = 0, \tag{3.7}$$

where

$$A_n = -\frac{1}{u_{n+3}}(3\triangle u_{n+2} - a_n u_{n+2}) = -\frac{1}{u_{n+3}}(3u_{n+3} - (a_n + 3)u_{n+2})$$

and

$$B_n = -\frac{1}{u_{n+3}}(3\triangle^2 u_{n+1} - 2a_n\triangle u_{n+1} - b_n u_{n+1})$$

$$= -\frac{1}{u_{n+3}}[3u_{n+3} - 2(a_n + 3)u_{n+2} + (3 + 2a_n - b_n)u_{n+1}]$$

Equation (3.4) may be written as

$$y_{n+3} = R_n y_{n+2} + Q_n y_{n+1} + P_n y_n, \tag{3.8}$$

where $R_n = 3 + a_n, Q_n = (b_n - 2a_n - 3)$ and $P_n = 1 + a_n + c_n - b_n$. Equation (3.8) can be solved in the closed form (see [6]) in terms of the coefficients P_n, Q_n and R_n and hence in terms of the known sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$. Then the solution $\{u_n\}$of (3.4) is known in terms of a_n, b_n and c_n. Thus we have the following theorems:

Theorem 3.6. *If $B_n \geq 0$ and $A_n - B_n + 1 \geq 0$, then all solutions of* (3.4) *are nonoscillatory.*

Proof. From Theorem 3.1 it follows that all solutions of (3.7) are nonoscillatory. Since $y_n = u_n x_n$ and $u_n > 0$, then all solutions of (3.4) are nonoscillatory. Thus the theorem is proved. □

Theorem 3.7. *If $B_n \leq 0$ and $A_n + B_n - 1 \geq 0$, then all solutions of* (3.4) *are nonoscillatory.*

The proof follows from Theorem 3.3 if we proceed as in the proof of Theorem 3.6.

Remark. We may note that

$$A_n - B_n + 1 = \frac{1}{u_{n+3}}[u_{n+3} - (a_n + 3)u_{n+2} + (3 + 2a_n - b_n)u_{n+1}]$$

and

$$A_n + B_n - 1 = -\frac{1}{u_{n+3}}[7u_{n+3} - 3(a_n + 3)u_{n+2} + (3 + 2a_n - b_n)u_{n+1}]$$

Remark.(i) In Example 7, $a_n = -\frac{23}{12}, b_n = -\frac{29}{24}$ and $c_n = -\frac{1}{4}$. For this example, we take $u_n = \frac{1}{3^n} > 0$. Then

$$3u_{n+3} - 2(a_n + 3)u_{n+2} + (3 + 2a_n - b_n)u_{n+1} = \frac{1}{3^{n+1}}[\frac{1}{3} - \frac{13}{8} + \frac{9}{24}]$$

$$= -\frac{1}{72 \times 3^{n+1}} < 0.$$

implies that $B_n > 0$. Further,

$$\begin{aligned} u_{n+3} - (a_n + 3)u_{n+2} + (3 + 2a_n - b_n)u_{n+1} &= \frac{1}{3^{n+1}}[\frac{1}{9} - \frac{13}{36} + \frac{9}{24}] \\ &= \frac{1}{8 \times 3^{n+1}} > 0 \end{aligned}$$

implies that $A_n - B_n + 1 > 0$. From Theorem 3.6 it follows that all solutions of (3.6) are nonoscillatory which is already established in two different ways in Example 7.

(ii) Considering Example 7 and $u_n = \frac{1}{4^n} > 0$, we obtain

$$\begin{aligned} 3u_{n+3} - 2(a_n + 3)u_{n+2} + (3 + 2a_n - b_n)u_{n+1} &= \frac{1}{4^{n+1}}[\frac{3}{16} - \frac{13}{24} + \frac{9}{24}] \\ &= \frac{1}{4^{n+1} \times 48} > 0 \end{aligned}$$

Hence $B_n < 0$. Further, $7u_{n+3} - 3(a_n + 3)u_{n+2} + (3 + 2a_n - b_n)u_{n+1} = \frac{1}{4^{n+1}}[\frac{7}{16} - \frac{13}{16} + \frac{9}{24}] = 0$ implies that $A_n + B_n - 1 = 0$. Hence all solutions of (3.6) are nonoscillatory by Theorem 3.7.

Remark. The substitution $y_n = u_n x_n$ transforms (3.4) into an equation of the form

$$\triangle^3 x_n - B_n \triangle x_n - C_n x_n = 0, \tag{3.9}$$

if $\{u_n\}$ is a positive solution of the equation

$$3\triangle z_{n+2} - a_n z_{n+2} = 0, \tag{3.10}$$

where $B_n = -\frac{1}{u_{n+3}}(3\triangle^2 u_{n+1} - 2a_n \triangle u_{n+1} - b_n u_{n+1})$ and $C_n = -\frac{1}{u_{n+3}}(\triangle^3 u_n - a_n \triangle^2 u_n - b_n \triangle u_n - c_n u_n)$ In (3.9), $\triangle^2 x_n$ term is absent. Sometimes it is useful to study such equations. Since $\{u_n\}$ is a solution of (3.10), then $3u_{n+3} - (a_n + 3)u_{n+2} = 0$. Hence $u_n = \frac{1}{3} u_{n_0} \prod_{i=n_0-2}^{n-3}(a_i + 3), n \geq n_0 + 1, n_0 > 0$.

Theorem 3.8. *If $b_n = 2(a_n + 2)$, then each of the equations* (3.1) *and*

$$x_{n+3} + (3 + a_n)x_{n+2} + (3 + 2a_n - b_n)x_{n+1} + (1 + a_n - b_n)x_n = 0 \tag{3.11}$$

admits both oscillatory and nonoscillatory solutions.

Proof. The substitution $y_n = (-1)^n x_n$ reduces (3.1) to (3.11). Hence $\{y_n\}$ is a solution of (3.1) if and only if $\{x_n\}$ is a solution of (3.11). Clearly, $\{y_n\} = \{1\}$ is a solution of (3.1).In view of the assumption $b_n = 2(a_n+2)$, $\{x_n\} = \{1\}$ is a solution of (3.11). Hence both (3.1) and (3.11) admit nonoscillatory solution $\{1\}$ and oscillatory solution $\{(-1)^n\}$. Thus the theorem is proved. □

Corollary 3.9. *If $p_n + r_n = 0$ and $q_n = -1$, then* (3.5) *admits both oscillatory and nonoscillatory solutions.*

Proof. Equation (3.5) can be put in the form (3.1) for $a_n = -3 - r_n$, $b_n = -(3 + 2r_n + q_n)$ and $p_n + q_n + r_n + 1 = 0$. As $p_n + r_n = 0$ and $q_n = -1$ imply that $b_n = 2(a_n + 2)$, then the corollary follows from Theorem 3.6. □

Example 8. The equation

$$y_{n+3} + y_{n+2} - y_{n+1} - y_n = 0$$

admits both oscillatory and nonoscillatory solutions by Corollary 3.7. Clearly, $\{1^n\}$ and $\{(-1)^n\}$ are solutions of the equation.

Corollary 3.10. *If all solutions of* (3.1) *are nonoscillatory, then $b_n \neq 2(a_n + 2)$.*

This follows from Theorem 3.8.

Corollary 3.11. *If all solutions of* (3.5) *are nonoscillatory, then $p_n + r_n \neq 0$ or $q_n \neq -1$.*

This follows from Corollary 3.7.

4. Summary

There are several problems which remain unresolved. The case that $\{p_n\}$ in Theorem 2.1 is sign-changing could not be handled by the present techniques. This would simplify the necessary and sufficient conditions for nonoscillation of linear second order difference equations. Sufficient conditions for nonoscillation of all solutions of linear third order difference equations (3.5) which could be of the type given in Propositions 1.1−1.4 in case of constant coefficients could not be formulated. Although necessary conditions as well as sufficient conditions are obtained for nonoscillaton of solutions of third order equations, neither conditions are both necessary and sufficient. It is desirable to obtain easily verifiable sufficient conditions for nonoscillation of (3.4). It seems that the techniques employed here are not adequate to handle nonoscillation of forced second/third order difference equations.

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