Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **6** (2011), No. 1, pp. 41-72

SOME LINKS OF BALANCING AND COBALANCING NUMBERS WITH PELL AND ASSOCIATED PELL NUMBERS

G. K. PANDA[1,a] AND PRASANTA KUMAR RAY[2,b]

[1]National Institute of Technology, Rourkela -769 008, Orissa, India.

[a]E-mail: gkpanda_nit@rediffmail.com

[2]College of Arts Science and Technology, Bandomunda, Rourkela -770 032, Orissa, India.

[b]E-mail: rayprasanta2008@gmail.com

Abstract

Links of balancing and cobalancing numbers with Pell and associated Pell numbers are established. It is proved that the n^{th} balancing number is product of the n^{th} Pell number and the n^{th} associated Pell number. It is further observed that the sequences of balancing and cobalancing numbers are very closely related to the Pell sequence whereas, the sequences of Lucas-balancing and Lucas-cobalancing numbers constitute the associated Pell sequence. The solutions of some Diophantine equations including Pythagorean and Pythagorean-type equations are obtained in terms of these numbers.

1. Introduction

The study of number sequences has been a source of attraction to the mathematicians since ancient times. From that time many mathematicians have been focusing their attention on the study of the fascinating triangular numbers (numbers of the form $n(n+1)/2$ where $n \in \mathbb{Z}^+$ are known as triangular numbers). Behera and Panda [1], while studying the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$ on triangular numbers, obtained an interesting relation of the numbers n in

Received April 28, 2009 and in revised form September 25, 2009.

AMS Subject Classification: 11B37, 11B83, 11D04, 11D09.

Key words and phrases: Triangular numbers, Pronic numbers, balancing numbers, cobalancing numbers, Lucas-balancing numbers, Lucas-cobalancing numbers, balancers, cobalancers, Pell numbers, associated Pell numbers, recurrence relations, Pell's equation.

the solutions (n, r), which they call *balancing numbers*, with square triangular numbers. The number r in (n, r) is called the *balancer* corresponding to n. They also explored many important and interesting results on balancing numbers. Later on, Panda [4] identified many fascinating properties of balancing numbers, some of which are equivalent to the corresponding properties of Fibonacci numbers, and some others are more interesting than those of Fibonacci numbers. Subsequently, Liptai [2] added another interesting result to the theory of balancing numbers by proving that the only balancing number in the Fibonacci sequence is 1.

Behera and Panda [1] proved that the square of any balancing number is a triangular number. Subramaniam [6, 7, 8] explored many interesting properties of square triangular numbers without linking them to balancing numbers. In [8], he considered almost square triangular numbers (triangular numbers that differ from squares by unity) and established relationships with square triangular numbers. Panda and Ray [3] studied another Diophantine equation $1+2+\cdots+n = (n+1)+(n+2)+\cdots+(n+r)$ on triangular numbers and call n a *cobalancing number* and r the *cobalancer* corresponding to n. The cobalancing numbers are associated with another category of triangular numbers that are expressible as product of two consecutive natural numbers or approximately as the arithmetic mean of squares of two consecutive natural numbers [3, p.1189]. It is worth mentioning that the numbers of the form $n(n+1)$ where $n \in \mathbb{Z}^+$ are called pronic numbers.

Panda [5] further enriched the literature on balancing and cobalancing numbers by introducing *sequence balancing* and *cobalancing numbers*, in which, the sequence of natural numbers used in the definition of balancing and cobalancing numbers is replaced by an arbitrary sequences of real numbers.

In this paper, we establish many important association of balancing numbers, cobalancing numbers, and other numbers associated balancing and cobalancing numbers with Pell and associated Pell numbers. We also study some simple Diophantine equations whose solutions are closely associated with balancing numbers, cobalancing numbers, Pell numbers and associated Pell numbers.

2. Auxilliary Results

We need the following definitions and results for proving some important results in the subsequent sections.

For $n = 1, 2, \ldots$, let P_n be the n^{th} Pell number and Q_n, the n^{th} associated Pell number. It is well known that

$$P_1 = 1, \quad P_2 = 2, \quad P_{n+1} = 2P_n + P_{n-1}, \tag{1}$$

$$Q_1 = 1, \quad Q_2 = 3, \quad Q_{n+1} = 2Q_n + Q_{n-1}, \tag{2}$$

and their Binet forms are

$$P_n = \frac{\alpha_1^n - \alpha_2^n}{2\sqrt{2}}, \quad Q_n = \frac{\alpha_1^n + \alpha_2^n}{2}, \tag{3}$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

Further, as usual, for $n = 1, 2, \ldots$, let B_n be the n^{th} balancing number and b_n, the n^{th} cobalancing number. The following are the linear recurrence relations for balancing and cobalancing numbers [1, 3].

$$B_1 = 1, \quad B_2 = 6, \quad B_{n+1} = 6B_n - B_{n-1}, \tag{4}$$

$$b_1 = 0, \quad b_2 = 2, \quad b_{n+1} = 6b_n - b_{n-1} + 2. \tag{5}$$

The nonlinear recurrences are [1, 3]

$$B_1 = 1, \quad B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1}, \tag{6}$$

and

$$b_1 = 0, \quad b_{n+1} = 3b_n + \sqrt{8b_n^2 + 8b_n + 1} + 1. \tag{7}$$

Also

$$B_{n-1} = 3B_n - \sqrt{8B_n^2 + 1}, \tag{8}$$

and

$$b_{n-1} = 3b_n - \sqrt{8b_n^2 + 8b_n + 1} + 1. \tag{9}$$

Their Binet forms are

$$B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}}, \quad b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}\,. \tag{10}$$

Using (8) one can easily get $B_0 = 0$.

The following theorem connects balancing numbers, cobalancing numbers, balancers and cobalancers [3, Theorems 6.1 and 6.2].

Theorem 2.1. *Every balancing number is a cobalancer and every cobalancing number is a balancer. More specifically,* $B_n = r_{n+1}$ *and* $R_n = b_n$ *for* $n = 1, 2, \ldots,$ *where* R_n *is the* n^{th} *balancer and* r_n *is the* n^{th} *cobalancer.*

We call

$$C_n = \sqrt{8B_n^2 + 1},$$

the n^{th} Lucas-balancing number and

$$c_n = \sqrt{8b_n^2 + 8b_n + 1},$$

the n^{th} Lucas-cobalancing number. The interested readers are advised to refer [4] for the justification of these names.

The following theorem establishes the similarity of Lucas-balancing and Lucas-cobalancing numbers with balancing numbers in terms of their recurrence relations.

Theorem 2.2. *The sequences of Lucas-balancing and Lucas-cobalancing numbers satisfy recurrence relations identical with balancing numbers. More precisely,* $C_1 = 3$, $C_2 = 17$, $C_{n+1} = 6C_n - C_{n-1}$ *and* $c_1 = 1$, $c_2 = 7$, $c_{n+1} = 6c_n - c_{n-1}$ *for* $n = 2, 3, \ldots$.

Proof. From (6) we have

$$\begin{aligned} C_{n+1}^2 &= 8B_{n+1}^2 + 1 \\ &= 8\Big(3B_n + \sqrt{8B_n^2 + 1}\Big)^2 + 1 \\ &= \Big(3\sqrt{8B_n^2 + 1} + 8B_n\Big)^2 \\ &= (3C_n + 8B_n)^2. \end{aligned}$$

Hence

$$C_{n+1} = 3C_n + 8B_n. \tag{11}$$

Similarly using (8) one can easily show that

$$C_{n-1} = 3C_n - 8B_n. \tag{12}$$

Adding (11) and (12) we obtain

$$C_{n+1} = 6C_n - C_{n-1}.$$

In a fashion similar to the derivation of (11) and (12) one can have

$$c_{n+1} = 3c_n + 8b_n + 4, \tag{13}$$

and

$$c_{n-1} = 3c_n - 8b_n - 4. \tag{14}$$

Combining (13) and (14) we get

$$c_{n+1} = 6c_n - c_{n-1}. \tag{15}$$

This ends the proof. □

Remark 2.3. Using the recurrence relations for C_n and c_n, the Binet forms for C_n and c_n are given as follows.

$$C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2}, \quad c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}. \tag{16}$$

3. Some Important Links

In this section we establish many important links of balancing and cobalancing numbers with Pell and associated Pell numbers. Pell and associated Pell numbers not only have direct relations with balancing numbers and cobalancing numbers, they also occur in the factorization of these numbers. Even the greatest common divisors of balancing numbers and cobalancing numbers of same order and of consecutive cobalancing numbers are also Pell or associated Pell numbers.

Throughout this section $\alpha_1 = 1+\sqrt{2}$, $\alpha_2 = 1-\sqrt{2}$ and the greatest common divisor of two positive integers m and n is denoted by (m, n). We observe that $\alpha_1\alpha_2 = -1$, and this will be used as and when necessary without any further mention.

We start with the following important theorem, which gives a direct relation of balancing numbers with Pell and associated Pell numbers.

Theorem 3.1. *For $n = 1, 2, \ldots$, the n^{th} balancing number is product of the n^{th} Pell number and the n^{th} associated Pell number.*

Proof. Using the Binet forms of P_n and Q_n from (3) and B_n from (10), we obtain

$$\begin{aligned} B_n &= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \\ &= \frac{\alpha_1^n - \alpha_2^n}{2\sqrt{2}} \cdot \frac{\alpha_1^n + \alpha_2^n}{2} = P_n Q_n. \end{aligned}$$ □

This is not the only relationship among balancing numbers, Pell numbers and associated Pell numbers. Truly speaking, the sequences of balancing and cobalancing numbers are contained in a sequence obtained from the Pell sequence dividing each term by 2, whereas, the sequences of Lucas-balancing and Lucas-cobalancing numbers are absorbed by the associated Pell sequence.

The following theorem closely relates the balancing and cobalancing numbers with Pell numbers.

Theorem 3.2. *If P is a Pell number, then $[P/2]$ is either a balancing number or a cobalancing number, where $[\cdot]$ denotes the greatest integer function. More specifically, $P_{2n}/2 = B_n$ and $[P_{2n-1}/2] = b_n$, $n = 1, 2, \ldots$.*

Proof. Using the Binet form for P_n from (3) and B_n and b_n from (10) we get

$$\frac{P_{2n}}{2} = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} = B_n,$$

and since P_{2n-1} is odd,

$$\left[\frac{P_{2n-1}}{2}\right] = \frac{P_{2n-1}}{2} - \frac{1}{2}$$

$$= \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} = b_n. \qquad \square$$

The following theorem establishes the fact that the union of the sequences of Lucas-balancing and Lucas-cobalancing numbers is nothing but the sequence of associated Pell numbers.

Theorem 3.3. *Every associated Pell number is either a Lucas-balancing number or a Lucas-cobalancing number. More specifically,* $Q_{2n} = C_n$ *and* $Q_{2n-1} = c_n$, $n = 1, 2, \ldots$.

Proof. The proof of the first part follows directly from the Binet forms for Q_n and C_n from (3) and (16) respectively and the proof of the second part follows directly from the Binet forms for Q_n and c_n from (3) and (16) respectively. $\square$

It is known that if n is a balancing number with balancer r, then the $(n+r)^{th}$ triangular number is n^2 [1, p.98]. The following theorem demonstrates the association of this number $n + r$ with the partial sums of Pell numbers.

Theorem 3.4. *The sum of first* $2n - 1$ *Pell numbers is equal to the sum of* n^{th} *balancing number and its balancer.*

Proof. Using the Binet forms for P_n from (3), and B_n and b_n from (10) we get

$$\begin{aligned}
P_1 + P_2 + \cdots + P_{2n-1} &= \frac{\alpha_1 - \alpha_2}{2\sqrt{2}} + \frac{\alpha_1^2 - \alpha_2^2}{2\sqrt{2}} + \cdots + \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{2\sqrt{2}} \\
&= \frac{\alpha_1(\frac{\alpha_1^{2n-1}-1}{\alpha_1-1}) - \alpha_2(\frac{\alpha_2^{2n-1}-1}{\alpha_2-1})}{2\sqrt{2}} \\
&= \frac{\alpha_1(\alpha_1^{2n-1} - 1) + \alpha_2(\alpha_2^{2n-1} - 1)}{4} \\
&= \frac{\alpha_1^{2n} + \alpha_2^{2n}}{4} - \frac{1}{2} \\
&= \frac{\alpha_1^{2n}(1 - \alpha_2) - \alpha_2^{2n}(1 - \alpha_1)}{4\sqrt{2}} - \frac{1}{2} \\
&= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} + \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \\
&= B_n + b_n.
\end{aligned}$$

By virtue of Theorem 2.1, $b_n = R_n$ and the proof is complete. □

It is also known that if n is a cobalancing number with cobalancer r, then the $(n+r)^{th}$ triangular number is the n^{th} pronic number [3, p.1190]. The following theorem demonstrates the association of this number $n+r$ with the partial sums of Pell numbers.

Theorem 3.5. *The sum of first $2n$ Pell numbers is equal to the sum of $(n+1)^{st}$ cobalancing number and its cobalancer.*

Proof. Using the Binet forms for P_n from (3), B_n and b_n from (10) we get

$$\begin{aligned}
P_1 + P_2 + \cdots + P_{2n} &= \frac{\alpha_1 - \alpha_2}{2\sqrt{2}} + \frac{\alpha_1^2 - \alpha_2^2}{2\sqrt{2}} + \cdots + \frac{\alpha_2^{2n} - \alpha_2^{2n}}{2\sqrt{2}} \\
&= \frac{\alpha_1(\frac{\alpha_1^{2n}-1}{\alpha_1-1}) - \alpha_2(\frac{\alpha_2^{2n}-1}{\alpha_2-1})}{2\sqrt{2}} \\
&= \frac{\alpha_1(\alpha_1^{2n}-1) + \alpha_2(\alpha_2^{2n}-1)}{4} \\
&= \frac{\alpha_1^{2n+1} + \alpha_2^{2n+1}}{4} - \frac{1}{2} \\
&= \frac{\alpha_1^{2n+1}(1-\alpha_2) - \alpha_2^{2n+1}(1-\alpha_1)}{4\sqrt{2}} - \frac{1}{2} \\
&= \frac{\alpha_1^{2n+1} - \alpha_2^{2n+1}}{4\sqrt{2}} - \frac{1}{2} + \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \\
&= b_{n+1} + B_n.
\end{aligned}$$

By virtue of Theorem 2.1, $B_n = r_{n+1}$ and the proof is complete. □

The last two theorems establishes the links among sums of Pell numbers up to odd and even order with balancing and cobalancing numbers. The next two theorems provides relationships among partial sums of odd ordered and even ordered Pell numbers with balancing and cobalancing numbers respectively.

The following theorem establishes direct link between partial sums of odd ordered Pell numbers and balancing numbers.

Theorem 3.6. *The sum of first n odd ordered Pell numbers is equal to the n^{th} balancing number ($(n+1)^{st}$ cobalancer).*

Proof. Using the Binet forms for P_n from (3) and B_n, from (10) we get

$$\begin{aligned}
P_1 + P_3 + \cdots + P_{2n-1} &= \frac{\alpha_1 - \alpha_2}{2\sqrt{2}} + \frac{\alpha_1^3 - \alpha_2^3}{2\sqrt{2}} + \cdots + \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{2\sqrt{2}} \\
&= \frac{\alpha_1(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}) - \alpha_2(\frac{\alpha_2^{2n}-1}{\alpha_2^2-1})}{2\sqrt{2}} \\
&= \frac{(\alpha_1^{2n} - 1) - (\alpha_2^{2n} - 1)}{4\sqrt{2}} \\
&= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} = B_n. \qquad \square
\end{aligned}$$

The following theorem establishes direct link between partial sums of even ordered Pell numbers and cobalancing numbers.

Theorem 3.7. *The sum of first n even ordered Pell numbers is equal to the $(n+1)^{st}$ cobalancing number (balancer).*

Proof. Using the Binet forms for P_n from (3) and b_n from (10) we get

$$\begin{aligned}
P_2 + P_4 + \cdots + P_{2n} &= \frac{\alpha_1^2 - \alpha_2^2}{2\sqrt{2}} + \frac{\alpha_1^4 - \alpha_2^4}{2\sqrt{2}} + \cdots + \frac{\alpha_1^{2n} - \alpha_2^{2n}}{2\sqrt{2}} \\
&= \frac{\alpha_1^2(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}) - \alpha_2^2(\frac{\alpha_2^{2n}-1}{\alpha_2^2-1})}{2\sqrt{2}} \\
&= \frac{\alpha_1(\alpha_1^{2n} - 1) - \alpha_2(\alpha_2^{2n} - 1)}{4\sqrt{2}} \\
&= \frac{\alpha_1^{2n+1} - \alpha_2^{2n+1}}{4\sqrt{2}} - \frac{1}{2} = b_{n+1}.
\end{aligned}$$

By virtue of Theorem 2.1, $b_{n+1} = R_{n+1}$ and the proof is complete. $\square$

The following theorem relates partial sums of odd ordered associated Pell numbers to sums of balancing numbers and their respective balancers.

Theorem 3.8. *The sum of first n odd ordered associated Pell numbers is equal to the sum of n^{th} balancing number and its balancer.*

Using the Binet forms for Q_n from (3), B_n and b_n from (10) we get

$$\begin{aligned} Q_1 + Q_3 + \cdots + Q_{2n-1} &= \frac{\alpha_1 + \alpha_2}{2} + \frac{\alpha_1^3 + \alpha_2^3}{2} + \cdots + \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} \\ &= \frac{\alpha_1(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}) + \alpha_2(\frac{\alpha_2^{2n}-1}{\alpha_2^2-1})}{2} \\ &= \frac{(\alpha_1^{2n} - 1) + (\alpha_2^{2n} - 1)}{4} \\ &= \frac{\alpha_1^{2n} + \alpha_2^{2n}}{4} - \frac{1}{2}\,. \end{aligned}$$

In the proof of Theorem 3.4, it has been shown that the last expression is equal to $B_n + R_n$. □

Similarly, the following theorem relates partial sums of even ordered associated Pell numbers with sums of cobalancing numbers and their respective cobalancers.

Theorem 3.9. *The sum of first n even ordered associated Pell numbers is equal to the $(n+1)^{st}$ cobalancing number and its cobalancer.*

Using the Binet forms for Q_n from (3) we get

$$\begin{aligned} Q_2 + Q_4 + \cdots + Q_{2n} &= \frac{\alpha_1^2 + \alpha_2^2}{2} + \frac{\alpha_1^4 + \alpha_2^4}{2} + \cdots + \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} \\ &= \frac{\alpha_1^2(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}) + \alpha_2^2(\frac{\alpha_2^{2n}-1}{\alpha_2^2-1})}{2} \\ &= \frac{\alpha_1(\alpha_1^{2n} - 1) + \alpha_2(\alpha_2^{2n} - 1)}{4} \\ &= \frac{\alpha_1^{2n+1} + \alpha_2^{2n+1}}{4} - \frac{1}{2}\,. \end{aligned}$$

In the proof of Theorem 3.5, it has been shown that the last expression is equal to $b_{n+1} + r_{n+1}$. □

The sums of associated Pell numbers up to even and odd order are also related to balancing and cobalancing numbers respectively.

The following theorem links partial sums of associated Pell numbers up to odd order with balancing numbers.

Theorem 3.10. *The sum of first $2n-1$ associated Pell numbers is equal to twice the n^{th} balancing number decreased by one.*

Proof. Using the Binet forms for Q_n from (3) and B_n from (10), we get

$$\begin{aligned}
Q_1+Q_2+\cdots+Q_{2n-1} &= \frac{\alpha_1+\alpha_2}{2}+\frac{\alpha_1^2+\alpha_2^2}{2}+\cdots+\frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}\\
&= \frac{\alpha_1(\frac{\alpha_1^{2n-1}-1}{\alpha_1-1})+\alpha_2(\frac{\alpha_2^{2n-1}-1}{\alpha_2-1})}{2}\\
&= \frac{\alpha_1(\alpha_1^{2n-1}-1)-\alpha_2(\alpha_2^{2n-1}-1)}{2\sqrt{2}}\\
&= \frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}}-1\\
&= 2B_n-1.
\end{aligned}$$

□

The following theorem links partial sums of associated Pell numbers up to even order with cobalancing numbers.

Theorem 3.11. *The sum of first $2n$ associated Pell numbers is equal to the twice the $(n+1)^{st}$ cobalancing number.*

Proof. Using the Binet forms for Q_n from (3) and b_n from (10) we get

$$\begin{aligned}
Q_1+Q_2+\cdots+Q_{2n} &= \frac{\alpha_1+\alpha_2}{2}+\frac{\alpha_1^2+\alpha_2^2}{2}+\cdots+\frac{\alpha_1^{2n}+\alpha_2^{2n}}{2}\\
&= \frac{\alpha_1(\frac{\alpha_1^{2n}-1}{\alpha_1-1})+\alpha_2(\frac{\alpha_2^{2n}-1}{\alpha_2-1})}{2}\\
&= \frac{\alpha_1(\alpha_1^{2n}-1)-\alpha_2(\alpha_2^{2n}-1)}{2\sqrt{2}}\\
&= \frac{\alpha_1^{2n+1}-\alpha_2^{2n+1}}{2\sqrt{2}}-1\\
&= 2b_{n+1}.
\end{aligned}$$

□

The following theorem establishes links between differences of Lucas-balancing numbers and cobalancing numbers.

Theorem 3.12. *The difference of n^{th} and $(n-1)^{st}$ Lucas-balancing numbers is equal to the difference of the $(n+1)^{st}$ and $(n-1)^{st}$ cobalancing numbers.*

Proof. From (11) we have

$$C_n = 8B_{n-1} + 3C_{n-1}.$$

Now using the recurrence relation (6) we get

$$\begin{aligned} C_n - C_{n-1} &= 8B_{n-1} + 2C_{n-1} \\ &= 2[B_{n-1} + (C_{n-1} + 3B_{n-1})] \\ &= 2(B_{n-1} + B_n). \end{aligned}$$

Since

$$2(B_1 + B_2 + \cdots + B_{n-1}) = b_n$$

[3, Theorem 4.1], it follows that

$$b_{n+1} - b_{n-1} = 2(B_{n-1} + B_n)$$

from which the result follows. □

The following corollary, the proof of which is contained in the proof of the above theorem, links differences of Lucas-balancing numbers and sums of balancing numbers.

Corollary 3.13. *The difference of n^{th} and $(n-1)^{st}$ Lucas-balancing numbers is equal to twice the sum of n^{th} and $(n-1)^{st}$ balancing numbers.*

Theorem 3.12 is a link between differences of Lucas-balancing numbers and cobalancing numbers. The following theorem establishes link between differences of Lucas-cobalancing numbers and balancing numbers.

Theorem 3.14. *The difference of n^{th} and $(n-1)^{st}$ the Lucas-cobalancing numbers is equal to the difference of n^{th} and the $(n-2)^{nd}$ balancing numbers.*

Proof. From (13) we have

$$c_n = 8b_{n-1} + 3c_{n-1} + 4.$$

Now using the recurrence relation (7) and Theorem 2.1, we get

$$c_n - c_{n-1} = 8b_{n-1} + 2c_{n-1} + 4$$

$$\begin{aligned} &= 2[b_{n-1} + (3b_{n-1} + c_{n-1} + 1) + 1] \\ &= 2(b_{n-1} + b_n + 1) \\ &= 2(R_{n-1} + R_n + 1) \\ &= (2R_{n-1} + 1) + (2R_n + 1). \end{aligned}$$

Since

$$R_n = \frac{-(2B_n + 1) + \sqrt{8B_n^2 + 1}}{2}$$

[1, p.98], we have

$$\begin{aligned} 2R_n + 1 &= -2B_n + \sqrt{8B_n^2 + 1} \\ &= -2B_n + C_n. \end{aligned}$$

Hence,

$$c_n - c_{n-1} = C_n + C_{n-1} - 2(B_n + B_{n-1}). \tag{17}$$

Use of Binet forms of B_n and C_n from (10) and (16) respectively gives

$$C_n + \sqrt{8}B_n = \alpha_1^{2n}$$

and

$$C_n - \sqrt{8}B_n = \alpha_2^{2n}.$$

Thus, for $n = 1$ we get

$$3 + \sqrt{8} = \alpha_1^2,$$

and replacement of n by $n - 1$ gives

$$C_{n-1} - \sqrt{8}B_{n-1} = \alpha_2^{2(n-1)}. \tag{18}$$

On the other hand,

$$\begin{aligned} (3 + \sqrt{8})(C_n - \sqrt{8}B_n) &= (3C_n - 8B_n) + \sqrt{8}(C_n - 3B_n) \\ &= \alpha_1^2(C_n - \sqrt{8}B_n) \\ &= \alpha_1^2\alpha_2^{2n} = \alpha_2^{2(n-1)}. \end{aligned} \tag{19}$$

Using (17) and (18), we get

$$C_{n-1} - \sqrt{8}B_{n-1} = (3C_n - 8B_n) + \sqrt{8}(C_n - 3B_n). \tag{20}$$

Comparison of rational and irrational parts from left hand and right hand sides of (20) yields

$$C_{n-1} = 3C_n - 8B_n, \tag{21}$$

and

$$B_{n-1} = 3B_n - C_n. \tag{22}$$

Now using (21) and (22), we find

$$\begin{aligned} B_{n-2} &= 3B_{n-1} - C_{n-1} \\ &= 3(3B_n - C_n) - (3C_n - 8B_n) \\ &= 17B_n - 6C_n. \end{aligned} \tag{23}$$

Inserting (21) and (22) into (17) and using (23) we get

$$\begin{aligned} c_n - c_{n-1} &= 6C_n - 16B_n \\ &= B_n - (17B_n - 6C_n) \\ &= B_n - B_{n-2}. \end{aligned}$$

□

The following theorem gives a relation between sums of Lucas-balancing and Lucas-cobalancing numbers of same order and differences of squares of two Pell numbers.

Theorem 3.15. *The sum of n^{th} Lucas-balancing and n^{th} Lucas-cobalancing number is equal to the difference of squares of the $(n+1)^{st}$ and $(n-1)^{st}$ Pell numbers.*

Proof. Using the Binet form for P_n from (3) we get

$$\begin{aligned} P_{n+1}^2 - P_{n-1}^2 &= \left[\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{2\sqrt{2}}\right]^2 - \left[\frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{2\sqrt{2}}\right]^2 \\ &= \frac{\alpha_1^{2n+2} + \alpha_2^{2n+2} - \alpha_1^{2n-2} - \alpha_2^{2n-2}}{8} \end{aligned}$$

$$= \frac{(\alpha_1^{2n} - \alpha_2^{2n})(\alpha_1^2 - \alpha_2^2)}{8} = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{\sqrt{2}}.$$

Observing that

$$1 - \alpha_2 = -(1 - \alpha_1) = \sqrt{2},$$

and using the Binet forms of C_n and c_n from (16), we get

$$\begin{aligned}\frac{\alpha_1^{2n} - \alpha_2^{2n}}{\sqrt{2}} &= \frac{\alpha_1^{2n}(1-\alpha_2) + \alpha_2^{2n}(1-\alpha_1)}{2} \\ &= \frac{\alpha_1^{2n} + \alpha_2^{2n} + \alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} \\ &= \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} + \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} \\ &= C_n + c_n. \end{aligned}$$

□

The following theorem establishes relations of Lucas-cobalancing numbers with sums of two consecutive balancing numbers.

Theorem 3.16. *The n^{th} Lucas-cobalancing number is equal to the sum of $(n-1)^{st}$ and n^{th} balancing numbers.*

Proof. Using the Binet form for B_n from (10) and c_n from (16) respectively, we find

$$\begin{aligned} B_{n-1} + B_n &= \frac{\alpha_1^{2(n-1)} - \alpha_2^{2(n-1)}}{4\sqrt{2}} + \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \\ &= \frac{\alpha_1^{2n}(1+\alpha_2^2) - \alpha_2^{2n}(1+\alpha_1^2)}{4\sqrt{2}} \\ &= \frac{\alpha_1^{2n}(-2\sqrt{2}\alpha_2) - \alpha_2^{2n}(2\sqrt{2}\alpha_1)}{4\sqrt{2}} \\ &= \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} = c_n. \end{aligned}$$

□

Remark 3.17. The following alternative forms are also available for c_n. Using (8) we can have

$$B_{n-1} = 3B_n - C_n,$$

and since $P_{2n} = 2B_n$ and $Q_{2n} = C_n$ by Theorems 3.2 and 3.3, it follows that

$$c_n \;=\; 4B_n - C_n = 2P_{2n} - Q_{2n}.$$

In some previous theorems, links among sums and differences of certain class of numbers are discussed. The following theorem links the arithmetic means of Pell and associated Pell numbers with balancing and cobalancing numbers respectively.

Theorem 3.18. *The arithmetic mean of n^{th} odd ordered Pell number and associated Pell number is equal to the n^{th} balancing number and the arithmetic mean of n^{th} even ordered Pell number and associated Pell number is $1/2$ more than the $(n+1)^{st}$ cobalancing number.*

Proof. Using the Binet forms for P_n, Q_n from (3) and b_n from (10) we get

$$\begin{aligned}\frac{P_{2n-1}+Q_{2n-1}}{2} &= \frac{1}{2}\left[\frac{\alpha_1^{2n-1}-\alpha_2^{2n-1}}{2\sqrt{2}} + \frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}\right]\\ &= \frac{-\alpha_1^{2n}\alpha_2(1+\sqrt{2})+\alpha_2^{2n}\alpha_1(1-\sqrt{2})}{4\sqrt{2}}\\ &= \frac{-\alpha_1^{2n}\alpha_1\alpha_2+\alpha_2^{2n}\alpha_1\alpha_2}{4\sqrt{2}}\\ &= \frac{\alpha_1^{2n}-\alpha_2^{2n}}{4\sqrt{2}} = B_n.\end{aligned}$$

Further,

$$\begin{aligned}\frac{P_{2n}+Q_{2n}}{2} &= \frac{1}{2}\left[\frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}} + \frac{\alpha_1^{2n}+\alpha_2^{2n}}{2}\right]\\ &= \frac{\alpha_1^{2n}(1+\sqrt{2})-\alpha_2^{2n}(1-\sqrt{2})}{4\sqrt{2}}\\ &= \frac{\alpha_1^{2n+1}-\alpha_2^{2n+1}}{4\sqrt{2}}\\ &= \left[\frac{\alpha_1^{2(n+1)-1}-\alpha_2^{2(n+1)-1}}{4\sqrt{2}} - \frac{1}{2}\right] + \frac{1}{2} = b_{n+1} + \frac{1}{2}\;. \qquad \square\end{aligned}$$

In Section 1, we have observed that n is a balancing number if and only if n^2 is a triangular number and n is a cobalancing number if and only if

$n(n+1)$ is a triangular number. It is worth mentioning here that if n^2 is a triangular number then $n+n=2n$ is an even ordered Pell number, and if $n(n+1)$ is a triangular number then $n+(n+1)=2n+1$ is an odd ordered Pell number. The following theorem demonstrates these assertions.

Theorem 3.19. *If n^2 is a triangular number (i.e. if n is a balancing number) then $2n$ is an even ordered Pell number, and if $n(n+1)$ is a triangular number (i.e. if n is a cobalancing number) then $2n+1$ is an odd ordered Pell number. Conversely, if P is an even ordered Pell number then $P^2/4$ is a square triangular number and if P is an odd ordered Pell number then $(P^2-1)/4$ is a pronic triangular number.*

Proof. If n^2 is a triangular number then n is a balancing number, say $n=B_k$ for some k. By virtue of Theorem 3.2,

$$2n = 2B_k = P_{2k}.$$

Conversely, for every even ordered Pell number P_{2k}, $P_{2k}/2$ is a balancing number and hence $P_{2k}^2/4$ is a square triangular number.

Further using the Binet form for P_n and b_n from (3) and (10) respectively we get

$$\begin{aligned} \frac{P_{2k-1}-1}{2} &= \frac{1}{2}\left[\frac{\alpha_1^{2k-1}-\alpha_2^{2k-1}}{2\sqrt{2}} - 1\right] \\ &= \frac{\alpha_1^{2k-1}-\alpha_2^{2k-1}}{4\sqrt{2}} - \frac{1}{2} = b_k. \end{aligned} \tag{24}$$

It is well known that [3, p.1189] n is a cobalancing number if and only if $n(n+1)$ is a triangular number. Hence if $n(n+1)$ is a triangular number then $n=b_k$ for some k and using the Binet form for b_n from (10) we find

$$2n+1 = 2b_k+1 = \frac{\alpha_1^{2k-1}-\alpha_2^{2k-1}}{2\sqrt{2}} = P_{2k-1}.$$

Conversely, if P is an odd ordered Pell number say $P=P_{2k-1}$ for some k, then by (24),

$$(P^2-1)/4 = \left[\frac{P-1}{2}\right]\left[\frac{P+1}{2}\right]$$

$$= b_k(b_k+1),$$

which is a pronic triangular number [3, p.1190]. □

The following theorem establishes the associations of Pell and associated Pell numbers in the factorization of cobalancing numbers (balancers).

Theorem 3.20. *For $n = 1, 2, \ldots$, the $2n^{th}$ balancer (cobalancing number) is equal to the product of $2n^{th}$ Pell number and the $(2n-1)^{st}$ associated Pell number and the $(2n+1)^{st}$ balancer is equal to the product of $2n^{th}$ Pell number and the $(2n+1)^{st}$ associated Pell number. More precisely $R_{2n} = P_{2n}Q_{2n-1}$ and $R_{2n+1} = P_{2n}Q_{2n+1}$.*

Proof. Using Theorem 2.1, the Binet forms of P_n and Q_n from (3) and that for b_n from (10), we obtain

$$\begin{aligned}
P_{2n}Q_{2n-1} &= \frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}}\cdot\frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}\\
&= \frac{\alpha_1^{4n-1}-\alpha_2^{4n-1}-\alpha_1+\alpha_2}{4\sqrt{2}}\\
&= \frac{\alpha_1^{4n-1}-\alpha_2^{4n-1}}{4\sqrt{2}}-\frac{1}{2} = R_{2n},
\end{aligned}$$

and

$$\begin{aligned}
P_{2n}Q_{2n+1} &= \frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}}\cdot\frac{\alpha_1^{2n+1}+\alpha_2^{2n+1}}{2}\\
&= \frac{\alpha_1^{4n+1}-\alpha_2^{4n+1}-\alpha_1+\alpha_2}{4\sqrt{2}}\\
&= \frac{\alpha_1^{2(2n+1)-1}-\alpha_2^{2(2n+1)-1}}{4\sqrt{2}}-\frac{1}{2} = R_{2n+1}.
\end{aligned}$$ □

Theorem 3.1 gives the basic relationships among balancing numbers, Pell numbers and associated Pell numbers in a nonlinear fashion. The following theorem establishes the presence of Pell and associated Pell numbers as the greatest common divisors of balancing and cobalancing numbers (balancers).

Theorem 3.21. *The greatest common divisor of a balancing number and a cobalancing number (balancer) of same order is either a Pell number or an* *associated Pell number of the same order. More precisely,* $(B_{2n-1}, R_{2n-1}) = Q_{2n-1}$ *and* $(B_{2n}, R_{2n}) = P_{2n}$.

Proof. By virtue of Theorems 3.1 and 3.20, and using the fact that for each n, P_n and P_{n-1} are relatively prime to each other, we obtain

$$\begin{aligned}(B_{2n-1}, R_{2n-1}) &= (P_{2n-1}Q_{2n-1}, P_{2n-2}Q_{2n-1}) \\ &= Q_{2n-1}(P_{2n-1}, P_{2n-2}) = Q_{2n-1}.\end{aligned}$$

Further using Theorem 3.1 and 3.20, once again and the fact that for each n, Q_n and Q_{n-1} are also relatively prime, we obtain

$$\begin{aligned}(B_{2n}, R_{2n}) &= (P_{2n}Q_{2n}, P_{2n}Q_{2n-1}) \\ &= P_{2n}(Q_{2n}, Q_{2n-1}) = P_{2n}.\end{aligned} \qquad \square$$

The following theorem, which is similar to Theorem 3.21, establishes the presence of Pell and associated Pell numbers with the greatest common divisors of consecutive cobalancing numbers (balancers).

Theorem 3.22. *The greatest common divisor of two consecutive cobalancing numbers (balancers) is either twice of an odd ordered associated Pell number or is an even ordered Pell number. More precisely,* $(R_{2n-1}, R_{2n}) = 2Q_{2n-1}$ *and* $(R_{2n}, R_{2n+1}) = P_{2n}$.

Proof. Using Theorem 3.20 and the fact that consecutive Pell numbers are relative primes and the greatest common divisor of consecutive even ordered Pell numbers is 2, we find

$$\begin{aligned}(R_{2n-1}, R_{2n}) &= (P_{2n-2}Q_{2n-1}, P_{2n}Q_{2n-1}) \\ &= Q_{2n-1}(P_{2n-2}, P_{2n}) = 2Q_{2n-1},\end{aligned}$$

and

$$\begin{aligned}(R_{2n}, R_{2n+1}) &= (Q_{2n-1}P_{2n}, P_{2n}Q_{2n+1}) \\ &= P_{2n}(Q_{2n-1}, Q_{2n+1}) \\ &= P_{2n}(Q_{2n-1}, 2Q_{2n} + Q_{2n-1}) \\ &= P_{2n}(Q_{2n-1}, 2Q_{2n}) = P_{2n}.\end{aligned} \qquad \square$$

4. Solutions of Some Diophantine Equations

In this section, we consider some simple Diophantine equations whose solutions are expressed as suitable combinations of balancing numbers, cobalancing numbers, Pell numbers and associated Pell numbers.

The following theorem deals with the solution of a beautiful Diophantine equation consists of finding two natural numbers such that the sum of all natural numbers from the smaller number to the larger is equal to the product of these two numbers.

Theorem 4.1. *The solutions of the Diophantine equation* $x + (x+1) + (x+2) + \cdots + (x+y) = x(x+y)$ *are* $x = R_n + 1$ *and* $y = B_n - R_n - 1$, $n = 1, 2, \ldots$.

Proof. B is a balancing number with balancer R if

$$1 + 2 + \cdots + (B-1) = (B+1) + (B+2) + \cdots + (B+R).$$

Thus,

$$\begin{aligned}&(R+1) + (R+2) + \cdots + (B-1) \\ &\quad = (B+1) + (B+2) + \cdots + (B+R) - (1 + 2 + \cdots + R) = RB.\end{aligned}$$

Adding B to both sides we get

$$(R+1) + (R+2) + \cdots + B = (R+1)B.$$

Thus,

$$x = R+1, \quad x + y = B,$$

from which the result follows.

An alternative proof of this theorem can be obtained using Pell's equation:

The Diophantine equation

$$x + (x+1) + \cdots + (x+y) = x(x+y)$$

is equivalent to

$$(2y+1)^2 - 2(2x-1)^2 = -1.$$

Setting $u = 2y+1$ and $v = 2x-1$, we get the Pell's equation

$$u^2 - 2v^2 = -1.$$

The fundamental solution of this equation is $u = 1$ and $v = 1$. Hence the totality of solutions is given by

$$u_n + \sqrt{2}v_n = (1+\sqrt{2})^n, \quad n = 1, 2, \ldots.$$

Since

$$u_n - \sqrt{2}v_n = (1-\sqrt{2})^n, \quad n = 1, 2, \ldots,$$

it follows that

$$u_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} = Q_n,$$

and

$$v_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} = P_n.$$

Since both u_n and v_n are odd and P_n is odd only when n is odd, we must have

$$u_n = Q_{2n-1}, \quad v_n = P_{2n-1}; \quad n = 1, 2, \ldots.$$

Thus,

$$2y+1 = Q_{2n-1}, \quad 2x-1 = P_{2n-1},$$

implying that

$$y = (Q_{2n-1} - 1)/2, \quad x = (P_{2n-1} + 1)/2.$$

Using the Binet forms in (3) and (10), it can be easily verified that

$$x = \frac{(P_{2n-1} + 1)}{2} = R_n + 1,$$

and using Theorem 3.18, we find

$$x+y=\frac{(P_{2n-1}+Q_{2n-1})}{2}=B_n,$$

from which the conclusion of the theorem follows. □

The following theorem, which resembles the previous theorem, deals with the solution of a Diophantine equation, consists of finding two natural numbers such that the sum of all natural numbers from next to the smaller number to the larger number is equal to the product of the two numbers.

Theorem 4.2. *The solutions of the Diophantine equation* $(x+1)+(x+2)+\cdots+(x+y)=x(x+y)$ *are* $x=B_n$ *and* $y=R_{n+1}-B_n$, $n=1,2,\ldots$.

Proof. If b is a cobalancing number with cobalancer r, then

$$1+2+\cdots+b=(b+1)+(b+2)+\cdots+(b+r),$$

which implies

$$\begin{aligned}
&(r+1)+(r+2)+\cdots+b\\
&\quad= (b+1)+(b+2)+\cdots+(b+r)-(1+2+\cdots+r)\\
&\quad= rb.
\end{aligned}$$

Thus, $x=r$ and $x+y=b$. Now using Theorem 2.1, we conclude that if $x=B_n$ then $x+y=R_{n+1}$.

Also, in this case, an alternative proof can be given using the Pell's equation:

The Diophantine equation

$$(x+1)+(x+2)+\cdots+(x+y)=x(x+y)$$

is equivalent to

$$(2y+1)^2-2(2x)^2=1,$$

Setting

$$u=2y+1,\quad v=2x,$$

we get the Pell's equation

$$u^2 - 2v^2 = 1, \quad u \text{ is odd}, \quad v \text{ is even}.$$

The fundamental solution of this equation is $u = 3$ and $v = 2$. Hence the totality of solutions is given by

$$u_n + \sqrt{2}v_n = (3 + 2\sqrt{2})^n, \quad n = 1, 2, \ldots.$$

This implies

$$u_n - \sqrt{2}v_n = (3 - 2\sqrt{2})^n, \quad n = 1, 2, \ldots.$$

By virtue of the last two equations and Theorems 3.2 and 3.3, we have

$$\begin{aligned} u_n &= \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n}{2} \\ &= \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} = C_n = Q_{2n}. \end{aligned}$$

and

$$\begin{aligned} v_n &= \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{2\sqrt{2}} \\ &= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{2} = 2B_n = P_{2n}. \end{aligned}$$

We observe that Q_{2n} is always odd and P_{2n} is always even. Thus

$$2y + 1 = C_n, \quad 2x = 2B_n,$$

implying that

$$y = (C_n - 1)/2, \quad x = B_n.$$

Thus

$$x + y = \frac{2B_n + C_n - 1}{2}.$$

But by virtue of Corollary 6.4 of [3],

$$\frac{2B_n + C_n - 1}{2} = R_{n+1}. \qquad \square$$

The following theorem deals with the solution of a Diophantine equation, consists of finding two natural numbers, the larger one being even, such that the sum of all natural up to the larger number is equal to square of the smaller number.

Theorem 4.3. *The solutions of the Diophantine equation* $1+2+\cdots+2x = y^2$ *are* $x = P_{2n}^2 = 4B_n^2$ *and* $y = P_{2n}Q_{2n} = B_{2n}$, $n = 1, 2, \ldots$.

Proof. The Diophantine equation

$$1 + 2 + \cdots + 2x = y^2$$

is equivalent to

$$x(2x+1) = y^2.$$

Since x and $2x+1$ are relatively prime to each other, both x and $2x+1$ must be squares. Letting

$$2x + 1 = (2l+1)^2,$$

we find

$$x = 4 \cdot \frac{l(l+1)}{2}.$$

Since x is a square it follows that $l(l+1)/2$ is a square triangular number. Hence

$$x = 4B_n^2 = P_{2n}^2, \quad n = 1, 2, \ldots.$$

[1, p.98], and by virtue of Theorem 3.1

$$\begin{aligned} y &= \sqrt{x(2x+1)} \\ &= \sqrt{4B_n^2(8B_n^2+1)} \\ &= \sqrt{4B_n^2C_n^2} = 2B_nC_n \\ &= B_{2n} = P_{2n}Q_{2n}. \end{aligned}$$

□

Like the previous theorem, the following theorem deals with the solution of a Diophantine equation, consists of finding two natural numbers, the larger one being odd, such that the sum of all natural numbers up to the larger number is equal square of the smaller number.

Theorem 4.4. *The solutions of the Diophantine equation* $1+2+\cdots+(2x-1)=y^2$ *are* $x=P_{2n-1}^2$ *and* $y=B_{2n-1}$, $n=1,2,\ldots$.

Proof. The Diophantine equation

$$1+2+\cdots+(2x-1)=y^2$$

is equivalent to

$$x(2x-1)=y^2.$$

Since x and $2x-1$ are relatively prime to each other, both x and $2x-1$ must be squares. Letting

$$2x-1=(2k+1)^2,$$

we get

$$x=2k^2+2k+1=k^2+(k+1)^2.$$

Letting $x=l^2$ the last equation takes the form

$$k^2+(k+1)^2=l^2.$$

The solutions of this Diophantine equation are [3, p.1199]

$$k=b_n+r_n=B_{n-1}+R_n, \quad n=1,2,\ldots,$$

and

$$l=\sqrt{2k^2+2k+1}.$$

Using Binet forms for B_n and b_n from (10) it can be seen that

$$l=P_{2n-1}.$$

Thus

$$x=P_{2n-1}^2,$$

and using Theorem 3.1 and the fact that

$$2P_{2n-1}^2-1=Q_{2n-1}^2,$$

we get

$$y = \sqrt{P_{2n-1}^2(2P_{2n-1}^2 - 1)} = P_{2n-1}Q_{2n-1} = B_{2n-1}. \qquad \square$$

In the last two theorems, if we keep the larger number unrestricted, then we have the following theorem.

Theorem 4.5. *The solutions of the Diophantine equation* $1+2+\cdots+x = y^2$ *are* $x = B_n + R_n$ *which is approximately equal to* Q_n^2, *and* $y = P_nQ_n = B_n$.

Proof. The Diophantine equation

$$1 + 2 + \cdots + x = y^2$$

is equivalent to

$$\frac{x(x+1)}{2} = y^2,$$

implying that y^2 is a square triangular number. Taking $y = B_n$ and using the Binet forms from (3) and (10) it can be easily verified that

$$x = B_n + R_n = \begin{cases} Q_n^2, & \text{if } n \text{ is odd,} \\ Q_n^2 - 1, & \text{if } n \text{ is even.} \end{cases} \qquad \square$$

In the following theorem we consider finding two natural numbers such that the sum of natural numbers up to the larger number decreased by the smaller number is equal to square of the smaller number.

Theorem 4.6. *The solutions of the Diophantine equation* $1+2+\cdots+(y-1)+(y+1)+\cdots+x = y^2$ *are* $x = b_n + r_n$ *and* $y = b_n$, $n = 1, 2, \ldots$.

Proof. The Diophantine equation

$$1 + 2 + \cdots + (y-1) + (y+1) + \cdots + x = y^2$$

is equivalent to

$$\frac{x(x+1)}{2} = y(y+1),$$

implying that $y(y+1)$ is a pronic triangular number. Taking $y = b_n$ [3, p.1190] and using Theorem 2.1 we get

$$x = \frac{-1+\sqrt{8b_n^2+8b_n+1}}{2} = b_n + r_n,$$

$n = 1, 2, \ldots$. □

Like Theorem 4.6, in the following theorem, we consider finding two natural numbers such that the larger one is even and the sum of natural numbers up to the larger number decreased by the smaller number is equal to square of the smaller number.

Theorem 4.7. *The Diophantine equation* $1+2+\cdots+(y-1)+(y+1)+\cdots+2x = y^2$ *has no solution if* x *is odd. If* x *is even, the solutions are given by* $x = (b_{2n}+r_{2n})/2$ *and* $y = b_{2n}$, $n = 1, 2, \ldots$.

Proof. The Diophantine equation

$$1+2+\cdots+(y-1)+(y+1)+\cdots+2x = y^2$$

is equivalent to

$$x(2x+1) = y(y+1).$$

If x is odd, then left hand side is also odd, but the right hand side is even. Hence in this case no solution exists. If x is even then solving the above equation for x we get

$$x = \frac{-1+\sqrt{8y^2+8y+1}}{4}.$$

Since $y(y+1)$ is a pronic triangular number, it follows that y is a cobalancing number [3, p.1190], say $y = b_n$ and since

$$r_n = \frac{-(2b_n+1)+\sqrt{8b_n^2+8b_n+1}}{2},$$

[3, p.1190], we have

$$x = (b_n + r_n)/2.$$

But $b_n + r_n$ is even only when n is even. Hence the totality of solutions is given by

$$x = (b_{2n} + r_{2n})/2, \quad y = b_{2n}, \quad n = 1, 2, \ldots. \qquad \square$$

Like Theorem 4.7, in the following theorem, we consider finding two natural numbers such that the larger one is odd and the sum of natural numbers up to the larger number decreased by the smaller number is equal to square of the smaller number.

Theorem 4.8. *The Diophantine equation* $1 + 2 + \cdots + (y-1) + (y+1) + \cdots + (2x-1) = y^2$ *has no solution if* x *is odd. If* x *is even, the solutions are given by* $x = (b_{2n-1} + r_{2n-1} + 1)/2$ *and* $y = b_{2n}$, $n = 1, 2, \ldots.$

Proof. The Diophantine equation

$$1 + 2 + \cdots + (y-1) + (y+1) + \cdots + (2x-1) = y^2$$

is equivalent to

$$x(2x-1) = y(y+1).$$

If x is odd, then left hand side is also odd, but the right hand side is even. Hence in this case no solution exists. If x is even, then solving the above equation for x we get

$$x = \frac{1 + \sqrt{8y^2 + 8y + 1}}{4}.$$

Since $y(y+1)$ is a pronic triangular number, it follows that y is a cobalancing number [3, p.1190], say $y = b_n$ and then

$$x = (b_n + r_n + 1)/2.$$

But $b_n + r_n + 1$ is even only when n is odd. Hence the solutions in this case are given by

$$x = (b_{2n-1} + r_{2n-1} + 1)/2, \quad y = b_{2n-1}, \quad n = 1, 2, \ldots. \qquad \square$$

The Diophantine equation $x^2 + y^2 = z^2$, where $x, y, z \in \mathbb{Z}^+$ is known as the Pythagorean equation and is available extensively in the literature. The particular case $x^2 + (x+1)^2 = y^2$ has been studied in [1, 3], wherein, the solutions are obtained in terms of balancing and cobalancing numbers. Let us call the equation $x^2 + y^2 = z^2 \pm 1$, an *almost Pythagorean equation*. In the following theorem we consider the Diophantine equations $x^2+(x+1)^2 = y^2 \pm 1$, which is a particular case of the almost Pythagorean equation.

Theorem 4.9. *The almost Pythagorean equation $x^2+(x+1)^2 = y^2+1$ has the solutions $x = B_n + b_n$ and $y = 2B_n = P_{2n}$, $n = 1, 2, \ldots$, whereas, the equation $x^2 + (x+1)^2 = y^2 - 1$ has no solution.*

Proof. The Diophantine equation

$$x^2 + (x+1)^2 = y^2 + 1$$

is equivalent to

$$\frac{x(x+1)}{2} = \frac{y^2}{4},$$

showing that $\frac{x(x+1)}{2}$ is a square triangular number. Hence

$$\frac{x(x+1)}{2} = B_n^2,$$

[see 1, p. 99]. Thus $y = 2B_n = P_{2n}$ and then

$$x = \frac{-1 + \sqrt{8B_n^2 + 1}}{2}.$$

Since from the definition of balancing numbers and balancers [1, p.99]

$$R_n = \frac{-1 + \sqrt{8B_n^2 + 1}}{2},$$

and by Theorem 2.1, $R_n = b_n$ for each n, it follows that $x = B_n + b_n$.

The equation

$$x^2 + (x+1)^2 = y^2 - 1$$

is equivalent to

$$2(x^2 + x + 1) = y^2.$$

Thus, y^2 is an even number and hence is divisible by 4. This indicates that $x^2 + x + 1$ is also even. But the pronic number $x^2 + x$ is always even, hence $x^2 + x + 1$ is always odd. Hence, in this case, no solution exists. □

The following theorem links the solutions of a Pythagorean equation with balancing numbers, cobalancing numbers and Pell numbers.

Theorem 4.10. *The Pythagorean equation $x^2 + (x + 2)^2 = y^2$ has the solutions $x = 2(B_{n-1} + b_n) = c_n - 1$ and $y = 2P_{2n+1}$, $n = 1, 2, \ldots$.*

Proof. The Diophantine equation

$$x^2 + (x + 2)^2 = y^2$$

is equivalent to

$$2(x^2 + 2x + 2) = y^2,$$

which indicates that y is even and hence x^2+2x+2 is also even, and thereby x is also even. Taking $x = 2u$ and $y = 2v$ the above equation reduces to

$$2u^2 + 2u + 1 = v^2,$$

which is the Pythagorean equation

$$u^2 + (u + 1)^2 = v^2.$$

The solutions of this equation are given by [3, p.1199],

$$u = b_n + r_n, \quad v = \sqrt{2u^2 + 2u + 1}.$$

Using the Binet forms of b_n, r_n and P_n, it can be easily verified that

$$2(b_n + r_n)^2 + 2(b_n + r_n) + 1 = P_{2n+1}^2.$$

Since $r_n = B_{n-1}$ by Theorem 2.1, it follows that the solutions of

$$u^2 + (u + 1)^2 = v^2$$

are given by

$$u = B_{n-1} + b_n, \quad v = P_{2n+1}, \quad n = 1, 2, \ldots.$$

Thus, the solutions of the Diophantine equation $x^2 + (x+2)^2 = y^2$ are given by

$$x = 2(B_{n-1} + b_n), \quad y = 2P_{2n+1}, \quad n = 1, 2, \ldots.$$

Using the Binet form for b_n, r_n and c_n, it can be easily seen that

$$2(b_n + r_n) + 1 = c_n.$$

Hence x can be alternatively given by $x = c_n - 1$. □

Replacing x by $x - 1$ in the above theorem, we get the following interesting result:

Corollary 4.11. *The Pythagorean equation* $(x-1)^2 + (x+1)^2 = y^2$ *has the solutions* $x = c_n = Q_{2n-1}$ *and* $y = 2P_{2n+1}$, $n = 1, 2, \ldots$.

The following theorem establishes a link of solutions of a Pythagorean equation involving consecutive triangular numbers with balancing numbers, associated Pell numbers and Lucas-cobalancing numbers.

Theorem 4.12. *The Pythagorean equation* $\left[\frac{x(x-1)}{2}\right]^2 + \left[\frac{x(x+1)}{2}\right]^2 = y^2$ *has the solutions* $x = Q_{2n-1} = c_n$ *and* $y = B_{2n-1}$, $n = 1, 2, \ldots$.

Proof. The Pythagorean equation

$$\left[\frac{x(x-1)}{2}\right]^2 + \left[\frac{x(x+1)}{2}\right]^2 = y^2$$

is equivalent to

$$\frac{x^2(x^2+1)}{2} = y^2, \tag{25}$$

indicating that y^2 is a square triangular number, hence y is a balancing number [1, p.99], say $y = B_n$ for some n. Now solving (25) for x^2 and using the relationships between B_n and R_n [1, p.99], we get

$$x^2 = \frac{-1 + \sqrt{8B_n^2 + 1}}{2} = B_n + R_n.$$

In the proof of Theorem 4.5, we have shown that $B_n + R_n$ is a perfect square only when n is odd and is equal to Q_n^2. Hence the totality of solutions of the Pythagorean equation

$$\left[\frac{x(x-1)}{2}\right]^2 + \left[\frac{x(x+1)}{2}\right]^2 = y^2$$

are given by $x = Q_{2n-1}$ and $y = B_{2n-1}$, $n = 1, 2, \ldots$. Indeed, by virtue of Theorem 3.3, $Q_{2n-1} = c_n$. □

References

1. A. Behera and G. K. Panda, On the square roots of triangular numbers, *The Fibonacci Quarterly*, **37** (1999), No. 2, 98-105.

2. K. Liptai, Fibonacci balancing numbers, *The Fibonacci Quarterly*, **42** (2004), No. 4, 330-340.

3. G. K. Panda and P. K. Ray, Cobalancing numbers and cobalancers, *International Journal of Mathematics and Mathematical Sciences*, **2005** (2005), No. 8, 1189-1200.

4. G. K. Panda, Some fascinating properties of balancing numbers, *Congressus Numerantium, Proceedings of the Eleventh International Conference on Fibonacci Numbers and Their Applications*, (Willian Webb, Ed.), Vol. 194, 2009.

5. G. K. Panda, Sequence balancing and cobalancing numbers, *The Fibonacci Quarterly*, **45** (2007), No. 3, 265-271.

6. K. B. Subramaniam, A simple computation of square triangular numbers, *International Journal of Mathematics Education in Science and Technology*, **23** (1992), No. 5, 790-793.

7. K. B. Subramaniam, A divisibility property of square triangular numbers, *International Journal of Mathematics Education in Science and Technology*, **26** (1995), No. 2, 284-286.

8. K. B. Subramaniam, Almost square triangular numbers, *The Fibonacci Quarterly*, **37** (1999), No. 3, 194-197.