Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **8** (2013), No. 3, pp. 351-358

$\overline{\partial}$-NEUMANN AND RELATED QUESTIONS

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Abstract

I shall discuss some of the reasons for the study of the $\overline{\partial}$-Neumann problem and speculate about its nonelliptic analogue the so called $\overline{\partial}_b$-Neumann problem, plus the possibility of a nonelliptic index. The talk will point to the necessity of understanding CR-geometry more precisely than is understood at present.

1. Given a (0,1)-form

$$f = f_1 d\overline{z}_1 + \cdots + f_{n+1} d\overline{z}_{n+1} \tag{1}$$

find a function u such that

$$f = \overline{\partial} u = \frac{\partial u}{\partial \overline{z}_1} d\overline{z}_1 + \cdots + \frac{\partial u}{\partial \overline{z}_{n+1}} d\overline{z}_{n+1}. \tag{2}$$

Here $z_j = x_j + i x_{j+n+1}$, $\partial/\partial z_j = (1/2)(\partial/\partial x_j - i\partial/\partial x_{j+n+1})$, $j = 1, 2, \ldots, n+1$. Since second derivatives commute, one has the following compatibility conditions on f:

$$\frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\partial \overline{z}_j} \tag{3}$$

which one writes in the following form

$$\overline{\partial} f = \sum_{j<k} \Big(\frac{\partial f_k}{\partial \overline{z}_j} - \frac{\partial f_j}{\partial \overline{z}_k} \Big) d\overline{z}_j \wedge d\overline{z}_k = 0; \tag{4}$$

Received December 5, 2012.
AMS Subject Classification: 58A30, 53C22.
Key words and phrases: Subelliptic heat kernels, subRiemannian geometry.
Research partially supported by NSERC Grant OGP0003047.

in words,

"f is a $\overline{\partial}$-closed form."

According to Dolbeault local solutions u exist.

We are interested in global solutions in a bounded domain. The $\overline{\partial}$-operator has a nullspace when acting on functions, namely all holomorphic functions. If f is square integrable it is natural to look for a square integrable u with minimal L^2-norm, that is for a u which is orthogonal to the nullspace of $\overline{\partial}$ which is the space of holomorphic functions. This is the canonical solution. To find it, set

$$\Lambda^{(0,0)} \underset{\theta}{\overset{\overline{\partial}}{\rightleftarrows}} \Lambda^{(0,1)} \underset{\theta}{\overset{\overline{\partial}}{\rightleftarrows}} \Lambda^{(0,2)} \cdots, \quad \theta = \overline{\partial}^*; \tag{5}$$

$\overline{\partial}$ is an exterior derivative and θ is a contraction. In view of Dolbeault's Lemma,

$$\theta + \overline{\partial} : \Lambda^{(0,1)} \longrightarrow \Lambda^{(0,0)} \oplus \Lambda^{(0,2)} \tag{6}$$

is locally 1-1, hence so is

$$\Box = (\theta + \overline{\partial})^*(\theta + \overline{\partial}) = \overline{\partial}\theta + \theta\overline{\partial} : \Lambda^{(0,1)} \longrightarrow \Lambda^{(0,1)}. \tag{7}$$

In particular, $\Box v = f$ and $\overline{\partial} f = 0$ imply that $\theta\overline{\partial} f = 0$ and then $u = \theta v$ is the canonical solution of $\overline{\partial} u = f$. According to (7) one needs

$$v \in \mathrm{Dom}(\theta), \qquad \overline{\partial} v \in \mathrm{Dom}(\theta). \tag{8}$$

Integrating by parts these requirements may be transferred to the boundary and yield the $\overline{\partial}$-Neumann boundary conditions; (8) characterizes the space of functions on which $\Box$ is selfadjoint. In our terminology $\Box = -\triangle$, where $\triangle$ is the Laplace-Beltrami operator in the induced metric, so $\Box$ is elliptic, and its nullspace may be identified with the boundary values of its harmonic forms via the Poisson kernel. Let Ω denote a bounded domain and let r represent the geodesic distance of its points from the boundary $b\Omega$. Let

$$\frac{1}{\sqrt{2}}\frac{\partial}{\partial r} + i\frac{1}{\sqrt{2}}T, \qquad \mathbb{Z}_1, \ldots, \mathbb{Z}_n. \tag{9}$$

denote an orthonormal basis of $T^{(1,0)}\Omega$ in some neightbourhood of the boundary. The $\overline{\partial}$-Neumann boundary operator applied to the Poisson kernel yields

$$\Box_+ = iT + \sqrt{\triangle_b} : C^\infty(b\Omega) \to C^\infty(b\Omega), \tag{10}$$

where $\triangle_b$ represents the Laplace-Beltrami operator on $b\Omega$. Now $\triangle_b = -\Box_b + T^2$ with

$$\Box_b = \frac{1}{2}\sum_{j=1}^{n}(Z_jZ_j^* + \overline{Z}_j\overline{Z}_j^*)I_n + \text{first order}. \tag{11}$$

$\Box_+$ is not elliptic. Indeed, when $\sigma(\Box_b) = 0$ one has $\sigma(\triangle_b) = -\sigma(T)^2$, so

$$\sigma(\Box_+) = i\sigma(T) + i|\sigma(T)| \tag{12}$$

vanishes on the half line $\sigma(\Box_b) = 0$, $\sigma(T) < 0$. Still, with $\Box_- = iT - \sqrt{\triangle_b}$ one has

$$\Box_+^{-1} = \Box_b^{-1}\Box_-, \tag{13}$$

and $\Box_b^{-1}$ has been constructed when $b\Omega$ is strongly pseudoconvex; $(r_{z_j\overline{z}_k}) > 0$ on tangential holomorphic vectorfields. Now the formula for $\Box_+^{-1}$ leads to the explicit $\overline{\partial}$-Neumann kernel.

Remark. A sum of squares of vectorfields is invertible if the vectorfields are bracket generating. Similarly, $\Box_b$, which is a system, is invertible when Kohn's nonvanishing ideal sheaf condition holds.

2. $\Box_b$ of (11) is not an elliptic operator. It is selfadjoint on a bounded domain M if $\mathrm{Dom}(\Box_b)$ is given by the $\overline{\partial}_b$-boundary conditions; again, these are found by integration by parts. The question of existence of solutions to this problem is the $\overline{\partial}_b$-Neumann problem; it played a role in the proof of Kuranishi's imbedding theorem. Since $\Box_b$ is not elliptic even the existence of the Poisson kernel is a problem. I shall restrict my attention to the Heisenberg group H_n, which is a prototype for strongly pseudoconvex CR-structures, and suggest two possible approaches to finding the $\overline{\partial}_b$-Neumann kernel. The Heisenberg group H_n is $\mathbb{C}_n \times \mathbb{R}$ with the underlying geometry induced by the vectorfields

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j\frac{\partial}{\partial y}, \qquad j = 1, \ldots, n. \tag{14}$$

(i) On H_1 let M denote the unit Korányi ball, $|z|^4 + y^2 \leq 1$. Set $y + i|z|^2 = \rho^2 e^{i\varphi}$. Following tradition one searches for spherical harmonics, i.e. for the boundary restriction of Heisenberg homogeneous polynomials in the nullspace of $\Box_b$; the Heisenberg dilation is given by $\lambda(z,y) = (\lambda z, \lambda^2 y)$. Separating variables one finds that the critical factor in these spherical harmonics of degree m is $H^{(\ell)}_{(m-|\ell|)/2}(e^{i\varphi})$, $\ell \in \mathbb{Z}$, generated by

$$(1 - re^{i\varphi})^{-\gamma}(1 - re^{-i\varphi})^{-\gamma+\ell} = \sum_{k=0}^{\infty} r^k H^{(\ell)}_k(e^{i\varphi}), \tag{15}$$

where $\gamma = (|\ell|+1)/2 + \ell/2$, $\ell \in \mathbb{Z}$ such that $(m-|\ell|)/2 = 0, 1, 2, \dots$, $H^{(\ell)}_k(e^{i\varphi})$ are Laurent polynomials, or twisted Legendre polynomials. They are complete but not orthogonal; orthogonality may suggest that $\Box_b$ is elliptic which it is not. Dunkl found a sequence of functions expressed in terms of Meixner-Pollaczek polynomials which is biorthogonal to $H^{(\ell)}_k(e^{i\varphi})$, $k = 0, 1, 2, \dots$. This yields a Poisson kernel and, in principle, the $\overline{\partial}_b$-Neumann kernel. All this works on H_n, $n \geq 1$, too.

(ii) More geometrically on H_n one has

$$\begin{aligned} \triangle_H &= \frac{1}{2}\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_j Z_j) \\ &= \sum_{j=1}^{n}\Big\{\frac{\partial^2}{\partial z_j \partial \overline{z}_j} + |z_j|^2\frac{\partial^2}{\partial y^2} + \frac{\partial}{i\partial y}\Big(z_j\frac{\partial}{\partial z_j} - \overline{z}_j\frac{\partial}{\partial \overline{z}_j}\Big)\Big\}. \end{aligned} \tag{16}$$

$\triangle_H$ is not elliptic at $z = 0$, but

$$\triangle_\varepsilon = \triangle_H + \varepsilon^2\frac{\partial^2}{\partial y^2} \tag{17}$$

is. $\triangle_\varepsilon$ is selfadjoint with respect to the ε-boundary conditions. Let B_ε denote the unit ball for the Riemannian distance induced by $\triangle_\varepsilon$. $\triangle_H = \triangle_0$ is subelliptic and induces a subRiemannian geometry which allows more than one geodesic between two given points, in general. There always exists a shortest geodesic whose length represents the Carnot-Carathéodory distance between the two points. The Carnot-Carathéodory unit ball is $B_0 = \lim_{\varepsilon\to 0} B_\varepsilon$; note that B_0 is singular. $\triangle_\varepsilon$ is elliptic and one should find its Neumann kernel by the method of 1, then letting $\varepsilon \to 0$ one obtains the $\overline{\partial}_b$-Neumann kernel on

B_0. In principle we reduced the boundary problem for a nonelliptic operator to the solution of a boundary problem for an elliptic operator modulo a limiting procedure.

3. So the construction of both the $\overline{\partial}$-Neumann kernel and the $\overline{\partial}_b$-Neumann kernel is reduced to inverting a $\Box_b$-like operator on the boundary. Since

$$\Box_b^{-1} = \int_0^\infty e^{-t\Box_b} dt, \tag{18}$$

we shall look for heat kernels. To discover the form of a subelliptic heat kernel let me discuss $p_c = \ker e^{t\triangle_c}$, where $\triangle_c$ is a subLaplacian on S^{2n+1}; we note that S^{2n+1} is a prototype of strongly pseudoconvex CR-structures, just like H_n is, but the heat kernel on S^{2n+1} yields more information about subelliptic heat kernels in general than the heat kernel on H_n. Let $\sqrt{2}Z_1, \ldots, \sqrt{2}Z_n$ denote an orthonormal basis of the subspace of holomorphic vectorfields in $\mathbb{C}^{n+1}$ which are tangent to S^{2n+1}. Then

$$\triangle_c = -2Re \sum_{j=1}^{n} Z_j^* Z_j. \tag{19}$$

Theorem. *p_c is a function of t and $z \cdot \overline{w} = z_1\overline{w}_1 + \cdots + z_{n+1}\overline{w}_{n+1} = \cos\theta e^{i\varphi}$ only,*

$$p_c = \frac{e^{\frac{n^2}{2}t}}{(2\pi t)^{n+1}} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{-\frac{f(u,\kappa+i2k\pi)}{2t}} V_n(\kappa + i2k\pi, t) du, \tag{20}$$

where

$$f(u,\kappa) = u^2 - \kappa^2 = u^2 - (\cosh^{-1}(\cos\theta\cosh(u+i\varphi)))^2, \tag{21}$$

$$V_n(x,t) = \sum_{j=0}^{n-1} V_{n,j}(x)t^j, \qquad V_{n,0}(x) = \Big(\frac{x}{\sinh x}\Big)^n, \ldots \tag{22}$$

It is interesting to compare p_c to the heat kernel p_s of the Laplace-Beltrami operator $\triangle_s$ on S^{2n+1}:

$$p_s = \frac{e^{\frac{n^2}{2}t}}{(2\pi t)^{n+1/2}} \sum_{k\in\mathbb{Z}} e^{-\frac{(\gamma+2k\pi)^2}{2t}} V_n(i\gamma + i2k\pi, t). \tag{23}$$

γ is the angle subtended by the arc on S^{2n+1} which joins the two points in p_s on a great circle, therefore $|\gamma+2k\pi|$, $k \in Z$, are geodesic lengths. So are the values of $f(u,\kappa)^{1/2}$ at the critical points of f as a function of u. p_c and p_s are global formulas. In general one finds local heat kernels where $k=0$ and $0 \leq j < \infty$. When $k=0$, p_s has one distance while p_c has many. The most interesting fact is that the p_c integrand is found in a manner completely analogous to the construction of p_s, word-for-word. This suggests that heat kernels of second order subelliptic operators, in general, may be constructed the same way as one normally derives heat kernels for elliptic operators.

4. The expansion of the heat kernel in powers of t also yields geometric information about the underlying manifold other than geodesic lengths, assuming that the differential operator is in some sense induced by the underlying geometry. The index of an operator L defined by

$$\operatorname{ind} L = \dim(\text{Null } L) - \dim(\text{Null } L^*), \tag{24}$$

is a topological invariant of L, meaning that it is invariant under small perturbations of L; the Gauss-Bonnet Theorem and the Riemann Roch formula are indices of geometrically defined differential operators. For an example consider the quadric ϕ_ε given as an intersection,

$$\phi_\varepsilon = \{z_1^2 + \cdots + z_{n+1}^2 = \varepsilon^2\} \cap \{|z_1|^2 + \cdots + |z_{n+1}|^2 \leq 1\}. \tag{25}$$

ϕ_ε is an n-dimensional complex analytic variety with a strongly pseudoconvex boundary $b\phi_\varepsilon$. When $\varepsilon = 0$, Stephen Yau calculated the $\overline{\partial}_b$-cohomology on $b\phi_0$. The alternating sum is the index of $\overline{\partial}_b$ modulo holomorphic functions. The index can also be found from

$$\operatorname{ind} L = \dim\ker L^*L - \dim\ker LL^* = \operatorname{trace}(e^{-tL^*L} - e^{-tLL^*}), \tag{26}$$

where

$$L \;=\; \overline{\partial}_b \oplus \theta_b : \sum_{q\,\text{odd}} \oplus \widetilde{H}^{p,q} \to \sum_{q\ \text{even}} \oplus \widetilde{H}^{p,q}, \tag{27}$$

$$\widetilde{H}^{p,q} \;=\; \begin{cases} H^{p,q}, & 0 < q \leq n, \\ I - C^{p,0}H^{p,0}, & q = 0; \end{cases} \tag{28}$$

$C^{p,0}$ is the Cauchy-Szegö projection. Note that the boundary is a twisted sphere so the work in 3 may be useful.

ϕ_0 has no interior cohomology. ϕ_ε, $\varepsilon > 0$, has interior cohomology, the so called vanishing classes, but no $\overline{\partial}_b$-cohomology on the boundary. For $\varepsilon > 0$ the interior Euler characteristic agrees with the index of $\overline{\partial}_b$ on $b\varphi_0$. Consequently this number is independent of $\varepsilon \geq 0$, so it is the index of the $\overline{\partial}$-Neumann problem on ϕ_ε. As a Riemann-Roch formula for a bounded domain it should be derived by local heat kernel calculations. Similar results should hold for the intersection

$$\{z_1^{2m_1} + \cdots + z_{n+1}^{2m_{n+1}} = \varepsilon^2\} \cap \{|z_1|^2 + \cdots + |z_{n+1}|^2 \leq 1\}. \tag{29}$$

5. In Riemannian geometry one has control of the whole tangent space, while working with subRiemannian geometry we control only a subspace of the tangent space, which, of course needs to be bracket generating. A natural question is how to find the missing directions geometrically.

On the other side of the coin, in Riemannian geometry an "origin" always has a sufficiently small neighbourhood in which every point is connected to the "origin" by a unique geodesic. In subRiemannian geometry every point has at least one geodesic connection to the "origin", but no matter how small the neighbourhood is it contains points which may be joined to the "origin" by a finite number of geodesics, more than one, and it will also contain points which have an infinite number of geodesic connections to the "origin"; at least in all the examples worked out so far. In these examples, the Heisenberg group and the geometries induced by the Grusin operator and the subLaplacian on S^{2n+1}, one has one missing direction and the points which can be joined to an "origin" by an infinite number of geodesics of different lengths are parameterized by a curve through the "origin", the "canonical curve". A somewhat more interesting example is $\mathbb{R}^3$ equipped with the geometry induced by the step 3 vectorfields

$$X = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y}; \tag{30}$$

again one direction is missing. One has

Theorem. $y_0 > 0$. *Every point* $P(x, y, t)$, $y > 0$, *can be joined to* $P(0, y_0, 0)$ *by at least one local geodesic.*

(a) *The number of the geodesics is finite if and only if* (i) $y \neq y_0$, *or* (ii) $y = y_0$ *but* $t + y_0^2 x \neq 0$.

(b) *When* $y = y_0$ *and* $t + y_0^2 x = 0$, $P(x, y_0, t)$ *can be joined to* $P(0, y_0, 0)$ *by a discrete infinity of local geodesics.*

For the missing directions in general we make the following

Conjecture. *Given* m *bracket generating vectorfields on an* n*-dimensional manifold* M_n, *for every point* $p \in M_n$ *there is an* $n-m$ *dimensional canonical submanifold* S_p, $p \in S_p$, *characterized by having all its points connected to* p *by infinite number of geodesics.*

Clearly, the tangent space of S_p at P yields canonically defined missing directions.

Remark. I venture to suggest that in the long run heat kernels will be derived by path integration.

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