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NOTES ON DENSITY OF THE ORDINARY LOCUS

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Abstract

The present are the notes for the author's talk given in the Conference on Diophantine Problems and Arithmetic Dynamics held at Academia Sinica, Taipei, June 24–28, 2013. The purposes are two-folds. The first is to introduce some background on the problem of density of the ordinary locus of the modulo p of certain Shimura varieties. The second aim is to show new results about the ordinary locus in reduction modulo p of the Hilbert-Siegel moduli spaces. This is a survey article of the current status of the aforementioned problem.

1. Introduction (the statement of the main result)

Throughout this section let p denote a prime number. Let F be a totally real number field of degree d and let O_F denote the ring of integers. A *polarized abelian O_F-variety* is a triple (A, λ, ι), where

- A is an abelian variety,
- $\iota : O_F \to \mathrm{End}(A)$ is a ring monomorphism, and
- $\lambda : A \to A^t$ is a polarization such that

$$\lambda \circ \iota(a) = \iota(a)^t \circ \lambda, \quad \forall\, a \in O_F. \tag{1.1}$$

A pair (A, ι) as above is called an *abelian O_F-variety*. A polarization λ on an abelian O_F-variety (A, ι) that satisfies the condition (1.1) is said to be an *O_F-linear* polarization. Clearly, one has the notion of families of polarized

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abelian O_F-varieties and we can formulate the moduli problem for these objects.

Notice that any abelian O_F-variety always has dimension divisible by $[F:\mathbb{Q}]$. Let $m \in \mathbb{Z}_+$ be a positive integer and put $g = md$. Let $\mathbf{M}$ be the moduli scheme over $\operatorname{Spec} \mathbb{Z}_{(p)}$ of g-dimensional prime-to-p degree polarized abelian O_F-varieties (A, λ, ι) together with a prime-to-p level structure and satisfying the Kottwitz determinant condition. $\mathbf{M}$ is called the Hilbert-Siegel moduli scheme of degree m associated to the totally real field F.

We have explained the objects polarized abelian O_F-varieties. Let us describe explicitly the prime-to-p level structure and determinant condition. This formulation is a special case of more general PEL-type Shimura varieties.

Let (V, ψ) be a vector space over F of dimension $2m$, together with a non-degenerate alternating pairing $\psi : V \times V \to \mathbb{Q}$ such that $\psi(ax, y) = \psi(x, ay)$ for all $a \in F$ and $x, y \in V$. The pair (V, ψ) is uniquely determined by m up to isomorphism. Choose an O_F-lattice L_0 in V so that $\psi(L_0, L_0) \subset \mathbb{Z}$ and its tensor $L_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ over $\mathbb{Z}_p$ is self-dual with respect to the pairing ψ. Denote by $\hat{\mathbb{Z}} = \lim \mathbb{Z}/n\mathbb{Z} = \prod_\ell \mathbb{Z}_\ell$ the profinite completion of $\mathbb{Z}$ and $\hat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$ its prime-to-p component. Let $N \geq 3$ be a prime-to-p integer. Put

$$U := \ker(G(\hat{\mathbb{Z}}) \to G(\mathbb{Z}/N\mathbb{Z})),$$

where G is the reductive group over $\mathbb{Q}$ defined by

$$G(\mathbb{Q}) = \{x \in \mathrm{GL}_F(V) \,|\, x'x \in \mathbb{Q}^\times \}, \tag{1.2}$$

where $x \mapsto x'$ is the adjoint with respect to the pairing ψ. Write $U = U_p U^p$, where $U_p = G(\mathbb{Z}_p)$ and $U^p \subset G(\mathbb{A}^{(p)})$ is an open compact subgroup. Here $\mathbb{A}^{(p)} = \hat{\mathbb{Z}}^{(p)} \otimes \mathbb{Q}$ denotes the prime-to-p adele ring of $\mathbb{Q}$.

For the prime-to-p level structure we refer to in the definition of $\mathbf{M}$ is an U^p-orbit $\bar{\eta}$ of $O_F \otimes \hat{\mathbb{Z}}^{(p)}$-linear isomorphisms

$$\eta : L_0 \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{(p)} \simeq T^{(p)}(A) := \prod_{\ell \neq p} T_\ell(A) \tag{1.3}$$

that preserve the pairings up to a scalar. There are two reasons of adding prime-to-p level structures. The first one is to rigidify the objects so that there is no non-trivial automorphisms of the objects and hence that the moduli functor becomes representable. The other one is to restrict the scope of the objects so that their prime-to-p Tate modules lie in the same "genus" of a given prime-to-p lattice $L_0 \otimes \hat{\mathbb{Z}}^{(p)}$. In particular, we have fixed a prime-to-p polarization type of objects and the moduli scheme $\mathbf{M}$ is of finite type.

We come to explain the Kottwitz determinant condition. This is a closed condition, which is necessary for the defined integral model $\mathbf{M}$ to be flat (but not sufficient for some PEL-type moduli spaces; see Pappas [25]). Choose a homomorphism of $\mathbb{R}$-groups $h : \mathbb{C}^\times \to G_\mathbb{R}$ so that $\psi(h(z)x, y) = \psi(x, h(\bar{z})y)$ for all $z \in \mathbb{C}$ and $x, y \in V_\mathbb{R} := V \otimes_\mathbb{Q} \mathbb{R}$ and that the pairing $(x, y) := \psi(x, h(i)y)$ is positive (or negative, but fixed for all) definite. The element $h(i)$ gives rise to a complex structure on $V_\mathbb{R}$ which commutes with the O_F-action. Let $V_\mathbb{C} = V^{-1,0} \oplus V^{0,-1}$ be the decomposition of eigenspaces so that $h(z)$ acts by z on $V^{-1,0}$. The determinant condition then is an equality of the following two characteristic functions:

$$\operatorname{char}(a|\operatorname{Lie}(A/S)) = \operatorname{char}(a|V^{-1,0}) \quad \text{in } O_S[\underline{X}] \tag{1.4}$$

for all $a \in O_F$, where S is the base scheme of the object (A, λ, ι).

We now state our main result. Let $\mathcal{M} := \mathbf{M} \otimes \overline{\mathbb{F}}_p$ be the reduction modulo p of the Hilbert-Siegel moduli scheme $\mathbf{M}$. Let $\mathcal{M}^{\mathrm{ord}} \subset \mathcal{M}$ be the ordinary locus of $\mathcal{M}$, which parameterizes the objects $(A, \lambda, \iota, \bar{\eta})$ in $\mathcal{M}$ whose underlying abelian variety A is *ordinary.* Recall that an abelian variety A over a field k of characteristic $p > 0$ is said to be *ordinary* if

$$A[p](\bar{k}) \simeq (\mathbb{Z}/p\mathbb{Z})^{\dim A}, \tag{1.5}$$

where $\bar{k}$ is an algebraic closure of k.

Theorem 1.1. *The ordinary locus $\mathcal{M}^{\mathrm{ord}}$ is open and dense in $\mathcal{M}$.*

Remark 1.2. In the case where the prime p does not divide the discriminant $\Delta_{F/\mathbb{Q}}$ of F over $\mathbb{Q}$, that is, p is unramified in F, the moduli space $\mathbf{M}$ is smooth over $\operatorname{Spec} \mathbb{Z}_{(p)}$, and Theorem 1.1 is known due to T. Wedhorn. The only new part of this theorem deals with the case where p divides the discriminant $\Delta_{F/\mathbb{Q}}$, the case where the moduli space has singularities. In the special case where $m = 1$, that is, $\mathbf{M}$ is a Hilbert moduli scheme, Theorem 1.1 is proved in [35].

In Section 4 we explain the strategy of the proof of Theorem 1.1. There are two main ingredients and we choose to explain the first part on the KR stratification. Some detailed proofs are given in Section 6.

2. Background

In this section we give an overview of some background of density of the ordinary locus in certain moduli spaces modulo p.

2.1. Siegel modular varieties

Let $g \geq 1$ be a positive integer and δ be a positive integer not necessarily divisible by p. Let $\mathcal{A}_{g,\delta}$ denote the moduli space over $\overline{\mathbb{F}}_p$ of g-dimensional polarized abelian varieties (A, λ) with $\deg \lambda = \delta^2$, and let $\mathcal{A}_{g,\delta}^{\text{ord}} \subset \mathcal{A}_{g,\delta}$ denote the ordinary locus. We have the following fundamental result.

Theorem 2.1 (Grothendieck-Messing, Mumford, Norman-Oort)**.** *The ordinary locus $\mathcal{A}_{g,\delta}^{\text{ord}}$ is open and dense in $\mathcal{A}_{g,\delta}$.*

Grothendieck and Messing [6, 16] established the deformation theory for abelian varieties and polarized abelian varieties. Mumford and Norman (see [17, 20]) established the tools for constructions of deformations of polarized abelian varieties through displays and the Cartier-Dieudonné theory. In [17] Mumford outlined a program for lifting polarized abelian varieties to characteristic zero by showing the density of the ordinary locus and using the canonical lifting of ordinary abelian varieties (see Katz [12]). Finally Norman and Oort showed the complete result in [22] by introducing more involved techniques. As is already pointed out, a main application of Theorem 2.1 is that any polarized abelian varieties in positive characteristic can be lifted to a polarized abelian variety in characteristic zero. Norman [21] gave another proof this lifting result; Norman's method was extended to lift abelian varieties with additional structures by the author [34]. Another application of Theorem 2.1 computes the dimension of the moduli space $\mathcal{A}_{g,\delta}$:

$$\dim \mathcal{A}_{g,\delta} = \frac{g(g+1)}{2}. \tag{2.1}$$

Generalizing the results of Theorem 2.1 J. Achter showed the density of the ordinary locus in Hilbert-Siegel moduli spaces under the assumption that p is unramified in F and the polarizations have mild degree in p. See [1] for detailed statements of Achter's results.

2.2. Stamm's example

Let F be a real quadratic field and assume that p is inert in F. Consider the Hilbert modular surface $\mathcal{M}$ over $\overline{\mathbb{F}}_p$ associated to the real quadratic field F (i.e. $m = 1$ and $d = 2$ in our case); this is a smooth quasi-projective surface. Let $\mathcal{M}_I \to \mathcal{M}$ be the Hilbert modular surface with Iwahori level structure at p. This moduli space is associated to the open compact subgroup $U = U_p U^p$ with

$$U_p = \left\{ \begin{pmatrix} * & * \\ p* & * \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathcal{O}_p), \quad \mathcal{O}_p := O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Stamm [31] showed that the ordinary locus $\mathcal{M}_I^{\mathrm{ord}}$ is not dense in $\mathcal{M}_I$. Notice that the ordinary locus $\mathcal{M}_I^{\mathrm{ord}}$ is nonempty as $\mathcal{M}^{\mathrm{ord}}$ is open and dense in $\mathcal{M}$. Stamm's example shows that in general the density of the ordinary locus may fail in the bad reduction cases.

In fact the moduli space $\mathcal{M}_I$ is equi-dimensional of dimension 2. There are 4 types of components; among them two are ordinary (whose generic points are all ordinary) and the other two are supersingular (namely they are entirely contained in the supersingular locus of $\mathcal{M}_I$). Moreover, the moduli space $\mathcal{M}_I$ has

$$2 + 2\#G(\mathbb{Z}/N\mathbb{Z}) \frac{\zeta_F(-1)}{4}$$

irreducible components, where $\zeta_F(s)$ is the Dedekind zeta function of the totally real field F; see [37, 39]. The appearance of these 4 types of components may be best explained by the Kottwitz-Rapoport stratification; see Kottwitz-Rapoport [15] and Ngô-Genestier [18] also cf. the expository article [42].

2.3. Siegel modular varieties with Iwahori level structure

Let $\mathcal{A}_I$ be Siegel modular varieties over $\overline{\mathbb{F}}_p$ with Iwahori level structure at p. The moduli space $\mathcal{A}_I$ parametrizes isomorphism classes of the following objects

$$(A_0 \xrightarrow{\alpha} A_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} A_g, \lambda_0, \lambda_g),$$

where

- each A_i is a g-dimensional abelian variety,
- each α is an isogeny of degree p,
- λ_0 and λ_g are principal polarizations on A_0 and A_g, respectively, such that $(\alpha^g)^*\lambda_g = p\lambda_0$.

Ngô and Genestier [18] showed that the moduli space $\mathcal{A}_I$ is equi-dimensional of dimension $g(g+1)/2$ and that the ordinary locus $\mathcal{A}_I^{\rm ord}$ is open and dense in $\mathcal{A}_I$. The first statement follows from the second statement.

As an application of density of the ordinary locus, the author [36] showed that the moduli space $\mathcal{A}_I$ has 2^g irreducible components. This was previously a conjecture of de Jong [4] where the case $g=2$ was proven. The idea of the proof is to apply the p-adic monodromy theorem of Faltings and Chai [5] on the ordinary locus and show that each type of maximal KR stratum is irreducible. For further studies of the KR stratification on the moduli space $\mathcal{A}_I$; see Görtz and the author [8, 9]. The p-adic monodromy theorem of Faltings and Chai is generalized to any p-rank stratum; see [40].

2.4. A PEL-type good reduction case

We consider good reductions of the Picard modular surface associated to the unitary group $G=\mathrm{GU}(2,1)$. Let K be an imaginary quadratic field with ring of integers O_K. Assume that p is unramified in K. Let $\mathcal{M}_{(2,1)}^{O_K}$ denote the moduli space over $\overline{\mathbb{F}}_p$ of principally polarized abelian three folds (A,λ,ι) with an action by O_K with signature $(2,1)$ on the Lie algebra $\mathrm{Lie}(A)$. The occurrence of Newton polygons in the Picard modular surface $\mathcal{M}_{(2,1)}^{O_K}$ depends on the behavior of p in K.

(a) If p is *inert* in K, then all possible slope sequences are

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \quad \left(0,0,\frac{1}{2},\frac{1}{2},1,1\right).$$

(b) If p *splits* in K, then all possible slope sequences are

$$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3}\right), \quad \left(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right), \quad (0,0,0,1,1,1).$$

Let us verify this. For example in case (a) one has embeddings of semi-simple $\mathbb{Q}_p$-algebras $K\otimes\mathbb{Q}_p \hookrightarrow \mathrm{Mat}_3(D_{1/2})$ and $K\otimes\mathbb{Q}_p \hookrightarrow \mathrm{Mat}_2(\mathbb{Q}_p)\times D_{1/2}\times \mathrm{Mat}_2(\mathbb{Q}_p)$, where $D_{1/2}$ is the unique quaternion division algebra over $\mathbb{Q}_p$. Case (b) is a Lubin-Tate case; one can easily determine the slope sequences of one-dimensional p-divisible groups of height 3. One sees that in case (a) the ordinary sequence does not appear, namely, the ordinary locus of $\mathcal{M}^{O_K}_{(2,1)}$ is empty in the inert case. In case (b) the ordinary sequence does appear; however, there is no supersingular slope sequence. That is, the supersingular locus of $\mathcal{M}^{O_K}_{(2,1)}$ is empty in the split case.

The smallest Newton stratum corresponding to $(1/3, 1/3, 1/3, 2/3, 2/3, 2/3)$ in case (b) is called the *basic Newton stratum* in (a good reduction of) a general PEL-type moduli space .

In any case one expects that the lowest (largest) Newton stratum (strata) should be open and dense in good reduction of PEL-type moduli space. This problem was settled by T. Wedhorn [32] where he introduced the notion of μ-ordinary locus in terms of group theory. We will describe his results on density of μ-ordinary locus in the next section.

3. Density of μ-Ordinary Locus (following Wedhorn [32])

In this section we explain μ-ordinary points in a good reduction of a PEL-type moduli space. Our reference is Wedhorn [32]. As in Introduction, to define a moduli space of abelian with additional structures we need to specify a datum – called the *PEL datum*. Here the letter P stands for polarizations, E for endomorphisms, and L for level structures.

3.1. p-integral PEL datum

Let p be a prime as before. Let $\mathcal{D} := (B, *, V, \psi, O_B, L_0, h)$ be a tuple, called a *p-integral PEL datum*, where

- B is a finite-dimensional semi-simple algebra over $\mathbb{Q}$ together with a positive involution $*$, that is, one has $\mathrm{tr}_{B/\mathbb{Q}}(bb^*) > 0$ for all $b \neq 0 \in B$;
- V is a finite faithful B-module together with a non-degenerate $\mathbb{Q}$-valued skew-Hermitian form ψ. That is, $\psi : V \times V \to \mathbb{Q}$ is a non-degenerate alternating form such that $\psi(bx, y) = \psi(x, b^*y)$ for all $x, y \in V$ and

$b \in B$.

- $O_B \subset B$ is an order which is stable under $*$ and is maximal at p, that is, $O_B \otimes \mathbb{Z}_p$ is a maximal order in $B \otimes \mathbb{Q}_p$;
- $L_0 \subset V$ is an O_B-lattice such that $\psi(L_0, L_0) \subset \mathbb{Z}$ and $L_0 \otimes \mathbb{Z}_p$ is self-dual with respect to the pairing ψ;
- $h : \mathbb{C}^\times \to G_\mathbb{R}$ is a homomorphism of $\mathbb{R}$-groups such that $\mathrm{Int}(h(i))$ is a Cartan involution on the adjoint group $G_\mathbb{R}^{\mathrm{ad}}$.

Here the reductive group G over $\mathbb{Q}$ is defined by

$$G(\mathbb{Q}) := \{x \in GL_B(V) \,|\, x'x \in \mathbb{Q}^\times \}, \tag{3.1}$$

where $x \mapsto x'$ is the adjoint with respect to the pairing ψ. The datum $(B, *, V, \psi)$ is called a (rational) *PEL datum*. I think given a pair $(B, *)$, it is not always possible to admit an order O_B which is both stable under $*$ maximal at p. We need to assume that such an order exists. Similarly the existence of the O_B-lattice L_0 as above may also need some condition; we need to modify ψ, if necessary, so that such lattice L_0 exists. We also remark that the datum h is part of information determined by $(B, *, V, \psi)$ (up to conjugate by an element in $G(\mathbb{R})$; see Kottwitz [14, Section 4]).

3.2. μ-ordinarity

We assume that the group $G_{\mathbb{Q}_p}$ is unramified. That is, $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified finite field extension of $\mathbb{Q}_p$. Equivalently, $B \otimes \mathbb{Q}_p \simeq \prod_i \mathrm{Mat}_{n_i}(F_i)$, where each $F_i/\mathbb{Q}_p$ is an unramified field extension. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. This gives rise to a prime $\mathfrak{p} \subset O_E \subset E(G, X)$, where $E(G, X)$ is the reflex field of the data. Using the same formulation and the moduli interpretation as defined in Section 1, we define the moduli space $\mathcal{M}_\mathcal{D}$ of abelian varieties with the additional structures arising from the datum $\mathcal{D}$.

Let $k = \bar{k}$ be an algebraically closed field of characteristic $p > 0$. Kottwitz defines a Newton map

$$\nu : \mathcal{M}_\mathcal{D}(k) \to X_*(T)_\mathbb{Q}/\Omega_0, \tag{3.2}$$

where to each point $x = (A, \lambda, \iota, \bar{\eta}) \in \mathcal{M}_\mathcal{D}(k)$, one associates the Newton vector $\nu(x)$. Here $T \subset G$ is a maximal torus over $\mathbb{Q}_p$ and Ω_0 is the Weyl group of T.

Now let $\mu : \mathbb{G}_{\mathrm{m}\mathbb{C}} \to G_{\mathbb{C}}$ be the Hodge co-character ($\mu := h_{\mathbb{C}}(z, 1)$). The $G(\mathbb{C})$-conjugacy class of μ gives rise to element in $X_*(T)/\Omega_0$, which we denote by μ again. Denote by $\bar{\mu}$ the $\mathrm{Gal}(E_{\mathfrak{p}}/\mathbb{Q}_p)$-Galois average of μ:

$$\bar{\mu} := \frac{1}{\#\,\mathrm{Gal}(E\mathfrak{p}/\mathbb{Q}_p)} \sum_{\sigma \in \mathrm{Gal}(E_{\mathfrak{p}}/\mathbb{Q}_p)} \sigma\mu \in X_*(T)_{\mathbb{Q}}/\Omega_0. \tag{3.3}$$

Note that in general $E_{\mathfrak{p}}$ is not Galois over $\mathbb{Q}_p$. This means that

$$\mathrm{Gal}(E_{\mathfrak{p}}/\mathbb{Q}_p) = \mathrm{Gal}(\widetilde{E}_{\mathfrak{p}}/\mathbb{Q}_p)/\,\mathrm{Gal}(\widetilde{E}_{\mathfrak{p}}/E_{\mathfrak{p}}),$$

where $\widetilde{E}_{\mathfrak{p}}$ is any finite field extension containing $E_{\mathfrak{p}}$ which is Galois over $\mathbb{Q}_p$. Then the definition $\bar{\mu}$ in (3.3) makes sense.

In the case where G is connected, one has the Hodge-Newton inequality (in the Bruhat order)

$$\nu(x) \preceq \bar{\mu}, \quad \forall\, x \in \mathcal{M}_{\mathfrak{D}}(k). \tag{3.4}$$

In this case (that is, the group does not contain a $\mathbb{Q}$-simple factor which is of type D in the Dynkin classification), one defines the open subset

$$\mathcal{M}_{\mathfrak{D}}^{\mu\text{-ord}} := (x \in \mathcal{M}_{\mathfrak{D}} \mid \nu(x) = \bar{\mu}\}, \tag{3.5}$$

called the *μ-ordinary locus* of $\mathcal{M}_{\mathfrak{D}}$.

Now consider the case where the defining group G is not connected, that is, G contains a $\mathbb{Q}$-simple factor which is of type D. Let $\mu : \mathbb{G}_{\mathrm{m}\mathbb{C}} \to G_{\mathbb{C}}$ and let

$$\mu^{(1)} = \mu, \quad \mu^{(2)}, \ldots, \mu^{(m)} \tag{3.6}$$

be all Hodge co-characters obtained by $G(\mathbb{C})$-conjugates. For each i, let $\bar{\mu}^{(i)}$ be the Galois average of $\mu^{(i)}$ defined by (3.3). Then the Hodge-Newton inequality states that for any $x \in \mathcal{M}_{\mathfrak{D}}(k)$, one has

$$\nu(x) \preceq \bar{\mu}^{(i)}, \quad \text{for some } i = 1, \ldots, m. \tag{3.7}$$

Now one defines the *μ-ordinary locus* $\mathcal{M}_{\mathfrak{D}}^{\mu\text{-ord}}$ by

$$\mathcal{M}_{\mathfrak{D}}^{\mu\text{-ord}} := (x \in \mathcal{M}_{\mathfrak{D}} \mid \nu(x) = \bar{\mu}^{(i)} \text{ for some } i = 1, \ldots, m\}. \tag{3.8}$$

In other words, one defines the μ-ordinary locus to be union of all maximal Newton strata. Now we can state the main result of Wedhorn in [32].

Theorem 3.1 (Wedhorn). *Assume that $G_{\mathbb{Q}_p}$ is unramified and that $p \neq 2$. Then the μ-ordinary locus $\mathcal{M}_{\mathfrak{D}}^{\mu-ord} \subset \mathcal{M}_{\mathfrak{D}}$ is open and dense.*

In other words, the moduli space $\mathcal{M}_{\mathfrak{D}}$ is the union of the Zariski closure of maximal Newton strata.

In the case where G is not connected, we can define for each $i = 1, \dots, m$, the $\mu^{(i)}$-ordinary locus:

$$\mathcal{M}_{\mathfrak{D}}^{\mu^{(i)}\text{-ord}} := (x \in \mathcal{M}_{\mathfrak{D}} \mid \nu(x) = \bar{\mu}^{(i)} \ \}. \tag{3.9}$$

Then one has the disjoint decomposition

$$\mathcal{M}_{\mathfrak{D}}^{\mu\text{-ord}} = \coprod_{i=1}^{m} \mathcal{M}_{\mathfrak{D}}^{\mu^{(i)}\text{-ord}}.$$

Is it true that for each $i = 1, \dots, m$, the $\mu^{(i)}$-ordinary locus $\mathcal{M}_{\mathfrak{D}}^{\mu^{(i)}\text{-ord}}$ non-empty?

4. Ingredients of the Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into two parts.

1. We study the Kottwitz-Rapoport (KR) stratification on the moduli space $\mathcal{M}$. From this we show that the smooth locus $\mathcal{M}^{\rm sm} \subset \mathcal{M}$ is open and dense in $\mathcal{M}$.
2. Deform any point x in the smooth locus $\mathcal{M}^{\rm sm}$ to a point y in the ordinary locus $\mathcal{M}^{\rm ord}$. That is, the ordinary locus $\mathcal{M}^{\rm ord}$ is open and dense in the smooth locus $\mathcal{M}^{\rm sm}$.

We point out that the step (1) may be a non-trivial fact. This of course follows immediately if we knew that the special fiber $\mathcal{M}$ is reduced. However, the latter is not known yet. The step (1) of using the KR stratification is a new ingredient from the proof of Wedhorn's Theorem (Theorem 3.1). The method to construct deformations in the step (2) is the same as that in Wedhorn's proof. There is no new difficulty in this step though one still needs to treat the ramification.

U. Görtz showed the following result, which is in the content of the Rapoport-Zink conjecture [30, p. 95] about the flatness of local models .

Theorem 4.1 (Görtz [7]). *The structure morphism* $\mathbf{M} \to \operatorname{Spec} \mathbb{Z}_{(p)}$ *is topologically flat. That is, every generic points in the special fiber can be lifted to characteristic zero.*

Theorem 4.1 provides an evidence that the integral model $\mathbf{M}$ could be *flat* over $\operatorname{Spec} \mathbb{Z}_{(p)}$. It follows from Görtz's Theorem (Theorem 4.1) that the integral model $\mathbf{M} \to \operatorname{Spec} \mathbb{Z}_{(p)}$ would be flat provided that the special fiber $\mathcal{M}$ is reduced (or that the moduli scheme $\mathbf{M}$ is reduced).

Note that the flatness of an integral model usually does not say anything about whether or not the special fiber is reduced. There are already some examples given by curves: The special fiber of the minimal regular integral model for a projective smooth curve over a complete discrete valuation field K may have non-reduced irreducible components.

Let $\mathbf{M}^{\mathrm{can}}$ denote the scheme-theoretic closure of the general fiber $\mathbf{M}_{\mathbb{Q}}$ in $\mathbf{M}$. The closed subscheme $\mathbf{M}^{\mathrm{can}} \subset \mathbf{M}$ is defined by the ideal leaf in $\mathcal{O}_{\mathbf{M}}$ generated by the p-power torsions. By definition $\mathbf{M}^{\mathrm{can}} \to \operatorname{Spec} \mathbb{Z}_{(p)}$ is flat. Since the generic fiber $\mathbf{M}_{\mathbb{Q}}$ is reduced, so is the scheme $\mathbf{M}^{\mathrm{can}}$. Let $\mathcal{M}^{\mathrm{can}} := \mathbf{M}^{\mathrm{can}} \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$ denote the reduction modulo p of $\mathbf{M}^{\mathrm{can}}$. The following is a special case of results of Pappas and Xinwen Zhu [29].

Theorem 4.2 (Pappas-Zhu). *The reduced subscheme* $(\mathcal{M}^{\mathrm{can}})_{\mathrm{red}}$ *of* $\mathcal{M}^{\mathrm{can}}$ *is a union of normal subvarieties.*

In fact, they proved much more results concerning the closure of each KR stratum. The methods of Xuhua He in [11] may lead to conclude that $\mathbf{M}^{\mathrm{can}}$ is Cohen-Macaulay and normal, which is still under investigation (noticing that p could be ramified in the totally real field here).

Let I be the ideal sheaf of the closed subscheme $\mathbf{M}^{\mathrm{can}} \subset \mathbf{M}$. Since the inclusion map $\mathbf{M}^{\mathrm{can}} \to \mathbf{M}$ is a homeomorphism, the ideal I is nilpotent, that is $\mathbf{M}^{\mathrm{can}} = \mathbf{M}_{\mathrm{red}}$. Let

$$\begin{aligned}\operatorname{Supp}(I) &:= \{x \in \mathbf{M} \,|\, I_x \neq 0\} \\ &= \{x \in \mathbf{M} \,|\, \text{the surjective map } \mathcal{O}_{\mathbf{M},x} \to \mathcal{O}_{\mathbf{M}^{\mathrm{can}},x} \\ &\qquad\qquad \text{is not an isomorphism}\}. \qquad (4.1)\end{aligned}$$

It would be interesting to know what the support $\operatorname{Supp}(I)$ is. We know that $\operatorname{Supp}(I) \subset \mathcal{M}$ has co-dimension at least two.

Through the study of the KR stratification, Theorems 4.1 and 4.2 and together with a recent result of Kai-Wen Lan [24], we obtain the following result. A detailed proof of this result will be given elsewhere.

Theorem 4.3. *The moduli space $\mathcal{M}$ is irreducible (after fixing an N-th root of unity ζ_N).*

5. The Kottwitz-Rapoport Stratification

5.1. Local models

Now we come to explain the first part of the proof of Theorem 1.1. We show this through studying the Kottwitz-Rapoport (KR) stratification. Let F, V, ψ, L_0 be as in Section 1. Put

$$\mathcal{O}_p := O_F \otimes \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_v. \tag{5.1}$$

Let e_v and f_v be the ramification index and inert degree of v, respectively. Put $\Lambda := L_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p$; this is a free $\mathcal{O}_p$-module of rank $2m$ together with a $\mathbb{Z}_p$-valued skew-Hermitian form $\psi : \Lambda \times \Lambda \to \mathbb{Z}_p$. Let $\mathcal{G} := \mathrm{GU}(\Lambda, \psi)$ be the automorphism group scheme of (Λ, ψ) over $\mathbb{Z}_p$; this is a group scheme over $\mathbb{Z}_p$ which represents the following functor: For any $\mathbb{Z}_p$-algebra R, $\mathcal{G}(R) := \mathrm{GU}(\Lambda \otimes_{\mathbb{Z}_p} R, \psi)$, which consists of $\mathcal{O}_p \otimes R$-automorphisms on $\Lambda \otimes R$ that preserve the pairing ψ up to a scalar in $R^\times$. Denote by $\mathbf{M}_\Lambda$ the associated local model. This is a projective scheme over $\operatorname{Spec} \mathbb{Z}_p$ of finite type which represents the following functor. For any $\mathbb{Z}_p$-scheme S, $\mathbf{M}_\Lambda(S)$ is the set of all locally free $\mathcal{O}_S$-submodules $\mathcal{F} \subset \Lambda \otimes \mathcal{O}_S$ of rank md such that

- locally in the Zariski topology in S, $\mathcal{F}$ is a direct summand of $\Lambda \otimes \mathcal{O}_S$;
- $\mathcal{F}$ is $\mathcal{O}_p$-invariant and is isotropic with respect to the pairing ψ;
- $\mathcal{F}$ satisfies the determinant condition; cf. (1.4).

Let $\mathcal{M}_\Lambda := \mathbf{M}_\Lambda \otimes \overline{\mathbb{F}}_p$ be the reduction modulo p of the local model $\mathbf{M}_\Lambda$.

5.2. Local model diagrams

Let $\widetilde{\mathcal{M}}$ denote the moduli space over $\overline{\mathbb{F}}_p$ parameterizing equivalence classes of objects $(\underline{A}, \xi)$, where $\underline{A} = (A, \lambda, \iota, \bar{\eta})$ is an object in $\mathcal{M}$ and

$\xi : H_1^{\mathrm{DR}}(A/S) \simeq \Lambda \otimes \mathcal{O}_S$ is an $\mathcal{O}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-linear isomorphism that preserves the pairings up to a scalar in $\mathcal{O}_S^\times$. We have the local model diagram (see de Jong [4] and Rapoport-Zink [30]):

$$\begin{array}{ccc} & \widetilde{\mathcal{M}} & \\ {\scriptstyle \varphi^{\mathrm{mod}}} \swarrow & & \searrow {\scriptstyle \varphi^{\mathrm{loc}}} \\ \mathcal{M} & & \mathcal{M}_\Lambda, \end{array} \tag{5.2}$$

In the above diagram φ^{loc} is the morphism that sends each object $(\underline{A}, \xi)$ to the image $\xi(\omega')$ of the Hodge submodule $\omega' \subset H_1^{\mathrm{DR}}(A)$, and φ^{mod} is the morphism that forgets the trivialization ξ.

The special fiber $\mathcal{G} \otimes \overline{\mathbb{F}}_p$ of $\mathcal{G}$ acts on $\widetilde{\mathcal{M}}$ and on $\mathcal{M}_\Lambda$ from the left. One has that

- the morphism φ^{loc} is $\mathcal{G} \otimes \overline{\mathbb{F}}_p$-equivalent, surjective and smooth, and
- the morphism $\varphi^{\mathrm{mod}} : \widetilde{\mathcal{M}} \to \mathcal{M}$ is a $\mathcal{G} \otimes \overline{\mathbb{F}}_p$-torsor.

5.3. Kottwitz-Rapoport stratification

Let

$$\mathcal{M}_\Lambda = \coprod_{\underline{e} \in \mathrm{Adm}(\mu)} \mathcal{M}_{\Lambda, \underline{e}} \tag{5.3}$$

be the decomposition of $\mathcal{M}_\Lambda$ into the $\mathcal{G} \otimes \overline{\mathbb{F}}_p$-orbits, where $\mathrm{Adm}(\mu)$ is the finite index set for the orbit spaces. The set $\mathrm{Adm}(\mu)$ is also called the μ-admissible set, which is originally defined by group theory and shown to agree with the set of geometric orbits; see Section 5.4. For this moment, we just regard it as the orbit set, which is defined by geometry. Pull back the $\mathcal{G} \otimes \overline{\mathbb{F}}_p$-orbits in (5.3) to the moduli space $\widetilde{\mathcal{M}}$ and get a stratification of $\widetilde{\mathcal{M}}$ into locally closed subsets:

$$\widetilde{\mathcal{M}} = \coprod_{\underline{e} \in \mathrm{Adm}(\mu)} \widetilde{\mathcal{M}}_{\underline{e}}. \tag{5.4}$$

Since φ^{mod} is a $\mathcal{G} \otimes \overline{\mathbb{F}}_p$-torsor, the stratification of $\widetilde{\mathcal{M}}$ descends to a stratification of $\mathcal{M}$:

$$\mathcal{M} = \coprod_{\underline{e} \in \mathrm{Adm}(\mu)} \mathcal{M}_{\underline{e}}, \tag{5.5}$$

which is called the *KR stratification*. Notice that the morphism φ^{loc} is surjective if and only if each KR stratum $\mathcal{M}_{\underline{e}}$ is nonempty for $\underline{e} \in \mathrm{Adm}(\mu)$.

5.4. Description of $\mathrm{Adm}(\mu)$

Let k be an algebraically closed field of characteristic p, $W=W(k)$ the ring of Witt vectors over k and $L:=\mathrm{Frac}(W)$ the fraction field. Let $\mathcal{O}_v^{\mathrm{nr}}$ be the etale extension of $\mathbb{Z}_p$ in $\mathcal{O}_v$ and $\mathcal{O}_p^{\mathrm{nr}}:=\prod_{v|p}\mathcal{O}_v^{\mathrm{nr}}$. Set

$$\Sigma:=\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_p^{\mathrm{nr}},W)=\coprod_{v|p}\Sigma_v, \quad \text{and} \quad \Sigma_v:=\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_p^{\mathrm{nr}},W).$$

Regard $G(L)\subset\prod_{v|p}\prod_{\alpha\in\Sigma_v}\mathrm{GSp}_{2m}(F_vL)$ as a subgroup. Let $T\subset B\subset G$ be the diagonal maximal torus over $\mathbb{Q}_p$ and the upper triangular Borel subgroup. Denote by $X_*(T)_L$ the group of co-characters defined over L. Let $X_*(T)_{L,+}\subset X_*(T)_L$ be the subset of dominant co-characters with respect to B. The Cartan decomposition gives the following natural bijection

$$X_*(T)_{L,+}\xrightarrow{\mathrm{ev}} G(W)\backslash G(L)/G(W), \quad t\mapsto[(t(\varpi_v))_{v|p,\alpha\in\Sigma_v})], \tag{5.6}$$

where ϖ_v is an uniformizer of $\mathcal{O}_v\subset F_v$ and the bracket $[x]$ denotes the double coset $G(W)xG(W)$.

Let $\mu:=\mu_h:\mathbb{G}_{\mathrm{m}\mathbb{C}}\to G_{\mathbb{C}}$ be the Hodge co-character defined by h. The $G(\mathbb{C})$-conjugacy class of μ defines a dominant co-character still denoted by $\mu\in X_*(T)_+$. Then there is an unique element $t_\mu\in X_*(T)_{L,+}$ such that $\mathrm{ev}(t_\mu)=[\mu(p)]$. Explicitly,

$$t_\mu=(\underline{e}_\alpha)_\alpha, \quad \underline{e}_\alpha=(e_v,\dots,e_v,0,\dots,0) \quad (\text{each multiplicity } m),$$

where α ranges elements in Σ and v is the place for which Σ_v contains α.

The original group-theoretic definition of the μ-admissible set is given by

$$\mathrm{Adm}(\mu):=\{\underline{e}\in X_*(T)_{L,+}\,|\,\underline{e}\preceq t_\mu\}, \tag{5.7}$$

where $\preceq$ denotes the Bruhat order. It is shown in Haines and Ngô [10, Theorems 1, 4 and Proposition 5] (also see [7, Theorem 7.2]) that it coincides with the set $\mathrm{Adm}(\mu)$ defined in (5.3) by geometry[1].

---

[1]There are actually three finite subsets that are shown latter to be identical: the set of Schubert cells in the special fiber $\mathcal{M}_\Lambda$ of the local model (seemingly no name for it), the set $\mathrm{Perm}(\mu)$ of μ-permissible elements and the set $\mathrm{Adm}(\mu)$ of the μ-admissible elements (whose original definition is given by (5.7) in the present paper). The set $\mathrm{Perm}(\mu)$ is the translation of the first one in group theory though they are defined a priori in two different sources. The equality of $\mathrm{Perm}(\mu)$ and $\mathrm{Adm}(\mu)$ is the precise statement proved in Haines and Ngô. After $\mathrm{Perm}(\mu)=\mathrm{Adm}(\mu)$ is proved, we use the term $\mathrm{Adm}(\mu)$ for the set of KR types.

We have the following result.

Theorem 5.1.

1. *For each element* $\underline{e} \in \mathrm{Adm}(\mu)$*, the KR stratum* $\mathcal{M}_{\underline{e}}$ *is nonempty. If* $\overline{\mathcal{M}}_{\underline{e}}$ *denotes the Zariski closure of* $\mathcal{M}_{\underline{e}}$ *in* $\mathcal{M}$*, then one has*

$$\overline{\mathcal{M}}_{\underline{e}} = \bigcup_{\underline{e}' \preceq \underline{e}} \mathcal{M}_{\underline{e}'}. \tag{5.8}$$

2. *Each KR stratum* $\mathcal{M}_{\underline{e}}$ *is smooth, equi-dimensional, and*

$$\dim \mathcal{M}_{\underline{e}} = \langle 2\rho, \underline{e} \rangle, \tag{5.9}$$

where ρ *is the half sum of all positive roots.*

3. *There is a unique minimal KR stratum and a unique maximal KR stratum* $\mathcal{M}_{t_\mu}$*. Moreover, the maximal KR stratum* $\mathcal{M}_{t_\mu}$ *is exactly the smooth locus* $\mathcal{M}^{\mathrm{sm}}$*.*

An immediate consequence of Theorem 5.1 is the following.

Corollary 5.2. *The smooth locus* $\mathcal{M}^{\mathrm{sm}} \subset \mathcal{M}$ *is open and dense.*

This finishes the first step of the proof. For the second step, we construct the universal deformation for every point x in the smooth locus $\mathcal{M}^{\mathrm{sm}}$. Then we show that the generic fiber of the universal deformation is indeed ordinary. For details, we refer to the forthcoming paper.

This finishes the sketch of the proof of our main result Theorem 1.1.

6. Proof of Theorem 5.1

We keep the notations of the previous section.

6.1. Relation with the Lie stratification

We shall compare the KR stratification with the Lie stratification on $\mathcal{M}$; the latter is introduced in [35] in the Hilbert-Blumenthal case. Let k, $W = W(k)$, Σ and Σ_v be as in the previous section. We say a place v of F over p is *associated to* an element $\alpha \in \Sigma$ if $\alpha \in \Sigma_v$.

Using the decomposition $\Lambda \otimes_{\mathbb{Z}_p} W = \prod_\alpha \Lambda_\alpha$, one has the decomposition of the local model $\mathbf{M}_\Lambda \otimes_{\mathbb{Z}_p} W = \prod_\alpha \mathbf{M}_{\Lambda_\alpha}$. The set $\mathrm{Adm}(\mu)$ of μ-admissible elements also decomposes as the product:

$$\mathrm{Adm}(\mu)=\prod_{\alpha}\mathrm{Adm}(\mu)_{\alpha},$$

where

$$\mathrm{Adm}(\mu)_{\alpha}=\{(e_1,\dots,e_{2m})\in\mathbb{Z}^{2m}\,|\,0\le e_i\le e_v,\ e_i\ge e_{i+1},\ e_i+e_{2m-i+1}=e_v\,\forall\, i\,\},$$

where v is the place associated to α. If one writes two elements $\underline{e}=(\underline{e}_{\alpha})$, $\underline{e}'=(\underline{e}'_{\alpha})\in\mathrm{Adm}(\mu)$ as

$$\underline{e}_{\alpha}=(e_{\alpha,1},\dots,e_{\alpha,2m}),\quad \underline{e}'_{\alpha}=(e'_{\alpha,1},\dots,e'_{\alpha,2m}),\quad \forall\,\alpha$$

then $\underline{e}\preceq\underline{e}'$ if and only if

$$e_{\alpha,i}\le e'_{\alpha,i}\quad\forall\,\alpha\in\Sigma \text{ and } i=1,\dots,m.$$

Thus, there is a unique maximal element t_μ in $\mathrm{Adm}(\mu)$ and a unique minimal element $\underline{e}_{\min}$, which is the following element

$$\underline{e}_{\min}=(\underline{e}_{\min,\alpha})_{\alpha},\quad \underline{e}_{\min,\alpha}=(c_v,\dots,c_v,e_v-c_v,\dots e_v-c_v),$$

where v is the place associated to α and $c_v=\lceil e_v/2\rceil$.

Suppose that $\mathcal{F}=\oplus_{\alpha}\mathcal{F}_{\alpha}\in\mathcal{M}_{\Lambda}(k)$ is a k-valued element. Then each $\mathcal{F}_{\alpha}\subset\Lambda_{\alpha}\otimes k$ is a $k[\pi_v]/(\pi_v^{e_v})$-module which is a maximal isotropic submodule with respect to ψ, where π_v is a uniformizer of F_v. The lift $\widetilde{\mathcal{F}}_{\alpha}$ of $\mathcal{F}_{\alpha}$ in Λ fits into the lattices

$$p\Lambda_{\alpha}\subset\widetilde{\mathcal{F}}_{\alpha}\subset\Lambda_{\alpha}.$$

According to the definition, the KR type of $\mathcal{F}_{\alpha}$ equals the relative position $\mathrm{inv}(\Lambda_{\alpha},\widetilde{\mathcal{F}}_{\alpha})$, which is $(e_1,\dots,e_{2m})\in\mathbb{Z}^{2m}_{\ge 0}$ if there is an isomorphism

$$\frac{\Lambda_{\alpha}}{\widetilde{\mathcal{F}}_{\alpha}}\simeq\bigoplus_{i=1}^{2m}k[\pi_v]/(\pi_v^{e_i})$$

of $k[\pi_v]/(\pi_v^{e_v})$-modules where the integers e_i are ordered decreasingly. In other words, if one has an isomorphism

$$\frac{\Lambda_{\alpha}\otimes k}{\mathcal{F}_{\alpha}}\simeq\bigoplus_{i=1}^{2m}k[\pi_v]/(\pi_v^{e_i})$$

of $k[\pi_v]/(\pi_v^{e_v})$-modules for some integers $e_1\ge\cdots\ge e_{2m}\ge 0$, then the KR type of $\mathcal{F}_{\alpha}$ is equal to $(e_1,\dots,e_{2m})$.

Suppose $\underline{A} = (A, \Lambda, \iota, \bar{\eta}) \in \mathcal{M}(k)$ is an object over k. Let M be the covariant Dieudonné module of A. Write

$$\mathrm{Lie}(A) = \bigoplus_{\alpha\in\Sigma} \mathrm{Lie}(A)_\alpha, \quad \text{and} \quad \mathrm{Lie}(A)_\alpha \simeq \bigoplus_{i=1}^{2m} k[\pi_v]/(\pi_v^{e_{\alpha,i}}),$$

for some non-negative integers $e_{\alpha,1} \geq \cdots \geq e_{\alpha,2m}$, where $\mathrm{Lie}(A)_\alpha$ is the α-eigenspace of $\mathrm{Lie}(A)$. The Lie type $\underline{e}(\underline{A})$ of $\underline{A}$ is defined to be

$$\underline{e}(\underline{A}) = (\underline{e}_\alpha)_{\alpha\in\Sigma}, \quad \underline{e}_\alpha = (e_{\alpha,1}, \ldots, e_{\alpha,2m}).$$

Under any choice of isomorphism $M \simeq \Lambda \otimes W$ of skew-Hermitian $\mathcal{O}_p \otimes_{\mathbb{Z}_p} W$-modules over W, one has an isomorphism

$$\mathrm{Lie}(A) \simeq (\Lambda \otimes k)/\mathcal{F}$$

of $\mathcal{O}_p \otimes k$-modules. Therefore, the KR type of $\underline{A}$ equals to the Lie type of $\underline{A}$ as a collection of $2m$-tuples of non-negative integers indexed by Σ.

Lemma 6.1. *The KR type of any object $\underline{A}$ in $\mathcal{M}(k)$ is equal to the Lie type of $\underline{A}$ as a collection of $2m$-tuples of non-negative integers indexed by Σ. In particular, the set of KR strata of $\mathcal{M}$ is that of Lie strata of $\mathcal{M}$.*

6.2. Nonemptiness of KR strata

Let $x \in \mathcal{M}(k)$ be a k-valued point and denote by $\mathcal{M}_x \subset \mathcal{M}$ the spectrum of the completion $\widehat{\mathcal{O}}_{\mathcal{M},x}$ of the local ring $\mathcal{O}_{\mathcal{M},x}$ at x. Choose any trivialization of the de Rham homology

$$H_1^{DR}(\underline{A}_{\mathrm{univ}}/\widehat{\mathcal{O}}_{\mathcal{M},x}) \simeq \Lambda \otimes \widehat{\mathcal{O}}_{\mathcal{M},x}.$$

This defines a section of $\varphi^{\mathrm{mod}} : \widetilde{\mathcal{M}} \to \mathcal{M}$ over $\mathcal{M}_x$ and get a morphism $\xi : \mathcal{M}_x \to \mathcal{M}_\Lambda$ after composing with the morphism φ^{loc}. If $y \in \mathcal{M}_\Lambda(k)$ is the corresponding closed point and $\mathcal{M}_{\Lambda,y}$ is the spectrum of the completed local ring of $\mathcal{M}_\Lambda$ at y, then one has the isomorphism $\xi : \mathcal{M}_x \to \mathcal{M}_{\Lambda,y}$, which follows from the Grothendieck-Messing deformation theory. Moreover, from the discussion above, the set of (non-closed) points with a fixed KR type $\underline{e}$ is sent to that of the same KR type.

In the local model, if y lies in the minimal Schubert cell of $\mathcal{M}_\Lambda$, then the local space $\mathcal{M}_{\Lambda,y}$ contains points of all KR types. Therefore, in order to show the non-emptiness of KR strata it suffices to show that the minimal KR stratum of $\mathcal{M}$ is nonempty. Since KR strata are Lie strata (Lemma 6.1), it amounts to show that the minimal Lie stratum is nonempty.

Lemma 6.2. *There is a superspecial point $\underline{A} \in \mathcal{M}$ so that $\underline{e}(\underline{A}) = \underline{e}_{\min}$.*

Proof. By Theorem 2.1 and Proposition 3.1 of [38], it is sufficient to construct a superspecial separably quasi-polarized Dieudonné $\mathcal{O}_p$-module M of rank $2dm$ whose Lie type is equal to $\underline{e}_{\min}$. One can reduce the construction to the case $m = 1$ as if such a Dieudonné module M_0 of rank $2d$ exists, then one takes $M = M_0^m$. For the case $m = 1$, i.e. the Hilbert-Blumenthal case, the quasi-polarized superspecial Dieudonné modules is classified in [37, Section 3]. In particular, there is a superspecial separably quasi-polarized Dieudonné $\mathcal{O}_p$-module M_0 of rank $2d$ whose Lie type is $\underline{e}_{\min}$. This completes the proof of the lemmas. □

Therefore, we obtain the following result.

Corollary 6.3. *For each $\underline{e} \in \mathrm{Adm}(\mu)$, the KR stratum $\mathcal{M}_{\underline{e}}$ is nonempty.*

6.3. Proof of Theorem 5.1

(1) We have proven the first statement. The second statement follows from the closure relation of Schubert cells in the local model $\mathcal{M}_\Lambda$.

(2) Since the KR stratum $\mathcal{M}_{\underline{e}}$ and the corresponding Schubert cell $\mathcal{M}_{\Lambda,\underline{e}}$ are smoothly equivalent of the same relative dimension, $\mathcal{M}_{\underline{e}}$ is smooth, equi-dimensional, and

$$\dim \mathcal{M}_{\underline{e}} = \dim \mathcal{M}_{\Lambda,\underline{e}} = \ell(\underline{e}),$$

where $\ell(\underline{e})$ is the length of $\underline{e}$. The length function $\ell(\underline{e})$ is given by (see Ngô and Polo [19], a dimension formula[2] before Lemma 2.2)

$$\ell(\underline{e}) = \langle 2\rho, \underline{e} \rangle.$$

(3) The partial ordered set $\mathrm{Adm}(\mu)$ has a unique minimal element $\underline{e}_{\min}$ and a unique maximal element $\underline{e}_{\max} = t_\mu$ as already described in Section 6.1. We calculate the tangent spaces (Lemma 6.5) and conclude that $\mathcal{M}_{\Lambda,\underline{e}_{\max}} = \mathcal{M}_\Lambda^{\mathrm{sm}}$. □

[2]The author was not aware where the length formula is first given.

6.4. Tangent space calculation

We shall compute the tangent spaces of $\mathcal{M}_{\Lambda_\alpha}$. For brevity, we suppress α from the notation and write e and π for e_v and π_v, respectively. Put $R = k[\pi]/(\pi^e)$. One writes $\Lambda = R^{2m}$ with the standard basis $x_1, \dots, x_{2m}$ and the non-degenerate alternating pairing $\psi : \Lambda \times \Lambda \to R$ with $\psi(x_i, x_{2m-i+1}) = 1$ for $i = 1, \dots, m$ and $\psi(x_i, x_j) = 0$ if $i + j \neq 2m + 1$. Let $\underline{e} = (e_1, \dots, e_{2m})$ be a KR type. Then

$$F(\underline{e}) := \langle \pi^{e-e_1} x_1, \dots, \pi^{e-e_{2m}} x_{2m} \rangle_R$$

is a special point in the Schubert cell $\mathcal{M}_{\Lambda,\underline{e}}$. We want to compute the tangent space $T_{F(\underline{e})}(\mathcal{M}_\Lambda)$ at the point $F(\underline{e})$.

The equi-characteristic first order deformations of $F(\underline{e})$ invariant under R-action are classified by the vector space

$$\mathcal{T}(\underline{e}) := \mathrm{Hom}_R(F(\underline{e}), \Lambda/F(\underline{e})).$$

For each $\varphi \in \mathcal{T}(\underline{e})$, the corresponding first order deformation is given by

$$\widetilde{F}(\underline{e})(\varphi) \;=\; \langle y_1 + \varphi(y_1)\varepsilon, \dots, y_{2m} + \varphi(y_{2m})\varepsilon \rangle_{R[\varepsilon]}.$$

where

$$y_i \;:=\; \pi^{e-e_i} x_i, \quad i = 1, \dots, 2m. \tag{6.1}$$

The submodule $\widetilde{F}(\underline{e})(\varphi)$ is isotropic with respect to ψ if and only if the following condition

$$(*) \quad \psi(y_i, \varphi(y_j)) = \psi(y_j, \varphi(y_i)), \quad \forall\, 1 \le i < j \le 2m$$

holds. Put

$$\mathcal{T}(\underline{e})^{\mathrm{sym}} := \{\varphi \in \mathcal{T}(\underline{e}) \,|\, (*) \text{ holds}\}.$$

Then we have the following

Lemma 6.4. *The tangent space $T_{F(\underline{e})}(\mathcal{M}_\Lambda)$ is isomorphic to the vector space $\mathcal{T}(\underline{e})^{\mathrm{sym}}$.*

Lemma 6.5. *One has $\dim T_{F(\underline{e})}(\mathcal{M}_\Lambda) \ge em(m+1)/2$. The equality holds if and only if $\underline{e} = \underline{e}_{\max} := (e, \dots, e, 0, \dots, 0)$.*

Proof. The case $e = 1$ is obvious and we may assume that $e \ge 2$. When $\underline{e} = \underline{e}_{\max}$, one has $F(\underline{e}) = \langle x_1, \dots, x_m \rangle_R$. Write

$$\varphi(x_j) \;=\; \sum_{i=1}^{m} a_{ij} \bar{x}_{2m-i+1} \quad a_{ij} \in R$$

$$= a_{1j}\bar{x}_{2m} + a_{2j}\bar{x}_{2m-1} + \cdots + a_{mj}\bar{x}_{m+1}, \tag{6.2}$$

where $\bar{x}_i$ is the image of x_i in $\Lambda/F(\underline{e})$. One easily computes

$$\psi(x_i, \varphi(x_j)) = a_{i,j}, \quad \forall\, 1 \le i, j \le m$$

and hence the condition $(*)$ is simply the condition $a_{i,j} = a_{j,i}$ for all $1 \le i \le j \le m$. Thus, $\dim_k \mathcal{T}(\underline{e}_{\max})^{\mathrm{sym}} = em(m+1)/2$. Since the dimension of tangent spaces is a upper semi-continuous function, one gets

$$\dim T_{F(\underline{e})}(\mathcal{M}_\Lambda) \ge \dim T_{F(\underline{e}_{\max})}(\mathcal{M}_\Lambda) = \frac{em(m+1)}{2}.$$

Put $\underline{e}' := (e, \dots, e, e-1, 1, 0 \dots, 0)$. Then $F(\underline{e}')$ is generated by $y_1, \dots, y_{m+1}$ over R, where y_i are in (6.1). For $d \le e-1$, denote by $R_d \subset R$ the subspace consisting of all elements of degree at most d in π; $\dim_k R_d = d+1$. For $\varphi \in \mathcal{T}(\underline{e}')$, write

$$\begin{aligned} \varphi(y_j) &= \sum_{i=1}^{m+1} a_{i,j}\, \bar{x}_{2m-i+1}, \\ &= a_{1,j}\, \bar{x}_{2m} + \cdots + a_{m-1,j}\, \bar{x}_{m+2} + a_{m,j}\, \bar{x}_{m+1} + a_{m+1,j}\, \bar{x}_m, \end{aligned} \tag{6.3}$$

where

$$a_{i,j} \in R \text{ for } 1 \le i \le m-1, \quad a_{m,j} \in R_{e-2} \text{ and } a_{m+1,j} \in R_0$$

(because $\pi^{e-1}\bar{x}_{m+1}$ and $\pi\bar{x}_m = 0$). It follows from $\pi^{e-1}\varphi(y_m) = 0$ that

$$a_{i,m} \in \pi R \text{ (for } 1 \le i \le m-1), \quad a_{m,m} \in R_{e-2} \ \text{ and } \ a_{m+1,m} \in R_0.$$

It follows from $\pi\varphi(y_{m+1}) = 0$ that

$$a_{i,m+1} \in \pi^{e-1}R \text{ (for } 1 \le i \le m-1), \quad a_{m,m+1} \in R_{e-2} \cap \pi^{e-2}R = \pi^{e-2}k$$

and $a_{m+1,m+1} \in R_0$. One computes

$$\psi(y_i, \varphi(y_j)) = \begin{cases} a_{i,j} & \text{for } 1 \le i \le m-1, \\ \pi a_{m,j} & \text{for } i = m, \\ \pi^{e-1} a_{m+1,j} & \text{for } i = m+1. \end{cases} \tag{6.4}$$

The symmetric condition $(*)$ is rephrased as

$$\begin{cases} a_{i,j} = a_{j,i} & \text{for } 1 \le i < j \le m-1, \\ a_{i,m} = \pi a_{m,i} & \text{for } 1 \le i < j = m, \\ a_{i,m+1} = \pi^{e-1} a_{m+1,i} & \text{for } 1 \le i \le m-1 \text{ and } j = m+1, \\ \pi a_{m,m+1} = \pi^{e-1} a_{m+1,m} & \text{for } (i,j) = (m, m+1). \end{cases} \tag{6.5}$$

We see that in the $(m+1)\times(m+1)$ matrix $(a_{i,j})$, the mth column and $m+1$th column are determined by the mth row and $m+1$th row, respectively. The left upper $(m-1)\times(m-1)$ block $(a_{i,j})$ has dimension $em(m-1)/2$. The mth row $(a_{m,1},\dots a_{m,m})$ of size m has dimension $(e-1)m$. Finally $m+1$th row $(a_{m+1,1},\dots,a_{m+1,m+1})$ has dimension $m+1$. Therefore,

$$\dim T_{F(\underline{e}')}(\mathcal{M}_\Lambda)=\frac{em(m-1)}{2}+(e-1)m+(m+1)=\frac{em(m+1)}{2}+1. \quad (6.6)$$

If $\underline{e}$ is not a maximal KR type, then $\underline{e}\preceq\underline{e}'$ and

$$\dim T_{F(\underline{e})}(\mathcal{M}_\Lambda)\geq\dim T_{F(\underline{e}')}(\mathcal{M}_\Lambda)>\frac{em(m+1)}{2}.$$

This completes the proof of the lemma. □

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