Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **10** (2015), No. 1, pp. 101-111

STRONG LAWS FOR RATIOS OF ORDER STATISTICS FROM EXPONENTIALS

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Abstract

The random variables examined are independent and identically distributed exponentials. The mean of these random variables can change from sample to sample. Within each sample we observe the ratio of various order statistics. Then we establish strong laws for these ratios. The most important statistic is the ratio of the two smallest order statistics within each sample. The distribution of this statistic is very interesting. It has infinite mean, but barely, which produces a very unusual strong law.

1. Introduction

This paper establishes unusual strong laws for resampling from exponential distributions. The underlying distribution is $f(x) = (1/\lambda)e^{-x/\lambda}I(x \geq 0)$, where $\lambda > 0$. However, the parameter λ can change from sample to sample. For example, we can sample lifetimes of a machine and we can change the equipment on a daily basis. Our sample size is $\{m_n, m_n \geq 2\}$. Next we order the data and then obtain the ratios of those order statistics.

Let X_{ni} be exponential random variables with mean λ_n, where $i = 1, \ldots, m_n$ and $n = 1, 2, 3, \ldots$ Our order statistics are $X_{n(1)} \leq X_{n(2)} \leq \cdots \leq X_{n(m_n)}$. The final random variable is, the ratio

$$R_{nij} = \frac{X_{n(j)}}{X_{n(i)}} \qquad 1 \leq i < j \leq m_n.$$

Received April 23, 2013 and in revised form March 31, 2014.

AMS Subject Classification: 60F15.

Key words and phrases: Almost sure convergence, order statistics, regular variation, strong laws of large numbers.

There are several interesting results here. First of all, the parameter can change as previously noted. Secondly, the interesting case is $i = 1$. As for j, the most telling situation is $j = 2$, but that's not as important as $i = 1$. The emphasis will be placed on the statistic R_{n12}, which is a nice measurement of the reliability of our system.

Thirdly, the sample size isn't that important either. In Theorem 2.1, we fix the sample size within each row while obtaining a strong law for R_{n12}. In that theorem we do see the parameter $m_n = m$. However, in Theorem 2.2, we see that by letting m_n go to infinity in ANY fashion, the sample size disappeared from the final result.

The most important statistic is R_{n12}. It measures the stability of our equipment and it shows whether or not our system is stable. Exponential random variables measure the lifetimes of equipment. The first order statistic is the measure of the failure of the first piece of our equipment. Comparing the smallest order statistic to the others tell us how stable our system is. Another truly fascinating result is that the distribution of R_{n13} isn't much different than R_{n12}. However, R_{n23} is a totally different animal. It turns out that $E(R_{n1j}) = \infty$ for all $j \geq 2$ and all of these densities, their means are barely infinite, see Klass and Teicher [3]. That is why it suffices to just observe R_{n12}. Meanwhile $E(R_{n23}) < \infty$, which doesn't tell us much about our system. Moreover, the densities of R_{n1j} permit unusual strong laws, see Adler [1], for all $j \geq 2$.

We need to mention that the constant C, used in the proofs, denotes a generic real number that is not necessarily the same in each appearance. It is used as an upper bound in order to establish the convergence of our various series.

2. Comparing our Two Smallest Order Statistics

For Theorems 2.1 and 2.2 we need the density of R_{n12}. We start with i.i.d. exponential random variables X_{ni}, with mean λ_n, where $i = 1, \ldots, m_n$. Next, we order them within each row. Thus $X_{n(1)} \leq X_{n(2)} \leq \cdots \leq X_{n(m_n)}$. So, the joint density of $X_{n(1)}$ and $X_{n(2)}$ is

$$f(x_1, x_2) = \frac{m_n(m_n-1)}{\lambda_n^2} e^{-x_1/\lambda_n} e^{-x_2/\lambda_n} \left(e^{-x_2/\lambda_n}\right)^{m_n-2} I(0 < x_1 < x_2)$$

which reduces to

$$f(x_1, x_2) = \frac{m_n(m_n-1)}{\lambda_n^2} e^{-[x_1+(m_n-1)x_2]/\lambda_n} I(0 < x_1 < x_2).$$

Next, we transform to the variables $w = x_1$ and $r = x_2/x_1$. The Jacobian is w and the joint density of w and r is

$$f(w, r) = \frac{m_n(m_n-1)}{\lambda_n^2} w e^{-w[1+r(m_n-1)]/\lambda_n} I(w > 0, r > 1).$$

Using the gamma function we obtain

$$\begin{aligned} f(r) &= \frac{m_n(m_n-1)}{\lambda_n^2} \int_0^\infty w e^{-w[1+r(m_n-1)]/\lambda_n} dw \\ &= \left(\frac{m_n(m_n-1)}{\lambda_n^2}\right)\left(\frac{\lambda_n}{1+r(m_n-1)}\right)^2 \\ &= \frac{m_n(m_n-1)}{[1+r(m_n-1)]^2}. \end{aligned}$$

Again, note that this density is free from our parameter λ_n. The first theorem examines strong laws for fixed samples sizes.

Theorem 2.1. *Let $X_{n(1)}$ and $X_{n(2)}$ be the first two order statistics from an exponential distribution with parameter λ_n and fixed sample size m. Then for all $\alpha > -2$*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{(\lg n)^\alpha X_{n(2)}}{n X_{n(1)}}}{(\lg N)^{\alpha+2}} = \frac{m}{(m-1)(\alpha+2)} \quad \text{almost surely.}$$

Proof. This is an Exact Strong Law. Our random variable R_{n12} has the following density

$$f(r) = \frac{m(m-1)}{[1+r(m-1)]^2}.$$

Thus

$$xP\{R_{n12} > x\} = x \int_x^\infty \frac{m(m-1)dr}{[1+r(m-1)]^2} = \frac{mx}{1+x(m-1)} \to \frac{m}{m-1}.$$

Using Example 2 from Adler [1], the conclusion is immediate. □

The next case doesn't immediately follow from Adler [1], but it does have the same properties. The tail distribution is regularly varying, so there is a strong law. To be precise, there is always an Exact Strong Law whenever $P\{R_{n12} > x\}$ is regularly varying with exponent -1. But the slight complication is due to the sample size.

Theorem 2.2. *Let $X_{n(1)}$ and $X_{n(2)}$ be the first two order statistics from an exponential distribution with parameter λ_n. For any $m_n \to \infty$ and all $\alpha > -2$*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{(\lg n)^\alpha X_{n(2)}}{nX_{n(1)}}}{(\lg N)^{\alpha+2}} = \frac{1}{\alpha+2} \quad \text{almost surely.}$$

Proof. Let $a_n = (\lg n)^\alpha/n$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = b_n/a_n = n(\lg n)^2$. Setting $R_{n12} = X_{n(2)}/X_{n(1)}$, we see that it has the density

$$f(r) = \frac{m_n(m_n-1)}{[1+r(m_n-1)]^2}.$$

We use the partition

$$\begin{aligned}
&\frac{1}{b_N}\sum_{n=1}^{N} a_n R_{n12} \\
&= \frac{1}{b_N}\sum_{n=1}^{N} a_n \big[R_{n12}I(1 \le R_{n12} \le c_n) - ER_{n12}I(1 \le R_{n12} \le c_n)\big] \\
&\quad + \frac{1}{b_N}\sum_{n=1}^{N} a_n R_{n12} I(R_{n12} > c_n) + \frac{1}{b_N}\sum_{n=1}^{N} a_n ER_{n12} I(1 \le R_{n12} \le c_n).
\end{aligned}$$

The first term vanishes almost surely by the Khintchine-Kolmogorov Convergence Theorem, see page 113 of Chow and Teicher [2], and Kronecker's lemma since

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{c_n^2} ER_{n12}^2 I(1 \le R_{n12} \le c_n) \\
&= \sum_{n=1}^{\infty} \frac{m_n(m_n-1)}{c_n^2} \int_1^{c_n} \frac{r^2 dr}{[1+r(m_n-1)]^2}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{m_n(m_n-1)}{c_n^2} \int_{m_n}^{1+c_n(m_n-1)} \left(\frac{u-1}{m_n-1}\right)^2 \left(\frac{1}{u^2}\right) \left(\frac{du}{m_n-1}\right) \\
&= \sum_{n=1}^{\infty} \frac{m_n}{c_n^2(m_n-1)^2} \int_{m_n}^{1+c_n(m_n-1)} \left(\frac{u-1}{u}\right)^2 du \\
&\le \sum_{n=1}^{\infty} \frac{m_n}{c_n^2(m_n-1)^2} \int_{m_n}^{1+c_n(m_n-1)} du \\
&= \sum_{n=1}^{\infty} \frac{m_n}{c_n^2(m_n-1)^2} \Bigg(1+c_n(m_n-1)-m_n\Bigg) \\
&\le C\sum_{n=1}^{\infty} \frac{m_n}{c_n^2(m_n-1)^2}(c_n m_n) \le C\sum_{n=1}^{\infty} \frac{1}{c_n} = C\sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.
\end{aligned}
$$

The second term in our partition vanishes, with probability one, by the Borel-Cantelli lemma, since

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\{R_{n12} > c_n\} &= \sum_{n=1}^{\infty} \int_{c_n}^{\infty} \frac{m_n(m_n-1)dr}{[1+r(m_n-1)]^2} \\
&= \sum_{n=1}^{\infty} \frac{m_n}{1+c_n(m_n-1)} \le C\sum_{n=1}^{\infty} \frac{1}{c_n} < \infty.
\end{aligned}
$$

As for the third term

$$
\begin{aligned}
&ER_{n12}I(1 \le R_{n12} \le c_n) = \int_1^{c_n} \frac{m_n(m_n-1)r\,dr}{[1+r(m_n-1)]^2} \\
&= \frac{m_n}{m_n-1} \int_{m_n}^{1+c_n(m_n-1)} \left(\frac{u-1}{u^2}\right) du \\
&= \frac{m_n}{m_n-1} \left[\lg(1+c_n(m_n-1)) - \lg m_n + \frac{1}{1+c_n(m_n-1)} - \frac{1}{m_n}\right] \\
&= \frac{m_n}{m_n-1} \left[\lg\left(\frac{1+c_n(m_n-1)}{m_n}\right) + o(1)\right] \\
&\sim \lg(c_n) \sim \lg n.
\end{aligned}
$$

Thus

$$
\frac{1}{b_N} \sum_{n=1}^{N} a_n ER_{n12} I(1 \le R_{n12} \le c_n) \sim \frac{\sum_{n=1}^{N} \frac{(\lg n)^{\alpha+1}}{n}}{(\lg N)^{\alpha+2}} \to \frac{1}{\alpha+2}
$$

which complete the proof. □

The most important statistic is R_{n12} and we will see that $X_{n(j)}/X_{n(1)}$ are quite similar for $j = 3, 4, \ldots, m$. While on the other hand, the statistics where we do not divide by our first order statistic loses it value. Comparing the lifetimes of ensuing equipment to the first failure tells us how stable our system is. With that in mind we next show that dividing by $X_{n(2)}$, these ratios lose their uniqueness.

3. Finite Expectations

Theorem 3.1. $E\left(\frac{X_{n(3)}}{X_{n(2)}}\right) = 1 + \frac{m(m-1)}{m-2}\lg\left(\frac{m}{m-1}\right)$ *for any* $m \geq 3$.

Proof. The joint density of the second and third order statistics from an i.i.d. sample of m exponentials with parameter λ is

$$f(x_2, x_3) = \frac{m!}{(m-3)!\lambda^2} e^{-x_2/\lambda} e^{-x_3(m-2)/\lambda}\left(1 - e^{-x_2/\lambda}\right) I(0 < x_2 < x_3).$$

As before, we transform to the variables $w = x_2$ and $r = x_3/x_2$. The Jacobian is w and the joint density of w and r is

$$f(w, r) = \frac{m!}{(m-3)!\lambda^2} w e^{-w[1+r(m-2)]/\lambda}\left(1 - e^{-w/\lambda}\right) I(w > 0, r > 1).$$

Thus the density of R_{n23} is

$$f(r) = \frac{m!}{(m-3)!\lambda^2}\left[\int_0^\infty w e^{-w[1+r(m-2)]/\lambda} dw - \int_0^\infty w e^{-w[2+r(m-2)]/\lambda} dw\right].$$

Using the gamma function twice, this reduces to

$$\begin{aligned}
&\frac{m!}{(m-3)!\lambda^2}\left[\left(\frac{\lambda}{1 + r(m-2)}\right)^2 - \left(\frac{\lambda}{2 + r(m-2)}\right)^2\right] \\
&\quad = \frac{m!}{(m-3)!}\left[\left(\frac{1}{1 + r(m-2)}\right)^2 - \left(\frac{1}{2 + r(m-2)}\right)^2\right].
\end{aligned}$$

Hence

$$\begin{aligned}
E\left(\frac{X_{n(3)}}{X_{n(2)}}\right) &= \frac{m!}{(m-3)!}\int_1^\infty\left[\frac{r}{[1+r(m-2)]^2}-\frac{r}{[2+r(m-2)]^2}\right]dr\\
&= \frac{m(m-1)}{(m-2)}\int_{m-1}^\infty (u-1)\left[\frac{1}{u^2}-\frac{1}{(u+1)^2}\right]du\\
&= \frac{m(m-1)}{(m-2)}\int_{m-1}^\infty \frac{(2u^2-u-1)du}{u^2(u+1)^2}\\
&= \frac{m(m-1)}{(m-2)}\int_{m-1}^\infty \left[\frac{1}{u}-\frac{1}{u^2}-\frac{1}{u+1}+\frac{2}{(u+1)^2}\right]du\\
&= \frac{m(m-1)}{(m-2)}\left[\lg\left(\frac{m}{m-1}\right)-\frac{1}{m-1}+\frac{2}{m}\right]\\
&= 1+\frac{m(m-1)}{m-2}\lg\left(\frac{m}{m-1}\right)
\end{aligned}$$

which is finite. □

As we just saw, the most telling statistic is R_{n12}. But any of the order statistics divided by the smallest one will work. And all of these distributions have infinite expectations, but barely, i.e., $xP\{X > x\}$ is slowly varying. Hence there will always be an Exact Strong Law. We conclude with a strong law for R_{n1j} for $j = 2, 3, \ldots, m$, where in this case m is fixed. Theorem 4.1 is a natural generalization of Theorem 2.1.

4. The General Infinite Case

Theorem 4.1. *Let $X_{n(1)}$ and $X_{n(j)}$ be the first and j^{th} order statistics from an exponential distribution with parameter λ_n and fixed sample size m. Then for all $\alpha > -2$*

$$\lim_{N\to\infty}\frac{\sum_{n=1}^N \frac{(\lg n)^\alpha X_{n(j)}}{nX_{n(1)}}}{(\lg N)^{\alpha+2}} = \frac{m!}{(j-2)!(m-j)!(\alpha+2)}\sum_{i=0}^{j-2}\frac{\binom{j-2}{i}(-1)^{j-2-i}}{(m-i-1)^2} \quad \text{almost surely.}$$

Proof. Let $a_n = (\lg n)^\alpha/n$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = b_n/a_n = n(\lg n)^2$. First we must obtain the density of $X_{n(j)}/X_{n(1)}$. We start with the joint density of $X_{n(1)}$ and $X_{n(j)}$

$$f(x_1,x_j)=\frac{m!}{(j-2)!(m-j)!\lambda_n^2}e^{-x_1/\lambda_n}e^{-x_j(m-j+1)/\lambda_n}\left[e^{-x_1/\lambda_n}-e^{-x_j/\lambda_n}\right]^{j-2}.$$

Once again, let $w=x_1$ and $r=x_j/x_1$. The Jacobian is w and the joint density of w and r is

$$f(w,r)=\frac{m!}{(j-2)!(m-j)!\lambda_n^2}we^{-w/\lambda_n}e^{-rw(m-j+1)/\lambda_n}\left[e^{-w/\lambda_n}-e^{-rw/\lambda_n}\right]^{j-2}.$$

The density of R_{n1j} is

$$\begin{aligned}
&\frac{m!}{(j-2)!(m-j)!\lambda_n^2}\int_0^\infty we^{-w[1+r(m-j+1)]/\lambda_n}\left[e^{-w/\lambda_n}-e^{-rw/\lambda_n}\right]^{j-2}dw\\
&=\frac{m!}{(j-2)!(m-j)!\lambda_n^2}\int_0^\infty we^{-w[1+r(m-j+1)]/\lambda_n}\\
&\quad\times\sum_{i=0}^{j-2}\binom{j-2}{i}e^{-wi/\lambda_n}\left(-e^{-rw/\lambda_n}\right)^{j-2-i}dw\\
&=\frac{m!}{(j-2)!(m-j)!\lambda_n^2}\sum_{i=0}^{j-2}\binom{j-2}{i}(-1)^{j-2-i}\int_0^\infty we^{-w[(i+1)+r(m-i-1)]/\lambda_n}dw\\
&=\frac{m!}{(j-2)!(m-j)!\lambda_n^2}\sum_{i=0}^{j-2}\binom{j-2}{i}(-1)^{j-2-i}\left[\frac{\lambda_n}{[(i+1)+r(m-i-1)]}\right]^2\\
&=\frac{m!}{(j-2)!(m-j)!}\sum_{i=0}^{j-2}\binom{j-2}{i}(-1)^{j-2-i}\frac{1}{[(i+1)+r(m-i-1)]^2}.
\end{aligned}$$

Using the typical partition

$$\begin{aligned}
\frac{1}{b_N}\sum_{n=1}^N a_nR_{n1j} &= \frac{1}{b_N}\sum_{n=1}^N a_n\big[R_{n1j}I(1\le R_{n1j}\le c_n)-ER_{n1j}I(1\le R_{n1j}\le c_n)\big]\\
&\quad+\frac{1}{b_N}\sum_{n=1}^N a_nR_{n1j}I(R_{n1j}>c_n)\\
&\quad+\frac{1}{b_N}\sum_{n=1}^N a_nER_{n1j}I(1\le R_{n1j}\le c_n).
\end{aligned}$$

The first term vanished because

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{c_n^2}ER_{n1j}^2 I(1\le R_{n1j}\le c_n)\\
&= \sum_{n=1}^{\infty}\frac{m!}{c_n^2(j-2)!(m-j)!}\sum_{i=0}^{j-2}\binom{j-2}{i}(-1)^{j-2-i}\int_1^{c_n}\frac{r^2dr}{[(i+1)+r(m-i-1)]^2}\\
&\le C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\sum_{i=0}^{j-2}\int_1^{c_n}\frac{r^2dr}{[(i+1)+r(m-i-1)]^2}\\
&\le C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\sum_{i=0}^{j-2}\int_1^{c_n}dr\le C\sum_{n=1}^{\infty}\frac{1}{c_n}=C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.
\end{aligned}$$

As for the second term

$$\begin{aligned}
&\sum_{n=1}^{\infty}P\{R_{n1j}>c_n\}\\
&= \sum_{n=1}^{\infty}\frac{m!}{(j-2)!(m-j)!}\sum_{i=0}^{j-2}\binom{j-2}{i}(-1)^{j-2-i}\int_{c_n}^{\infty}\frac{dr}{[(i+1)+r(m-i-1)]^2}\\
&\le C\sum_{n=1}^{\infty}\sum_{i=0}^{j-2}\int_{c_n}^{\infty}\frac{dr}{[(i+1)+r(m-i-1)]^2}\\
&\le C\sum_{n=1}^{\infty}\int_{c_n}^{\infty}\frac{dr}{r^2}\le C\sum_{n=1}^{\infty}\frac{1}{c_n}<\infty.
\end{aligned}$$

In order to compute the almost sure limit we need

$$\begin{aligned}
&ER_{n1j}I(1\le R_{n1j}\le c_n)\\
&= \frac{m!}{(j-2)!(m-j)!}\sum_{i=0}^{j-2}\binom{j-2}{i}(-1)^{j-2-i}\int_1^{c_n}\frac{rdr}{[(i+1)+r(m-i-1)]^2}\\
&= \frac{m!}{(j-2)!(m-j)!}\sum_{i=0}^{j-2}\binom{j-2}{i}(-1)^{j-2-i}\frac{1}{(m-i-1)^2}\\
&\quad\times\int_m^{(i+1)+c_n(m-i-1)}\left[\frac{1}{w}-\frac{i+1}{w^2}\right]dw.
\end{aligned}$$

This last integral is asymptotically equivalent to $\lg n$ since

$$\int_m^{(i+1)+c_n(m-i-1)}\left[\frac{1}{w}-\frac{i+1}{w^2}\right]dw$$

$$\begin{aligned}
&= \lg[(i+1)+c_n(m-i-1)]-\lg m+(i+1)\left[\frac{1}{(i+1)+c_n(m-i-1)}-\frac{1}{m}\right]\\
&\sim \lg[(i+1)+c_n(m-i-1)] \sim \lg[c_n(m-i-1)] \sim \lg[c_n]\\
&= \lg[n(\lg n)^2] \sim \lg n.
\end{aligned}$$

Putting this all together we have

$$\begin{aligned}
&\frac{1}{b_N}\sum_{n=1}^{N} a_n E R_{n1j} I(1 \le R_{n1j} \le c_n)\\
&\sim \frac{\frac{m!}{(j-2)!(m-j)!}\sum_{i=0}^{j-2}\frac{\binom{j-2}{i}(-1)^{j-2-i}}{(m-i-1)^2}\sum_{n=1}^{N}\frac{(\lg n)^{\alpha+1}}{n}}{(\lg N)^{\alpha+2}}\\
&\to \frac{m!}{(j-2)!(m-j)!(\alpha+2)}\sum_{i=0}^{j-2}\frac{\binom{j-2}{i}(-1)^{j-2-i}}{(m-i-1)^2}
\end{aligned}$$

which complete this proof. □

Remarks: When we let $j = 2$ in Theorem 4.1, we obtain the same result as Theorem 2.1. Even though Theorem 2.1 does follow from Theorem 4.1, it is a nice example of an Exact Strong Law and its proof is immediate based on past results. The emphasis here should be placed on $X_{n(2)}/X_{n(1)}$. In measuring the reliability of our system we should compare the lifetimes of those quickest to fail. Also, it is important to note that R_{n1j} has infinite expectation for all $j \ge 2$, regardless of the sample size and their densities are such that we can obtain these unusual strong laws. And moreover, all of these results are free of the parameter, λ_n. This allows us to change the equipment from sample to sample as long as the underlying distribution remains an exponential.

Acknowledgment

I would like to thank the referee for his/her kind remarks and for the careful reading of my manuscript.

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