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EKMANN BOUNDARY LAYER EXPANSIONS OF NAVIER-STOKES EQUATIONS WITH ROTATION

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Abstract

This paper concerns the validity of the boundary layers expansion for Navier-Stokes equations with rotation at a high frequency. We establish the error estimate for such an expansion in L^∞, which improves the result in [6].

1. Introduction

In this paper, we consider the geophysical fluid dynamics coupled with rotation in $(0,T)\times\Omega$ with $\Omega=\{x\in\mathbb{T},\ y\in\mathbb{T},\ 0<z<1\}$, which is governed by the following initial and boundary value problems

$$\begin{cases}\partial_t u^{\varepsilon,\mu}+u^{\varepsilon,\mu}\cdot\nabla u^{\varepsilon,\mu}+\frac{1}{\varepsilon}e\times u^{\varepsilon,\mu}+\frac{1}{\varepsilon}\nabla P^{\varepsilon,\mu}-\Delta_{x,y}u^{\varepsilon,\mu}-\mu\partial_z^2u^{\varepsilon,\mu}=F^{\varepsilon,\mu},\\ \nabla\cdot u^{\varepsilon,\mu}=0,\\ u^{\varepsilon,\mu}|_{z=0}=u^{\varepsilon,\mu}|_{z=1}=0,\\ u^{\varepsilon,\mu}|_{t=0}=u_0^{\varepsilon,\mu}(x,y,z)\ \text{ with } \nabla\cdot u_0^{\varepsilon,\mu}=0,\end{cases}\tag{1.1}$$

where $u^{\varepsilon,\mu}=(u_1^{\varepsilon,\mu},u_2^{\varepsilon,\mu},u_3^{\varepsilon,\mu})$, $P^{\varepsilon,\mu}$, $e=(0,0,1)$, ε^{-1} and μ denote the velocity, pressure, direction of rotation, frequency of rotation and viscosity in z of the incompressible flow, respectively. The parameter ε is also called the Rossby number, and $\frac{e\times u^{\varepsilon,\mu}}{\varepsilon}$ represents the Coriolis force created by rotation.

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Here, we assume the viscosities along the x, y directions to be 1, which is much larger than μ.

The global well-posedness for the Cauchy problem (or periodic boundary case) with uniformly positive viscosity was first proved by Grenier and his colleagues in [5, 6]. The weak limit of the solution is also considered as ε and μ go to. Whereas, the method fails in the initial and boundary value problems since the appearance of boundary layers and high frequency oscillation in the time direction, cf. [4] and reference therein. In particular, the oscillation in time does not occur for the well-prepared case, i.e. $u_0^{\varepsilon,\mu}$ and $F^{\varepsilon,\mu}$ are independent of z and their third components vanish. In this setting, the problem (1.1) was considered by Colin [2], Colin and Fabrie [3], Grenier and Masmoudi [6]. For completeness, it starts by giving a brief description of [6]. In what follows, it is convenient to adapt the notation u_h, $\nabla^h u$ and $\nabla^h \cdot u$ to represent the components, the gradient and the divergence of u in (x, y), respectively.

Assume μ equals to ε and denote $u^{\varepsilon,\mu}$ by u^ε, from [4] we know the thickness of boundary layers equals to ε and that

$$u^\varepsilon \sim \sum_{k=0}^{m} \varepsilon^k u^k(t, x, y, z, \frac{z}{\varepsilon}, \frac{1-z}{\varepsilon}) + o(\varepsilon^m), \tag{1.2}$$

where u^k consists of $u^{I,k}(t, x, y, z)$, $u^{B,k,0}(t, x, y, \frac{z}{\varepsilon})$ and $u^{B,k,1}(t, x, y, \frac{1-z}{\varepsilon})$, it satisfies the non-slip boundary condition at both $z = 0$ and $z = 1$. Meanwhile, the boundary layers $u^{B,k,0}$ and $u^{B,k,1}$ have fast decay in $\theta = \frac{z}{\varepsilon}$ and $\xi = \frac{1-z}{\varepsilon}$, respectively.

Replacing u^ε by (1.2) and letting ε goes to zero, then u^ε is proved in [6] to converge to some $(u_h^{I,0}, 0)$, in which $u_h^{I,0}$ is the solution to the Cauchy problem of the following two dimensional Navier-Stokes equations with a dispersive term in $(0, T) \times \mathbb{T}^2$.

$$\begin{cases} \partial_t u + u \cdot \nabla_{x,y} u - \Delta_{x,y} u + \nabla_{x,y} q + \sqrt{2} u = F^0, \\ \nabla_{x,y} \cdot u = 0, \\ u|_{t=0} = u_0(x, y). \end{cases} \tag{1.3}$$

It is emphasized that the pressure q could be different from P. In fact, they satisfy the following identity,

$$\sqrt{2}u_h^{I,0} + \nabla q = (e \times u^{I,1} + \nabla P^{I,1})_h. \tag{1.4}$$

Meanwhile, by using the multi-scale argument they deduced $u_3^{B,0,0} \equiv 0$, and that

$$\begin{cases} \partial_\theta^2 u_h^{B,0,0} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_h^{B,0,0} = 0, \\ u_h^{B,0,0}|_{\theta=0} = -u_h^{I,0}(x,y,0), \\ \lim\limits_{\theta\to\infty} u_h^{B,0,0} = 0, \end{cases} \tag{1.5}$$

which is given by

$$u_h^{B,0,0} = -e^{-\frac{\theta}{\sqrt{2}}} M(\frac{\theta}{\sqrt{2}}) u_h^{I,0}, \tag{1.6}$$

with $M(\tau) = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}$. For the reason of incompressibility and decay in θ, it implies

$$u_3^{B,1,0} = -e^{-\frac{\theta}{\sqrt{2}}} \sin(\frac{\theta}{\sqrt{2}} + \frac{\pi}{4}) Rotu_h^{I,0}. \tag{1.7}$$

In a similar way, $(u_h^{B,0,1}, u_3^{B,1,1})$ can be defined as

$$\begin{aligned} u_h^{B,0,1} &= -e^{-\frac{\xi}{\sqrt{2}}} M(-\frac{\xi}{\sqrt{2}}) u_h^{I,0}, \\ u_3^{B,1,1} &= e^{-\frac{\xi}{\sqrt{2}}} \sin(\frac{\xi}{\sqrt{2}} + \frac{\pi}{4}) Rotu_h^{I,0}. \end{aligned}$$

On the other hand, although (1.4) is a under-determined system, we can still define $u_h^{I,1}$ and $P^{I,1}$ in a natural way that

$$u_h^{I,1} = -\sqrt{2}(e \times u^{I,0})_h, \tag{1.8}$$

$$P^{I,1} = (q, 0). \tag{1.9}$$

Then, by the divergence free condition and the boundary condition at $z = 0$

we get

$$u_3^{I,1} = \frac{1}{\sqrt{2}}(1-2z)Rotu^{I,0}. \tag{1.10}$$

In order to cancel out the boundary values produced by $u^{I,1}$, $u^{B,0,0}$ and $u^{B,0,1}$, correctors B^3 and B^4 are constructed in [6]. Finally, they proved the convergence of u^ε to $u^{I,0}$ in the L^2 norm by applying standard energy estimate and the Gronwall's inequality as shown in Lemma 3.6.

However, it remains open whether u^ε can be estimated by $u^{B,0,0}$ and $u^{B,0,1}$ near the boundary $z=0$ and $z=1$, separatively in a more natural L^∞ space. To prove this, it is sufficient to obtain the estimates for higher order derivatives of the remained error. Thus, higher order expansion in (1.2) than the case in [6] is needed since the derivatives of the error in z could be of the order $O(1)$. In fact, one can observe that the expansion (1.2) can be derived explicitly for any $m \in \mathbb{N}$. Consequently, the high order derivative estimates can be obtained progressively by using Lemma 3.6. Our main result is as follows:

Theorem 1.1. *Assume u_0^ε and F^ε are well-prepared, $u^{I,0} \in L^2([0,T), H^7(\mathbb{T}^2))$ $\cap L^\infty([0,T), H^6(\mathbb{T}^2))$, $\partial_t^j F^0 \in L^2([0,T), H^{5-j}(\mathbb{T}^2)) \cap L^\infty([0,T), H^{4-j}(\mathbb{T}^2))$ for $0 \le j \le 2$, $\partial_t^k F^1 \in L^2([0,T), H^{4-k}(\mathbb{T}^2)) \cap L^\infty([0,T), H^{3-k}(\mathbb{T}^2))(k=0,1)$ and $F^2 \in L^2([0,T), H^3(\Omega))$ with some $T>0$, then*

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - u^{I,0} - u^{B,0,0} - u^{B,0,1}\|_{L^\infty([0,T)\times\Omega)} = 0. \tag{1.11}$$

Remark 1.2. Here, our purpose is only to investigate the validity of the boundary layer expansions (1.2), so the initial data and force term are assumed to be well-prepared to avoid the difficulty raised from the high frequency oscillation in t.

It is worth noting that Masmoudi [8] also considered the problem for the general ill-prepared case by applying a group introduced by Schochet[11] to filter the effect of the oscillation in time. The linear and nonlinear instability of Ekmann layer were considered by Rousset [10], Chemin, Desjardins [1], Lilly [7] and the references therein.

The rest of this paper is organized as follows: In Section 2, we will construct an approximation solution in (1.2) with $m=2$, and derive the equations for the remained terms. The error estimate and the proof of our main theorem are given in Section 3.

2. The Construction of Approximation Solution

In this part, we will construct an approximate solution u^{app} near the boundary $z=0$ (the case at $z=1$ is similar) in the form

$$u^{app} = u^{I,0} + u^{B,0} + \varepsilon(u^{I,1} + u^{B,1}) + \varepsilon^2(u^{I,2} + u^{B,2}) + u_c. \tag{2.1}$$

Here u_c denotes the corrector at $z=1$ such that u^{app} satisfies the non-slip boundary conditions, which has the order of $O(e^{-\frac{1}{\sqrt{2}\varepsilon}})$ thanks to the exponentially decay of boundary layers. Meanwhile, there is a similar expansion P^{app} for the pressure P^{ε}, and F^{ε} is assumed to be

$$F^{\varepsilon} = F^0(t,x,y) + \varepsilon F^1(t,x,y) + \varepsilon^2 F^2(\varepsilon,t,x,y,z). \tag{2.2}$$

Plugging (2.1) into in (1.1) directly, it reduces to

$$\partial_t u^{app} + u^{app}\cdot\nabla u^{app} + \frac{e\times u^{app}}{\varepsilon} + \frac{\nabla P^{app}}{\varepsilon} - \Delta_{x,y}u^{app} - \varepsilon\partial_z^2 u^{app} - F^{\varepsilon} = O(\varepsilon^2), \tag{2.3}$$

$$\nabla\cdot u^{I,i} = 0, \tag{2.4}$$

$$\nabla^h\cdot u_h^{B,i} + \partial_\theta u_3^{B,i+1} = 0. \tag{2.5}$$

In particular, it implies $u_3^{B,0}=0$ for the sake of (2.5).

As shown in [6], $u_h^{I,0}$ is the solution to the 2-dimensional Cauchy problem (1.3) and $u_3^{I,0}=0$. The quantities $u_h^{I,1}$, $P^{I,1}$, $u_h^{B,0}$ and $u_3^{B,1}$ are given by (1.8), (1.9), (1.6) and (1.7), respectively.

Through a similar way of the derivation of $(u_h^{B,0}, u_3^{B,1})$, $u_h^{B,1}$ satisfies

$$\begin{cases} \partial_\theta^2 u_h^{B,1} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_h^{B,1} = f, \\ u_h^{B,1}|_{\theta=0} = -u_h^{I,1}(x,y,0), \\ \lim\limits_{\theta\to\infty} u_h^{B,1} = 0, \end{cases}$$

where f is given as

$$\begin{aligned} f \;=\; & e^{-\frac{\theta}{\sqrt{2}}}M(\frac{\theta}{\sqrt{2}})(\nabla q+\sqrt{2}u^{I,0}+F^0)+(u^{I,0}+u^{B,0})\cdot\nabla u_h^{B,0}+u^{B,0}\cdot\nabla u_h^{I,0}\\ & +(u_3^{I,1}+u_3^{B,1})\partial_\theta u_h^{B,0}. \end{aligned}$$

Remark 2.1. Here, the term $(u^{I,0}+u^{B,0})\cdot\nabla u_h^{B,0}$ has the order of $O(1)$ since the vanish of the third components of $u^{I,0}$ and $u^{B,0}$. In addition, the last term of f is preserved although it can be expanded more specifically.

Therefore, from the definition of $u_h^{I,1}$ we know

$$u_h^{B,1}=-\sqrt{2}e^{-\frac{\theta}{\sqrt{2}}}M(\frac{\theta}{\sqrt{2}})(e\times u^{I,0})_h+Lf, \tag{2.6}$$

in which L is defined by

$$\begin{aligned} Lf(s) \;=\; & \frac{1}{2\sqrt{2}}\Big[-\int_0^s e^{-\frac{s-\tau}{\sqrt{2}}}M_1(\frac{s-\tau}{\sqrt{2}})f(\tau)d\tau-\int_s^\infty e^{-\frac{\tau-s}{\sqrt{2}}}M_1(\frac{\tau-s}{\sqrt{2}})fd\tau\\ & +\int_0^\infty e^{-\frac{s+\tau}{\sqrt{2}}}M_1(\frac{s+\tau}{\sqrt{2}})f(\tau)d\tau\Big], \end{aligned}$$

with $M_1(\tau)=\begin{pmatrix}\cos(\tau+\frac{\pi}{4}) & \sin(\tau+\frac{\pi}{4})\\ \sin(\tau+\frac{\pi}{4}) & -\cos(\tau+\frac{\pi}{4})\end{pmatrix}$. Consequently, derived from (2.5) we have

$$u_3^{B,2}=\int_\theta^\infty(\sqrt{2}e^{-\frac{\tau}{\sqrt{2}}}Rotu_h^{I,0}+L(\nabla^h\cdot f))(\tau)d\tau, \tag{2.7}$$

where $Rotu:=\partial_x u_2-\partial_y u_1$.

Meanwhile, the third equation in (2.3) with the order of $O(1)$ reads $\partial_\theta P^{B,2}=\partial_\theta^2 u_3^{B,1}$. Therefore, it can be solved by using (1.7) that

$$P^{B,2}=-e^{-\frac{\theta}{\sqrt{2}}}\sin\frac{\theta}{\sqrt{2}}Rotu_h^{I,0}. \tag{2.8}$$

As to the second order term $(u^{I,2},P^{I,2})$, the order of $O(\varepsilon)$ in (2.3) becomes

$$\begin{cases}\partial_t u^{I,1}+u^{I,0}\cdot\nabla u^{I,1}+u^{I,1}\cdot\nabla u^{I,0}+\nabla P^{I,2}+e\times u^{I,2}-\Delta_{x,y}u^{I,1}=F^1,\\ \nabla\cdot u^{I,2}=0.\end{cases} \tag{2.9}$$

In particular, the third component implies

$$\partial_z P^{I,2} = -(\partial_t u_3^{I,1} + u^{I,0}\cdot\nabla u_3^{I,1} - \Delta_{x,y} u_3^{I,1}).$$

Combining this with (1.3) and (1.10), it arrives at

$$P^{I,2} = \frac{1}{\sqrt{2}}(z^2 - z)(RotF^0 - \sqrt{2}Rotu_h^{I,0}). \tag{2.10}$$

Taking it into (2.9), and utilizing (1.8) and (1.3),

$$\begin{aligned} u_h^{I,2} &= \sqrt{2}(F^0 - \sqrt{2}u^{I,0} - \nabla q) - \sqrt{2}((e\times u^{I,0})\cdot\nabla(e\times u^{I,0}))_h - (e\times F^1)_h \\ &\quad + \frac{1}{\sqrt{2}}(z^2 - z)(e\times(\nabla^h RotF^0 - \sqrt{2}\nabla^h Rotu_h^{I,0}))_h. \end{aligned} \tag{2.11}$$

Based on this, using by now (2.4) and taking $\nabla^h\times$ to (2.9) we deduce that

$$\partial_z u_3^{I,2} = Rot(\partial_t u_h^{I,1} + u^{I,0}\cdot\nabla u_h^{I,1} + u^{I,1}\cdot\nabla u_h^{I,0} - \Delta_{x,y}u_h^{I,1} - F^1).$$

Note that $Rotu_h^{I,1} = 0$ and $u_3^{I,2}|_{z=0} = -u_3^{B,2}|_{\theta=0}$, hence

$$u_3^{I,2} = \int_0^\infty \sqrt{2}e^{-\frac{s}{\sqrt{2}}} ds Rotu_h^{I,0} + (\sqrt{2}u^{I,0}\cdot\nabla(Rotu_h^{I,0}) - RotF^1)z. \tag{2.12}$$

On the other hand, the order of $O(\varepsilon)$ in (2.3) which varies in θ is reduced to

$$\begin{cases} \partial_\theta^2 u_h^{B,2} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_h^{B,2} = g, \\ u_h^{B,2}|_{\theta=0} = -u_h^{I,2}(x,y,0), \\ \lim\limits_{\theta\to\infty} u_h^{B,2} = 0, \end{cases}$$

where

$$\begin{aligned} g &= \partial_t u_h^{B,1} + (u_h^{I,1} + u_h^{B,1})\cdot\nabla^h u_h^{B,0} + u^{B,1}\cdot\nabla u_h^{I,0} + (u^{I,0} + u^{B,0})\cdot\nabla u_h^{B,1} \\ &\quad + u^{B,0}\cdot\nabla u_h^{I,1} + (u_3^{I,1} + u_3^{B,1})\partial_\theta u^{B,1} + (u_3^{I,2} + u_3^{B,2})\partial_\theta u^{B,0} + \nabla^h P^{B,2} \\ &\quad - \Delta_{x,y} u_h^{B,1}. \end{aligned}$$

Therefore,

$$u_h^{B,2} = -e^{-\frac{s}{\sqrt{2}}} M(\frac{s}{\sqrt{2}})(\sqrt{2}(F^0 - \sqrt{2}u^{I,0} - \nabla q)$$

$$-\sqrt{2}((e\times u^{I,0})\cdot\nabla(e\times u^{I,0})-e\times F^1)_h)+Lg. \tag{2.13}$$

In addition, we can also obtain

$$\partial_\theta P^{B,3}+\partial_t u_3^{B,1}+(u^{I,0}+u^{B,0})\cdot\nabla u_3^{B,1}+u^{B,0}\cdot\nabla u_3^{I,1}+(u_3^{I,1}+u_3^{B,1})\partial_z u_3^{B,1}$$
$$-\Delta_{x,y}u_3^{B,1}-\partial_\theta^2 u_3^{B,2}=0.$$

Thus, we compute it directly as

$$\begin{aligned}P^{B,3} &= \partial_\theta u_3^{B,2}+\int_\theta^\infty [e^{-\frac{\tau}{\sqrt{2}}}\sin(\frac{\tau}{\sqrt{2}}+\frac{\pi}{4})(RotF^0-\sqrt{2}Rotu_h^{I,0})u_3^{B,1}\\ &\quad -u^{B,0}\cdot\nabla-(u_3^{I,1}+u_3^{B,1})\partial_z u_3^{B,1}]d\tau.\end{aligned} \tag{2.14}$$

At last, we need only to construct an corrector u_c of the order $O(e^{-\frac{1}{\sqrt{2}\varepsilon}})$ such that

$$u_c(t,x,y,1)=-u^{B,0}|_{\theta=\frac{1}{\varepsilon}}-\varepsilon u^{B,1}|_{\theta=\frac{1}{\varepsilon}}-\varepsilon^2 u^{B,2}|_{\theta=\frac{1}{\varepsilon}}, \tag{2.15}$$

and $u_c|_{z=0}=0$. Indeed, we can choose u_c as the solution to the Navier-Stokes equations in a domain with smooth boundary which contains $\{z=0\}\cup\{z=1\}$, and the initial and boundary conditions coincide with u_c at both $z=0$ and $z=1$.

Based on the approximation solution u^{app} established as above, it suffices to prove the smallness of $R=u^\varepsilon-u^{app}$, which satisfies the following problems

$$\begin{cases}\partial_t R+u^\varepsilon\cdot\nabla R+R\cdot\nabla u^{app}+\frac{1}{\varepsilon}P_R+\frac{1}{\varepsilon}e\times R-\Delta_{x,y}R-\varepsilon\partial_z^2 R=\sum\limits_{i=1}^4 I_i,\\ \nabla\cdot R=0,\\ R|_{t=0}=R|_{z=0}=R|_{z=1}=0,\end{cases} \tag{2.16}$$

where

$$\begin{aligned}I_1 &= \varepsilon^2F^2+\varepsilon^2(\nabla^h P^{B,3},0)-\varepsilon^2\Delta_{x,y}(u^{I,2}+u^{B,2})-\varepsilon^3\partial_z^2 u^{I,2},\\ I_2 &= -\varepsilon^2\Big(\partial_t(u^{I,2}+u^{B,2})+u^{I,0}\cdot\nabla u^{I,2}+u^{I,1}\cdot\nabla u^{I,1}+u^{I,2}\cdot\nabla u^{I,0}\Big)\\ &\quad -\varepsilon^3(u^{I,1}\cdot\nabla u^{I,2}+u^{I,2}\cdot\nabla u^{I,1})-\varepsilon^4 u^{I,2}\cdot\nabla u^{I,2},\\ I_3 &= -\varepsilon^2\Big((u^{I,0}+u^{B,0})\cdot\nabla u^{B,2}+u^{B,0}\cdot\nabla u^{I,2}+(u_h^{I,1}+u_h^{B,1})\cdot\nabla^h u^{B,1}\end{aligned}$$

$$
\begin{aligned}
& +u^{B,1}\cdot\nabla u^{I,1}+(u_h^{I,2}+u_h^{B,2})\cdot\nabla u^{B,0}+u^{B,2}\cdot\nabla u^0\Big) \\
& -\varepsilon^3\Big[(u^{I,1}+u^{B,1})\cdot\nabla u^{B,2}+u^{B,1}\cdot\nabla u^{I,2}+(u^{I,2}+u^{B,2})\cdot\nabla u^{B,1} \\
& +u^{B,2}\cdot\nabla u^{I,1}\Big]-\varepsilon^4\Big((u^{I,2}+u^{B,2})\cdot\nabla u^{B,2}+u^{B,2}\cdot\nabla u^{I,2}\Big), \\
I_4 \;=\; & -\partial_t u_c-u^{app}\cdot\nabla u_c-u_c\cdot\nabla u^{app}+u_c\cdot\nabla u_c+\Delta_{x,y}u_c+\varepsilon\partial_z^2 u_c.
\end{aligned}
$$

3. Estimates for R

In this section, we will derive the estimates for R and its derivatives. Henceforth, denote the L^2 norm in $(0,T)\times\Omega$ by $\|\cdot\|$, C_is and K_is are positive constants depending on $\|u^{I,0}\|_{L^\infty((0,T),H^4(\mathbb{T}^2))}$, $\|F^0\|_{L^\infty((0,T),H^2(\mathbb{T}^2))}$ and $\|F^1\|_{L^\infty((0,T),L^2(\mathbb{T}^2))}$.

First, the following energy estimate for the problem (2.16) can be obtained by a standard energy method.

Proposition 3.1. *Suppose $u^{I,0}$, F^0, F^1 and F^2 are assumed in Theorem* (1.1), *then there exist some $\varepsilon_0>0$ and $K_1>0$ such that*

$$
\|R\|_{L^\infty((0,T),L^2(\Omega))}+\|\nabla^h R\|+\sqrt{\varepsilon}\|\partial_z R\|\le K_1\varepsilon^{\frac{3}{2}}, \tag{3.1}
$$

provided $0\le\varepsilon\le\varepsilon_0$. In particular, by using Poincare's inequality

$$
\|R\|_{L^2}\le K_2\varepsilon. \tag{3.2}
$$

The proof of Proposition 3.1 is based on the following lemmas.

Lemma 3.1. *There exists some $K_3>0$ for sufficiently small $\varepsilon>0$ such that*

$$
\begin{aligned}
|(R\nabla\cdot u^{app},R) \;\le\; & \frac{1}{8}(\|\nabla^h R\|^2+\varepsilon\|R_z\|^2)+K_3(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^3}^2 \\
& +\|F_t^0\|_{H^1}^2+\|F^1\|_{H_2}^2)\|R\|^2.
\end{aligned} \tag{3.3}
$$

Proof. Integrating by parts, it is easy to prove

$$
\begin{aligned}
|(R\cdot\nabla u^{I,0},R)| \;=\; & |(R_h\cdot\nabla^h u_h^{I,0},R_h)| \\
\le\; & 2\|\nabla^h R\|\|R_h u^{I,0}\|
\end{aligned}
$$

$$
\begin{aligned}
&\le c\|\nabla^h R\|^2+\frac{1}{c}\|u^{I,0}\|_{L^\infty}^2\|\|R\|^2\\
&\le c\|\nabla^h R\|^2+\frac{1}{c}\|u^{I,0}\|_{H^2}^2\|\|R\|^2 \qquad (3.4)
\end{aligned}
$$

for any small $c>0$.

Meanwhile, $\varepsilon(R\cdot\nabla u^{I,1},R)=\varepsilon(R\cdot\nabla u_h^{I,1},R_h)+\varepsilon(R_h\cdot\nabla^h u_3^{I,1},R_3)+\varepsilon(R_3\partial_z u_3^{I,1},R_3)$, in which the first two terms are bounded by

$$
\begin{aligned}
&|\varepsilon(R\cdot\nabla u_h^{I,1},R_h)+\varepsilon(R_h\cdot\nabla^h u_3^{I,1},R_3)|\\
&\le c\varepsilon\|\nabla^h R\|^2+\frac{\varepsilon}{c}(\|\|u^{I,0}\|_{H^2}^2\|R\|^2+\|\nabla u^{I,0}R\|^2)\\
&\le c\varepsilon\|\nabla^h R\|^2+\frac{C_1\varepsilon}{c}(\|u^{I,0}\|_{H^2}^2\|R\|^2+\|\nabla u^{I,0}\|_{L^4}^2\|R\|_{L^4}^2)\\
&\le c\varepsilon(1+\|u^{I,0}\|_{H^2}^2)\|\nabla R\|^2+\frac{C_2\varepsilon}{c}\|u^{I,0}\|_{H^2}^2\|R\|^2.
\end{aligned}
$$

Here, Cauchy-Schwartz's inequality and the following Gagliardo-Nirenberg inequality are used in the last inequality,

$$
\|f\|_{L^4}^2\le C_3\|f\|_{L^2}\|f\|_{H^1}. \qquad (3.5)
$$

In addition, directly from the definition of $u_h^{I,1}$ and integrating by parts

$$
\begin{aligned}
|\varepsilon(R_3\partial_z u_3^{I,1},R_3)| &= \sqrt{2}\varepsilon\|Rotu_h^{I,0}R_3^2\|\\
&\le c\varepsilon\|\nabla^h R\|^2+\frac{\sqrt{2}\varepsilon}{c}\|Rotu_h^{I,0}R\|^2\\
&\le c\varepsilon(1+\|u^{I,0}\|_{H^2}^2)\|\nabla R\|^2+\frac{C_4\varepsilon}{c}\|u^{I,0}\|_{H^2}^2\|R\|^2.
\end{aligned}
$$

Thus, we deduce that

$$
\varepsilon(R\cdot\nabla u^{I,1},R)\le c\varepsilon(1+\|u^{I,0}\|_{H^2}^2)\|\nabla R\|^2+\frac{C_5\varepsilon}{c}\|u^{I,0}\|_{H^2}^2\|R\|^2. \qquad (3.6)
$$

In a similar way,

$$
\begin{aligned}
\varepsilon^2(R\cdot\nabla u^{I,2},R) &\le c\varepsilon^2(1+\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2)\|\nabla R\|^2\\
&\quad+\frac{C_6\varepsilon^2}{c}(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2)\|R\|^2. \qquad (3.7)
\end{aligned}
$$

On the other hand, $(R\cdot\nabla u^{B,0},R)=(R^h\cdot\nabla^h u_h^{B,0},R_h)+(R_3\partial_z u_h^{B,0},R_h)$.

The first term is estimated as the same as (3.4), whereas the second term is bounded as follows

$$\begin{aligned}
&(R_3\partial_z u^{B,0}, R_h)\\
&= -\int_{x,y}\int_0^1 dzdxdy\partial_z u^{B,0}R_h\int_s^1 ds\partial_s R_3\\
&\le C_7\|u^{I,0}\|_{L^\infty}\int_{x,y}\int_0^1|\partial_s R_3|(\varepsilon^{-1}e^{-\frac{s}{\sqrt{2}\varepsilon}})^{\frac12}(\int_0^1|R_h|^2dz)^{\frac12}dxdyds\\
&\le C_7\|u^{I,0}\|_{L^\infty}\|R\|\|\partial_z R_3\|\le c\|\nabla^h R\|^2+\frac{C_7}{c}\|u^{I,0}\|_{H^2}^2\|R\|^2, \qquad (3.8)
\end{aligned}$$

in which the divergence free condition is used.

In a similar way of the proof of (3.8), we get

$$\varepsilon(R\cdot\nabla u^{B,1},R) \le c\varepsilon\|\nabla R^h\|^2+\frac{C_8\varepsilon}{c}(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2)\|R\|^2, \qquad (3.9)$$

$$\begin{aligned}
\varepsilon^2(R\cdot\nabla u^{B,2},R) \le\ & C_9\varepsilon^3(\|u^0\|_{H^4}^2+\|F^0\|_{H^1}^2+\|F_t^0\|_{H^1}^2\\
&+\|F^1\|_{H^1}^2)(\|\nabla R\|^2+\|R\|^2). \qquad (3.10)
\end{aligned}$$

At last, the term related to the corrector u_c can be bounded as

$$\begin{aligned}
|(R\nabla u_c,R)| \le\ & C_{10}e^{-\frac{1}{\sqrt{2}\varepsilon}}(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2\\
&+\|F_t^0\|_{H^1}^2+\|F^1\|_{H^1}^2)(\|\nabla R\|^2+\|R\|^2). \qquad (3.11)
\end{aligned}$$

Thus, the inequality (3.3) follows immediately from summing all the above inequalities (3.4)-(3.11) up and choosing c sufficiently small. □

Lemma 3.2. *For some $K_4>0$ and sufficiently small $\varepsilon>0$, it satisfies*

$$\begin{aligned}
|(I_1,R)| \le\ & \frac18(\|\nabla^h R\|^2+\varepsilon\|R_z\|^2)+K_4\varepsilon^3(\|F^2\|^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2\\
&+\|F_t^0\|_{H^1}^2+\|u^{I,0}\|_{H^4}^2). \qquad (3.12)
\end{aligned}$$

Proof. First, by the Cauchy-Schwartz inequality and Poincare's inequality $\|R\|\le\|R_z\|$ we have

$$|\varepsilon^2(F^2,R)|\le c\varepsilon\|R_z\|^2+\frac{C_{11}\varepsilon^3}{c}\|F^2\|^2.$$

Meanwhile, from the definitions (2.7), (2.11), (2.12) and (2.13), it is easy to show

$$\begin{aligned}&|\varepsilon^2(\Delta_{x,y}(u^{I,2}+u^{B,2}),R)|\\&\leq\ c\|\nabla^h R\|^2+\frac{C_{12}\varepsilon^4}{c}(\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2+\|F_t^0\|_{H^1}^2+\|u^{I,0}\|_{H^4}^2).\end{aligned}$$

On the other hand, it follows from (2.11), (2.12) and (2.14) that

$$\begin{aligned}|\varepsilon^2(\nabla^h P^{B,3},R)|\ &\leq\ c\|\nabla^h R\|^2+\frac{C_{13}\varepsilon^4}{c}(\|F^0\|_{H^1}^2+\|u^{I,0}\|_{H^2}^2),\\|\varepsilon^3(\partial_z^2 u^{I,2},R)|\ &\leq\ c\varepsilon\|R_z\|^2+\frac{C_{14}\varepsilon^5}{c}(\|F^0\|_{H^2}^2+\|F^1\|_{H^2}^2+\|u^{I,0}\|_{H^2}^2).\end{aligned}$$

Thus, we conclude the proof by letting c be small enough. □

Lemma 3.3. *Assume ε is small enough, then the following inequality holds for some K_5,*

$$\begin{aligned}|(I_2,R)|\ \leq\ &\frac{1}{8}\|\nabla^h R\|^2+c\varepsilon\|R_z\|^2+K_5\varepsilon^2(\|u^{I,0}\|_{H^5}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2)\|R\|\\&+K_5\varepsilon^3(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2+\|F_t^0\|_{H^2}^2+\|F^1\|_{H^2}^2+\|F_t^1\|_{H^1}^2\\&+\|F_{tt}^0\|^2).\end{aligned}\tag{3.13}$$

Proof. From (1.3), it is straightly forward to obtain

$$\begin{aligned}|\varepsilon^2(u_t^{I,0},R)|\ \leq\ &c\|\nabla^h R\|^2+\frac{C_{15}\varepsilon^2}{c}(\|u^{I,0}\|+\|u^{I,0}\|_{H^2}^2+\|F^0\|)\|R\|\\&+\frac{C_{15}\varepsilon^4}{c}\|u^{I,0}\|_{H^2}^2,\end{aligned}$$

for any $c>0$. Thus, it follows by using Poincare's inequality that

$$\begin{aligned}\varepsilon^2|(u_t^{I,0},R)|\ \leq\ &c\varepsilon\|\partial_z R\|^2+c\|\nabla^h R\|^2+\frac{C_{16}\varepsilon^2}{c}\|u^{I,0}\|_{H^2}^2\|R\|\\&+\frac{C_{16}\varepsilon^2}{c}(\|u^{I,0}\|_{H^2}^2+\|F^0\|^2).\end{aligned}\tag{3.14}$$

Analogous to the proof of (3.14), by using (2.11), (2.12), (2.13) and (2.7) we get

$$\varepsilon^2|(u_t^{I,2}+u_t^{B,2},R)|\ \leq c\varepsilon\|\partial_z R\|^2+c\|\nabla^h R\|^2+\frac{C_{17}\varepsilon^2}{c}\|u^{I,0}\|_{H^5}^2\|R\|$$

$$+\frac{C_{17}\varepsilon^2}{c}(\|F^0\|_{H^2}^2+\|F_t^0\|_{H^2}^2+\|u^{I,0}\|_{H^4}^2+\|F_t^1\|_{H^1}^2+\|F_{tt}^0\|^2).$$

On the other hand, derived from the definition of $u^{I,2}$ that

$$\begin{aligned}|\varepsilon^4(u^{I,2}\cdot\nabla u^{I,2},R)| &= |\varepsilon^4(\nabla(u^{I,2}\otimes u^{I,2}),R)|\\ &\le c\varepsilon^4\|\nabla R\|^2+\frac{C_{18}\varepsilon^4}{c}(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2+\|F^1\|_{H^1}^2).\end{aligned}$$

As for the remained terms, we can obtain the following estimates directly

$$\begin{aligned}|\varepsilon^2(u^{I,0}\cdot\nabla u^{I,2},R)| &\le C_{19}\varepsilon^2(\|u^{I,0}\|_{H^3}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2)\|R\|,\\ |\varepsilon^2(u^{I,1}\cdot\nabla u^{I,1},R)| &\le C_{20}\varepsilon^2\|u^{I,0}\|_{H^2}^2\|R\|,\\ |\varepsilon^2(u^{I,2}\cdot\nabla u^{I,0},R)| &\le C_{21}\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2+\|F^1\|_{H^1}^2)\|R\|,\\ |\varepsilon^3(u^{I,1}\cdot\nabla u^{I,2},R)| &\le C_{22}\varepsilon^3(\|u^{I,0}\|_{H^3}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2)\|R\|,\\ |\varepsilon^3(u^{I,2}\cdot\nabla u^{I,1},R)| &\le C_{23}\varepsilon^3(\|u^{I,0}\|_{H^2}^2+\|F^0\|_{H^2}^2+\|F^1\|_{H^1}^2)\|R\|,\end{aligned}$$

In summary, (3.13) is proved. This concludes the proof of this lemma. □

Lemma 3.4. *For sufficiently small ε, there exists some $K_6>0$ such that*

$$|(I_3,R)|\le K_6\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^3}^2+\|F_t^0\|_{H^1}^2)\|R\|. \tag{3.15}$$

Proof. First, due to the vanish of the third component of $u^{I,0}$ and $u^{B,0}$ we know

$$\begin{aligned}&\varepsilon^2|((u^0+u^{B,0})\cdot\nabla u^{B,2},R)|\\ &\quad\le \varepsilon^2\|u^0+u^{B,0}\|_{L^\infty}\|\nabla^h u^{B,2}\|\|R\|\\ &\quad\le C_{24}\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^1}^2+\|F^1\|_{H^3}^2+\|F_t^0\|^2)\|R\|,\end{aligned}\tag{3.16}$$

where the definitions (2.13) and (2.7) are used.

In a similar way to the proof of (3.16), we can derive the following with some universal positive constant C_{25},

$$\begin{aligned}&\varepsilon^2|((u^{I,0}+u^{B,0})\cdot\nabla u^{B,2},R)|\\ &\quad\le C_{25}\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^1}^2+\|F^1\|_{H^3}^2+\|F_t^0\|^2)\|R\|,\\ &\varepsilon^2|(u^{B,0}\cdot\nabla^h u^{I,2},R)|\\ &\quad\le C_{25}\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2)\|R\|,\end{aligned}$$

$$\begin{aligned}
&\varepsilon^2|((u_h^{I,1}+u_h^{B,1})\cdot\nabla^h u^{B,1}+u^{B,1}\cdot\nabla u^{I,1},R)| \\
&\quad\leq\ C_{25}\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^1}^2)\|R\|, \\
&\varepsilon^2|((u_h^{I,2}+u_h^{B,2})\cdot\nabla^h u^{B,0},R)| \\
&\quad\leq\ C_{25}\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2+\|F^1\|_{H^1}^2+\|F_t^0\|^2)\|R\|, \\
&\varepsilon^2|(u^{B,2}\cdot\nabla u^{I,0},R)| \\
&\quad\leq\ C_{25}\varepsilon^2(\|u^{I,0}\|_{H^3}^2+\|F^0\|_{H^1}^2+\|F^1\|^2+\|F_t^0\|^2)\|R\|, \\
&\varepsilon^3|((u^{I,1}+u^{B,1})\cdot\nabla u^{B,2},R)| \\
&\quad\leq\ C_{25}\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2+\|F^1\|_{H^1}^2+\|F_t^0\|^2)\|R\|, \\
&\varepsilon^3|(u^{B,1}\cdot\nabla u^{I,2},R)| \\
&\quad\leq\ C_{25}\varepsilon^3(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2)\|R\|, \\
&\varepsilon^3|((u^{B,2}+u^{I,2})\cdot\nabla u^{B,1},R)| \\
&\quad\leq\ C_{25}\varepsilon^2(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2+\|F^1\|_{H^1}^2+\|F_t^0\|^2)\|R\|, \\
&\varepsilon^3|(u^{B,2}\cdot\nabla u^{I,1},R)| \\
&\quad\leq\ C_{25}\varepsilon^3(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^1}^2+\|F^1\|^2+\|F_t^0\|^2)\|R\|, \\
&\varepsilon^4|((u^{I,2}+u^{B,2})\cdot\nabla u^{B,2},R)| \\
&\quad\leq\ C_{25}\varepsilon^3(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^2}^2+\|F^1\|_{H^1}^2+\|F_t^0\|_{H_1}^2)\|R\|.
\end{aligned}$$

In conclusion, (3.15) is proved. □

Lemma 3.5. *There exists some $K_7>0$ such that*

$$\begin{aligned}
|(I_4,R)|\ \leq\ &\frac{1}{8}\varepsilon\|R_z\|^2+K_7\varepsilon^{-1}e^{-\frac{1}{\sqrt{2\varepsilon}}}(\|u^{I,0}\|_{H^5}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2+\|F_t^0\|_{H^1}^2 \\
&++\|F_t^1\|^2\|F_{tt}^0\|^2). \qquad (3.17)
\end{aligned}$$

Proof. Recalling that u_c is of the order $O(e^{-\frac{1}{\sqrt{2\varepsilon}}})$, from the definition we know

$$\begin{aligned}
|(\partial_t u_c,R)|\ &\leq\ C_{26}e^{-\frac{1}{\sqrt{2\varepsilon}}}(\|u^{I,0}\|_{H^5}+\|F_t^0\|_{H^1}+\|F_{tt}^0\|+\|F_t^1\|)\|R\| \\
&\leq\ c\varepsilon\|R_z\|^2+\frac{C_{26}\varepsilon^{-1}e^{-\frac{1}{\sqrt{2\varepsilon}}}}{c}(\|u^{I,0}\|_{H^5}^2+\|F_t^0\|_{H^1}^2+\|F_{tt}^0\|^2+\|F_t^1\|^2),
\end{aligned}$$

in which the Poincare's inequality is used in the last inequality.

Similarly, we can obtain

$$|(u^{app}\cdot\nabla u_c+u_c\cdot\nabla u^{app},R)|\ \leq\ C_{27}\varepsilon^{-1}e^{-\frac{1}{\sqrt{2\varepsilon}}}(\|u^{I,0}\|_{H^5}^2+\|F^0\|_{H^3}^2+\|F^0\|_{H^1}^2$$

$$
\begin{aligned}
&\quad +\|F^1\|_{H^2}^2)\|R\|,\\
|(\Delta_{x,y}u_c+\varepsilon\partial_z^2u_c,R)| &\le c\|\nabla^h R\|^2+c\varepsilon\|R_z\|^2+\frac{C_{27}e^{-\frac{1}{\sqrt{2\varepsilon}}}}{c}(\|u^{I,0}\|_{H^4}^2\\
&\quad +\|F^0\|_{H^3}^2+\|F_t^0\|_{H^1}^2+\|F^1\|_{H^2}).
\end{aligned}
$$

Thus, (3.17) is proved by choosing sufficiently small $c>0$. □

Now, we are ready to prove Proposition 3.3 with the help of the Gronwall's inequality developed by Masmoudi [6], stated as follows.

Lemma 3.6. *Assume f and a_i $(i=0,1,2)$ are non-negative functions satisfying*

$$
\partial_t(f^2)\le a_0(t)f^2+a_1(t)f+a_2(t), \tag{3.18}
$$

with $f(0)\le C\alpha$, $\int_0^T a_i(t)dt\le C\alpha^i$ $(i=0,1,2)$ for some positive constants α and C, then there exists some $M>0$ depending only on C such that $f\le M\alpha$ for all $0\le t\le T$.

Proof of Proposition 3.3. By using standard energy method to (2.16), it yields

$$
\frac{1}{2}\frac{d}{dt}\|R\|^2+(R\cdot\nabla u^a,R)+\|\nabla^h R\|^2+\varepsilon\|\partial_z R\|^2=(\sum_{i=1}^4 I_i,R). \tag{3.19}
$$

As a result of the Lemmas 3.1-3.5, we deduce that

$$
\begin{aligned}
&\frac{d}{dt}\|R\|^2+\|\nabla^h R\|^2+\varepsilon\|\partial_z R\|^2\\
&\le K_8(\|u^{I,0}\|_{H^4}^2+\|F^0\|_{H^3}^2+\|F_t^0\|_{H^1}^2+\|F^1\|_{H_2}^2)\|R\|^2\\
&\quad +K_8\varepsilon^2(\|u^{I,0}\|_{H^5}^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^3}^2+\|F_t^0\|_{H^1}^2)\|R\|\\
&\quad +K_8\varepsilon^3(\|F^2\|^2+\|F^0\|_{H^3}^2+\|F^1\|_{H^2}^2+\|F_t^0\|_{H^2}^2+\|F_{tt}^0\|^2\\
&\quad +\|u^{I,0}\|_{H^5}^2+\|F_t^0\|_{H^1}^2+\|F_t^1\|^2).
\end{aligned}
$$

Thus, (3.1) is proved by applying Lemma 3.5. □

Based on the estimate for $\|R\|$, we can derive the estimates for its derivatives of R by induction, and then prove our main theorem.

Proof of Theorem 1.1. Differentiating (2.16) in z, we can handle almost every term as the same as done in the proof of Proposition 3.1 except that

(1) $(u_z^{app}\cdot\nabla R, R_z)$. Thanks to the assumptions on $u^{I,0}$ and F^i (i=0, 1, 2), we can find some $C>0$ so that

$$\begin{aligned}(u_z^{app}\cdot\nabla R, R_z) &= (u_z^{app,I}+u_{cz}\cdot\nabla R, R_z)+\varepsilon^{-1}(u_\theta^{app,B}\cdot\nabla R, R_z)\\ &\le C\varepsilon^{-1}\|\nabla R\|^2 \qquad (3.20)\end{aligned}$$

where $u^{app,I}$ and $u^{app,B}$ denote the sum of $u^{I,i}$ and $u^{B,i}$, respectively. For the reason of (3.1), the integration of the right hand side of (3.20) in t is bounded by $C\varepsilon$ with C depending on $u^{I,0}$ and F^i (i=0, 1, 2).

(2) $(R_z\cdot\nabla R, R_z)$. By using (3.5), we can deduce that

$$\begin{aligned}(R_z\cdot\nabla R, R_z) &\le \|\nabla R\|\|R_z\|_{L^4}^2\\ &\le C_3\|\nabla R\|\|R_z\|(\|R_z\|+\|\nabla^h R_z\|+\|\partial_z R_z\|)\\ &\le c\varepsilon\|\partial_z R_z\|^2+\frac{C_3}{c}(\|\nabla R\|^2\|R_z\|+\|\nabla R\|^2\|R_z\|^2), \qquad (3.21)\end{aligned}$$

in which the coefficients of $\|R_z\|$ and $\|R_z\|^2$ are integrable in t and bounded by $C\varepsilon^2$ and $C\varepsilon$, respectively.

(3) $(R\cdot\nabla u_z^{app}, R_z)=(R\cdot\nabla(u_z^{app,I}+u_c), R_z)+(R\cdot\nabla u_z^{app,B}, R_z)$. The first term is bounded by $C(t)\|R_z\|$ with $\int_0^\infty C(t)dt\le C\varepsilon^2$. Whereas, the last term is bounded by

$$(R\cdot\nabla u_z^{app,B}, R_z)\le C\varepsilon^{-1}\|R\|\|R_z\| \qquad (3.22)$$

wihch is integrable in t and bounded by $C\varepsilon$.

(4) Since R_z does satisfies the no-slip boundary condition, we can not apply the Poincare inequality for terms like $(u^{I,0}, R_z)$ as before. However, they can be regarded as source terms, the integration of which in t are bounded by $C\varepsilon$.

As a result, it follows from Lemma 3.1 and Lemma 3.6 that

$$\|R_z\|_{L^\infty((0,T),L^2)}+\|(\nabla^h)^2R_z\|+\sqrt{\varepsilon}\|\partial_z^2 R\|\le C\varepsilon^{\frac{1}{2}}. \qquad (3.23)$$

Based on this, it is similar to obtain the higher order derivatives of R_z in (x, y) that

$$\|(\nabla^h)^2 R_z\|_{L^\infty((0,T),L^2(\Omega))} + \|(\nabla^h)^3 R_z\| + \sqrt{\varepsilon}\|(\nabla^h)^2 R_z\| \leq C\varepsilon^{\frac{1}{2}}. \quad (3.24)$$

Consequently, (1.11) follows from (3.24) by using Sobolev embedding theorem. This concludes the proof. □

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