A BRIEF HISTORY OF THE POSITIVITY CONJECTURE IN TENSOR CATEGORY THEORY

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Abstract

We show the existence of a finite group G having an irreducible character χ with Frobenius-Schur indicator $\nu_2(\chi)=+1$ such that χ^2 has an irreducible constituent φ with $\nu_2(\varphi)=-1$. This provides counterexamples to the positivity conjecture in rational CFT and a conjecture of Zhenghan Wang about pivotal fusion categories.

1. Introduction

There are many positivity conjectures in mathematics. The one that concerns us here appeared at the turn of the century in a paper of Peter Bantay ([1], eqn(2)). It states that

$$N_{pq}^r \nu_p \nu_q \nu_r \geq 0. \tag{1}$$

The context is nominally that of rational CFT: p, q, r are simple objects with Frobenius-Schur indicators ν_p, ν_q, ν_r respectively, and N_{pq}^r is the corresponding fusion rule. The conjecture is meaningful in tensor categories equipped with appropriate additional structures, and we shall view it in that context.

Soon after his paper appeared, Bantay brought the positivity conjecture to the author's attention. Framing it in a group-theoretic context, Bantay asked if one can find a finite group G and a pair of irreducible characters α, β of G, each with Frobenius-Schur indicator $+1$, such that $\alpha\beta$ has an

irreducible constituent with indicator -1? I was able to supply such a G, thereby falsifying (1). The construction was never published, but experts became aware of its existence.

In the meantime several variants of the positivity conjecture emerged. Already in 2001 Bantay had communicated a *modified positivity conjecture* to me, which he ascribed to 'string theorists'. This conjecture proposed that (1) holds whenever N_{pq}^r is *odd*. This form of the conjecture is true and, as Richard Ng has pointed out, may be deduced from the final Theorem in a 2002 paper by Jürgen Fuchs, Ingo Runkel and Christoph Schweigert [4], who work in the context of a semisimple, braided, sovereign tensor category.

More recently, Zhenghan Wang conjectured a version in his 2006 book on topological quantum computation ([5], Conjecture 4.26) which may be stated as follows (we are in a pivotal fusion category):

$$N_{p,p^\vee}^r > 0 \Rightarrow \nu_r = 1.$$

It transpires that the same counterexample to the positivity conjecture also serves as a counterexample to Wang's conjecture, and the purpose of this Note is to belatedly make this example public. We will show the following:

There is a pair (G, χ) consisting of a finite group G (of order 2^7) and an irreducible character χ of G such that χ has indicator $\nu_2(\chi) = +1$ and χ^2 has an irreducible constituent φ with indicator $\nu_2(\varphi) = -1$.

We are indebted to Eric Rowell for bringing Wang's conjecture to our attention and Richard Ng for helpful correspondence.

2. Construction

We use standard facts from representation theory, but include some details for the sake of completeness.

Notation: if G is a group and $a, b \in G$ then $[a, b] := a^{-1}b^{-1}ab$.
$\hat{G} := Hom(G, \mathbf{C}^\times)$ is the group of *characters* of G;
If $A \subseteq G$ is a subset of G, the *centralizer* of A is $C_G(A) := \{g \in G | ga = ag \ (a \in A)\}$.
In particular, the *center* of G is $Z(G) := C_G(G)$.
Q_8: quaternion group of order 8.

$GL_n(2)$: group of all $n \times n$ invertible matrices with entries in the field of 2 elements.

S_n: symmetric group of degree n; A_n:alternating group of degree n.

1) Take G to be any 2-group with a normal subgroup $H \trianglelefteq G$ satisfying $G/H \cong Q_8$, $H \cong \mathbf{Z}_2^4$ and $C_G(H) = H$. For example, we may take $G = H \rtimes Q$ where $Q \cong Q_8$ acts faithfully (by conjugation) on H. (The existence of such a G is shown below.)

2) Let $\lambda \in \hat{H}$ be such that for each coset $xH \neq H$ in G (alternatively, in the semi-direct product case, for $1 \neq x \in Q$) $\ker \lambda$ does *not* contain $[H, x]$. (The existence of λ is shown below. $[H, x]$ is defined below.)

Claim 1. The induced character $\chi := \lambda^G := Ind_H^G \lambda$ is *irreducible*.

Proof. One knows ([3], Corollary 10.20) that χ is irreducible, if, and only if, $\lambda \neq {}^x \lambda$ for all $x \in G \backslash H$. Here, ${}^x \lambda$ is defined by

$$^x\lambda(h) := \lambda(x^{-1}hx) \quad (h \in H).$$

Now we have

$$\lambda = {}^x\lambda \Leftrightarrow \lambda(h) = {}^x\lambda(h) \quad (h \in H)$$
$$\Leftrightarrow \lambda(h) = \lambda(x^{-1}hx) \Leftrightarrow \lambda(h^{-1}x^{-1}hx) = 1$$
$$\Leftrightarrow [H, x] \subseteq \ker\lambda,$$

where $[H, x] := \langle [h, x] | h \in H \rangle$.

By 2) this never happens. So indeed χ is irreducible. $\square$

Claim 2. χ^2 has an irreducible constituent φ with $\nu_2(\varphi) = -1$.

Proof. Mackey has made a general study of the decomposition of tensor squares of induced characters such as χ ([3], Theorem 12.17). In particular, it is known (loc. cit.) that there is a constituent equal to $(\lambda^2)^G$. Now λ is a character of H, which is a group of exponent 2. Therefore, $\lambda^2 = 1_H$. It follows that $(\lambda^2)^G = (1_H)^G = \rho_{G/H} = \rho_Q$ is the *regular representation* of G/H. This has every irreducible character of Q_8 as a constituent, and in particular it has a constituent with FS indicator equal to -1, namely the 2-dimensional irreducible φ for Q_8 (lifted to G). $\square$

Claim 3. G exists.

Regarding H as the natural module for $GL_4(2)$, it suffices to show that $Q_8 \subseteq GL_4(2)$. It is then enough to prove that $Q_8 \subseteq A_8$ on account of the exceptional isomorphism $GL_4(2) \cong A_8$. (For a proof of this well-known fact, see e.g., [2], P. 277.) Consider the left regular representation of Q_8. It provides an embedding $Q_8 \subseteq S_8$. In fact, each element of order 4 in Q_8 is represented as a product of a pair of cycles of length 4, and such a permutation is *even*, hence lies in A_8. Since Q_8 is generated by its elements of order 4 we obtain $Q_8 \subseteq A_8$.   □

Claim 4. Let $1 \neq z \in Z(Q)$, and set $H_0 := [H, z]$. Then $|H_0| = 2$.

This is the same as the assertion $|C_H(z)| = 8$. To see this, let $P \subseteq GL_4(2)$ be the stabilizer of some hyperplane of H, call it H_1. Then $P \cong D \rtimes L$ with $D \cong \mathbf{Z}_2^3$, $L \cong GL_3(2)$, moreover D acts trivially on H_1. Notice that P contains a Sylow 2-subgroup of $GL_4(2)$, so that by Sylow's theorem we may assume that $Q \subseteq P$. If $z \in D$ then our assertion follows, whereas if $z \notin D$ then $Q \cap D = 1$, so that Q is isomorphic to a Sylow 2-subgroup of $P/D \cong L$. But this is impossible because a Sylow 2-subgroup of $GL_3(2)$ is dihedral of order 8.   □

Claim 5. λ exists.

We assert that

$$H_0 = \bigcap_{1 \neq x \in Q} [H, x]. \tag{2}$$

It suffices to show that $H_0 \subseteq [H, x]$ whenever $x \in Q$ has order 4. To see this, suppose that x has order 4. Then $x^2 = z$. We have

$$
\begin{aligned}
[h, x^2] &= h^{-1}x^{-2}hx^2 = (h^{-1}x^{-1}hx)(x^{-1}h^{-1}x^{-1}hx^2) \\
&= [h, x][h, x]^x = [[h, x], x] \in [H, x],
\end{aligned}
$$

as needed.

To get the existence of λ, choose any hyperplane H_2 of H that does *not* contain H_0. H_2 exists thanks to Claim 4. Choose $\lambda \in \hat{H}$ so that $H_2 := \ker\lambda$. Since $H_0 \subseteq [H, x]$ for all $x \in G \backslash H$ by (2) then $[H, x] \not\subseteq \ker\lambda$.   □

Claim 6. $\nu_2(\chi) = +1$.

We have $\nu_2(\chi) = |G|^{-1} \sum_{g \in G} \chi(g^2)$. Because χ is induced from the normal subgroup H then χ *vanishes* on $G \backslash H$. Moreover, $g^2 \in H$ if, and only if,

$g \in H \langle z \rangle$. If $g \in H$ then $\chi(g^2) = \chi(1) = 8$. If $g \in Hz$, say $g = hz$, then $g^2 = [h, z] \in H_0$. Moreover, there are just 8 choices of h with $[h, z] = 1$, and for these we again have $\chi(g^2) = 8$. As for the 8 choices satisfying $g^2 = [h, z] \neq 1$, by construction we have $\lambda(g^2) = -1$, moreover $g^2 \in Z(G)$. Therefore

$$\chi(g^2) = |H|^{-1} \sum_{t \in G} \lambda(t^{-1} g^2 t) = |H|^{-1} |G| \lambda(g^2) = -8.$$

The Claim follows from these calculations.   □

References

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