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IMPLICIT FRACTIONAL DIFFERENTIAL EQUATION INVOLVING ψ–CAPUTO WITH BOUNDARY CONDITIONS

BOUTIARA ABDELLATIF[1,a] AND MAAMAR BENBACHIR[2,b]

[1]Laboratory of Mathematics And Applied Sciences, University of Ghardaia, 47000. Algeria.
[a]E-mail: boutiara_a@yahoo.com
[2]Faculty of Sciences, Saad Dahlab University, Blida, Algeria.
[b]E-mail: mbenbachir2001@gmail.com

Abstract

In this research article, we discuss the existence results and uniqueness of solutions for a class of boundary value problems of fractional differential equations with the ψ–Caputo fractional derivative. The reasoning is mainly based upon different types of classical fixed point theory such as the Banach contraction principle and Krasnoselskii's fixed point theorem. Besides, the Ulam-Hyers result is addressed for the proposed problem. We illustrate our main findings, with a particular case example included to show the applicability of our outcomes.

1. Introduction

Fractional calculus generalizes the integer-order integration and differentiation concepts to an arbitrary(real or complex) order. Fractional calculus is one of the most emerging areas of investigation. The fractional differential operators are used to model many biological and physical phenomena, mathematical modeling of engineering, etc. in a much better form as compared to ordinary differential operators, which are local. To get a couple of developments about the theory of fractional differential equations, one can allude to the monographs of Hilfer [17], Kilbas et al [22], Miller and Ross [23], Oldham [25], Pudlubny [26], Sabatier et al [28], Tarasov [31] and the references therein.

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At the present day, there are numerous results on the existence and uniqueness of solutions for fractional differential equations. For greater details, the readers are cited the previous research [22, 10, 15, 12, 4, 8] and the references therein. however, due to the fact that in lots of conditions, which include nonlinear analysis and optimization, locating the exact solution of differential equations is almost tough or impossible, we don't forget approximate solutions. it is essential to observe that only stable approximate solutions are proper. various approaches of stability analysis are adopted for this reason. The HU-type stability concept has been taken into consideration in the severa literature. The said stability analysis is an clean and easy manner on this regard. This type idea of stability become formulated for the primary time by means of Ulam [32], and then the next year it become elaborated with the aid of Hyers [18].

Inside the starting, this concept became implemented to ordinary differential equations after which extended to fractional diferential equation FDEs. We refer the readers to [19, 20, 21, 24, 9, 27].

In this paper deals with the existence and uniqueness of solutions for boundary-value problem of the nonlinear ψ–Caputo fractional differential equations

$$\begin{cases} {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}u(t) = f(t, u(t), {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}u(t)), \quad t \in J := [a, T], \\ \\ u(T) = \lambda u(\eta). \end{cases} \tag{1}$$

where ${}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $\alpha \in (0, 1]$, $f\colon [a, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function. λ is real constant and $\eta \in (a, T)$.

Here is a brief outline of the paper. The Section 2 provides the definitions and preliminary results that we will need to prove our main results and present an auxiliary lemma that provides solution representation for the solutions of Problem (1). In Section 3, we establish existence and uniqueness for fractional differential equations involving ψ–Caputo fractional differential operator. In Section 4, we discuss some types of fractional Ulam stability. In Section 5, we give an example to illustrate the obtained results.

2. Preliminaries and Lemmas

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later.

Let $C(J,\mathbb{R})$ the space of real and continuous functions with the norm

$$\|u\|_{\infty} = \sup\left\{\|u(t)\| : \quad t \in J\right\}.$$

Let $L^1(J,\mathbb{R})$ be the Banach space of Lebesgue integrable functions $u : J \to \mathbb{R}$, equipped with the norm

$$\|u\|_{L^1} = \int_J |u(t)|\, dt.$$

We begin by defining ψ-Riemann-Liouville fractional integrals and derivatives. In what follows,

Definition 1 ([4]). For $\alpha > 0$, the left-sided ψ–Riemann-Liouville fractional integral of order α for an integrable function u: J $\longrightarrow \mathbb{R}$ with respect to another function ψ: J $\longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in$ J is defined as follows

$$\mathcal{I}_{a^+}^{\alpha;\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s)\mathrm{ds}, \tag{2}$$

where Γ is the classical Euler Gamma function.

Definition 2 ([4]). Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(\mathrm{J},\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in$ J. The left-sided ψ–Riemann–Liouville fractional derivative of a function u of order α is defined by

$$\begin{aligned} \mathcal{D}_{a^+}^{\alpha;\psi} u(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_{a^+}^{n-\alpha;\psi} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} u(s)\mathrm{ds}, \end{aligned}$$

where $n = [\alpha] + 1$.

Definition 3 ([4]). Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(\mathrm{J},\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in$ J. The left-sided ψ-Caputo fractional derivative of u of order α is defined by

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}u(t) \;=\; \mathcal{I}_{a^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t), $$

where $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$, $n=\alpha$ for $\alpha\in\mathbb{N}$.
To simplify notation, we will use the abbreviated symbol

$$ u_{\psi}^{[n]}(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t). $$

From the definition, it is clear that

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}u(t)=\begin{cases}\displaystyle\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}u_{\psi}^{[n]}(s)\mathrm{ds} & ,\quad \text{if } \alpha\notin\mathbb{N},\\ u_{\psi}^{[n]}(t) & ,\quad \text{if } \alpha\in\mathbb{N}.\end{cases} \tag{3} $$

We note that if $u\in C^n(\mathrm{J},\mathbb{R})$ the ψ–Caputo fractional derivative of order α of u is determined as

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}u(t)=\mathcal{D}_{a^+}^{\alpha;\psi}\left[u(t)-\sum_{k=0}^{n-1}\frac{u_{\psi}^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k\right]. $$

(see, for instance, [4, Theorem 3]).

Lemma 1 ([6])**.** *Let $\alpha,\beta>0$, and $u\in L^1(\mathrm{J},\mathbb{R})$. Then*

$$ \mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\beta;\psi}u(t)=\mathcal{I}_{a^+}^{\alpha+\beta;\psi}u(t),\ \textit{a.e. } t\in\mathrm{J}. $$

In particular, if $u\in\mathcal{C}(\mathrm{J},\mathbb{R})$, then $\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\beta;\psi}u(t)=\mathcal{I}_{a^+}^{\alpha+\beta;\psi}u(t)$, $t\in\mathrm{J}$.

Lemma 2 ([6])**.** *Let $\alpha>0$, The following holds:*

If $u\in\mathcal{C}(\mathrm{J},\mathbb{R})$ then

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\alpha;\psi}u(t)=u(t),\ t\in\mathrm{J}. $$

If $u\in C^n(\mathrm{J},\mathbb{R})$, $n-1<\alpha<n$. Then

$$ \mathcal{I}_{a^+}^{\alpha;\psi}\ {}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u_{\psi}^{[k]}(a)}{k!}\left[\psi(t)-\psi(a)\right]^k,\quad t\in\mathrm{J}. $$

Lemma 3 ([6]). *Let $t > a$, $\alpha \geq 0$, and $\beta > 0$. Then*

- $\mathcal{I}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1}$,
- ${}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha-1}$,
- ${}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{k} = 0$, *for all* $k \in \{0,\ldots,n-1\}, n \in \mathbb{N}$.

3. Main Results

Before starting and proving our main result we introduce the following auxiliary lemma.

Lemma 4. *Let $0 < \alpha < 1, \rho > 0$ and $w \in \mathcal{C}(J,\mathbb{R})$. Then the linear anti-periodic boundary value problem*

$$\begin{aligned} {}^{C}\mathcal{D}^{\alpha,\psi}u(t) &= \sigma(t), \quad t \in J, \\ u(T) &= \lambda u(\eta), \end{aligned} \tag{4}$$

has a unique solution defined by

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\sigma(t)ds \\ &+ \frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^\eta \psi'(s)\left(\psi(\eta)-\psi(s)\right)^{\alpha-1}\sigma(t)ds\right. \\ &\left.+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}\sigma(t)ds\right\}. \end{aligned} \tag{5}$$

Proof. Assume u satisfies (4). Then Lemma 2 implies that

$$u(t) = \mathcal{I}^{\alpha;\psi}\sigma(t) + c_1. \tag{6}$$

The condition (4) implies that

$$\begin{aligned} u(T) =& \mathcal{I}^{\alpha;\psi}\sigma(T) + c_1 \\ u(\eta) =& \mathcal{I}^{\alpha;\psi}\sigma(\eta) + c_1. \end{aligned}$$

Thus,

$$c_1\left(1-\lambda\right) = \lambda\mathcal{I}^{\alpha;\psi}\sigma(\eta)) - \mathcal{I}^{\alpha;\psi}\sigma(T)$$

Consequently,

$$c_1 = \frac{1}{\Lambda}\left\{\lambda \mathcal{I}^{\alpha;\psi}\sigma(\eta)) - \mathcal{I}^{\alpha;\psi}\sigma(T)\right\}.$$

Where,

$$\Lambda = (1-\lambda).$$

Finally, we obtain the solution in the equation (5). □

Lemma 5. *Assume that $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. A function $u(t)$ solves the problem* (1) *if and only if it is a fixed-point of the operator $\mathcal{G} : \mathcal{C}(J,\mathbb{R}) \to \mathcal{C}(J,\mathbb{R})$ defined by*

$$\begin{aligned}\mathcal{G}u(t) = &\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi}u(s)\right) ds \\ &+ \frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^\eta \psi'(s)\left(\psi(\eta)-\psi(s)\right)^{\alpha-1} f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi}u(s)\right) ds\right. \\ &\left. + \frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1} f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi}u(s)\right) ds\right\}. \quad (7)\end{aligned}$$

In the following subsections, we establish the existence and uniqueness of solutions for the boundary value problem (1) by applying a variety of fixed point theorems.

3.1. Uniqueness result via banach fixed point theorem

Theorem 1. *Assume that $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition:*
(H1) *There exists $L_1 > 0$ and $0 < L_2 < 1$ such that:*

$$\left|f(t, u_1, u_2) - f(t, v_1, v_2)\right| \leq L_1 |u_1 - v_1| + L_2 |u_2 - v_2|,$$

for $t \in J$ and every $u_i, v_i \in \mathbb{R}$, $(i = 1, 2)$ If

$$\Lambda_1 = \frac{L_1}{1-L_2}\left\{\frac{(\psi(T)-\psi(s))^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Lambda}\left\{\frac{\lambda(\psi(\eta)-\psi(s))^\alpha}{\Gamma(\alpha+1)} + \frac{(\psi(T)-\psi(s))^\alpha}{\Gamma(\alpha+1)}\right\}\right\} < 1$$

then the boundary value problem (1) *has a unique solution on J.*

Proof. In the first step, we prove that $\mathcal{G}\mathcal{B}_r \subseteq \mathcal{B}_r$ where the operator $\mathcal{G} : \mathcal{C}(J,\mathbb{R}) \to \mathcal{C}(J,\mathbb{R})$ defined by (7) and

$$\mathcal{B}_r = \{u \in \mathcal{C}(J,\mathbb{R}), \|u\| \leq r\}$$

with choose $r \geq \frac{\Lambda_2}{1-\Lambda_1}$, where $\Lambda_1 < 1$ and

$$\Lambda_2 = \frac{\mu}{1-L_2}\left\{\frac{(\psi(T)-\psi(s))^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Lambda}\left\{\frac{\lambda\,(\psi(\eta)-\psi(s))^\alpha}{\Gamma(\alpha+1)} + \frac{(\psi(T)-\psi(s))^\alpha}{\Gamma(\alpha+1)}\right\}\right\}$$

and $\sup_{l\in J}|f(t,0,0)| := \mu < \infty$. Set $\mathcal{K}_u(t) := f\left(t, u(t), {}^C\mathcal{D}_{a^+}^{\alpha\rho} u(t)\right)$. For any $u \in \mathcal{S}_r$, we have

$$\begin{aligned}
|\mathcal{G}u(t)| \leq& \sup_{t\in J} |\mathcal{G}u(t)| \\
\leq& \sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1}\,|\mathcal{K}_u(s)|\,ds\right. \\
&+ \frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^\eta \psi'(s)\,(\psi(\eta)-\psi(s))^{\alpha-1}\,|\mathcal{K}_u(s)|\,ds\right. \\
&\left.\left.+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\,|\mathcal{K}_u(s)|\,ds\right\}\right\}.
\end{aligned}$$

From (H1), we get

$$\begin{aligned}
|\mathcal{K}_u(s)| &= \left|f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi} u(s)\right)\right| \\
&\leq \left|f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi} u(\tau)\right) - f(s,0,0)\right| + |f(\tau,0,0)| \\
&\leq L_1\,|u(s)| + L_2\left|{}^C\mathcal{D}_{a^+}^{\alpha,\psi} u(s)\right| + \mu \\
&= L_1 r + L_2\,|\mathcal{K}_u(s)| + \mu
\end{aligned}$$

which gives

$$|\mathcal{K}_u(s)| \leq \frac{(L_1 r + \mu)}{1-L_2} \tag{8}$$

Therefore,

$$\begin{aligned}
|\mathcal{G}u(t)| \leq& \sup_{t\in J}\left\{\frac{(L_1 r+\mu)}{1-L_2}\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1}\,ds\right. \\
&+ \frac{(L_1 r+\mu)}{1-L_2}\frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^\eta \psi'(s)\,(\psi(\eta)-\psi(s))^{\alpha-1}\,ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(L_1r+\mu)}{1-L_2}\frac{1}{\Gamma(\alpha)}\int_a^T\psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}ds\Bigg\}\Bigg\}\\
\leq&\frac{(L_1r+\mu)}{1-L_2}\left\{\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\Lambda}\left\{\frac{\lambda\left(\psi(\eta)-\psi(s)\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}\\
=&\frac{L_1r}{1-L_2}\left\{\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\Lambda}\left\{\frac{\lambda\left(\psi(\eta)-\psi(s)\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}\\
&+\frac{\mu}{1-L_2}\left\{\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\Lambda}\left\{\frac{\lambda\left(\psi(\eta)-\psi(s)\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}\\
=&\Lambda_1r+\Lambda_2<r
\end{aligned}
$$

$$
\|\mathcal{G}u\|<r \tag{9}
$$

which implies that $\mathcal{G}u\in\mathcal{B}_r$. Moreover, by (7), and Lammas 2, 3 we obtain

$$
{}^C\mathcal{D}_{a^+}^{\alpha\cdot\rho}\mathcal{G}u(t)={}^C\mathcal{D}_{a^+}^{\alpha;\rho}\mathcal{I}_{a^+}^{\alpha\rho}\mathcal{K}_u(t)=\mathcal{K}_u(t)
$$

since $\mathcal{K}_u(\cdot)$ is continuous on J, the operator ${}^C\mathcal{D}_{a^+}^{\alpha\rho}\mathcal{G}u(t)$ is continuous on J, that is $\mathcal{G}\mathcal{B}_r\subseteq\mathcal{B}_r$ Next, we apply the Banach fixed point theorem to prove that $\mathcal{G}$ has a fixed point. Indeed, it enough to show that $\mathcal{G}$ is contraction map. Let $u_1,u_2\in\mathcal{C}(J,\mathbb{R})$ and for $t\in J$. Then, we have

$$
\begin{aligned}
|\mathcal{G}u_1(t)-\mathcal{G}u_2(t)|\leq&\left\{\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\mathcal{K}_{u_1}(s)-\mathcal{K}_{u_2}(s)|\,ds\right.\\
&+\frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^\eta\psi'(s)\left(\psi(\eta)-\psi(s)\right)^{\alpha-1}|\mathcal{K}_{u_1}(s)-\mathcal{K}_{u_2}(s)|\,ds\right.\\
&\left.\left.+\frac{1}{\Gamma(\alpha)}\int_a^T\psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}|\mathcal{K}_{u_1}(s)-\mathcal{K}_{u_2}(s)|\,ds\right\}\right\}
\end{aligned}
$$

by (H_1), we get

$$
\begin{aligned}
|\mathcal{K}_{u_1}(s)-\mathcal{K}_{u_2}(s)|&=\left|f\left(s,u_1(s),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}u_1(s)\right)-f\left(s,u_1(s),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}u_2(s)\right)\right|\\
&\leq L_1|u_1-u_2|+L_2|{}^C\mathcal{D}_{a^+}^{\alpha;\psi}u_1(s)-{}^C\mathcal{D}_{a^+}^{\alpha;\psi}u_2(s)\,|\\
&=L_1|u_1-u_2|+L_2|\mathcal{K}_{u_1}(s)-\mathcal{K}_{u_2}(s)|
\end{aligned}
$$

which implies

$$
|\mathcal{K}_{u_1}(s)-\mathcal{K}_{u_2}(s)|\leq\frac{L_1}{1-L_2}|u_1-u_2|. \tag{10}
$$

Then

$$\begin{aligned}\|\mathcal{G}u_1-\mathcal{G}u_2\| \leq & \frac{L_1}{1-L_2}\left\{\frac{(\psi(T)-\psi(s))^\alpha}{\Gamma(\alpha+1)}\right.\\ &\left.+\frac{1}{\Lambda}\left\{\frac{\lambda\,(\psi(\eta)-\psi(s))^\alpha}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(s))^\alpha}{\Gamma(\alpha+1)}\right\}\right\}\|u_1-u_2\|.\end{aligned}$$

Consequently, $\|\mathcal{G}u_1-\mathcal{G}u_2\| \leq \Lambda_1\|u_1-u_2\|$. Since $\Lambda_1<1$, the operator $\mathcal{G}$ is contraction mapping. Hence, we deduce by Banach contraction mapping principle that the operator $\mathcal{G}$ has a unique fixed point, which corresponds to a unique solution of the problem in Equation (1) on J. The proof is completed. □

3.2. Existence result via Kransnoselskii's fixed point theorem

In the next existence result, we apply Krasnoselskii fixed point theorem [28].

Theorem 2. *Assume that* (H_1) *holds. If*

$$\Delta := \frac{L_1}{1-L_2}\left\{\frac{(\psi(T)-\psi(s))^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Lambda}\left\{\frac{\lambda\,(\psi(\eta)-\psi(s))^\alpha}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(s))^\alpha}{\Gamma(\alpha+1)}\right\}\right\} < 1$$

then the problem (1) *has at least one solution on J.*

Proof. Consider the operator $\mathcal{G}:\mathcal{C}(J,\mathbb{R})\to\mathcal{C}(J,\mathbb{R})$ defined by (7). Define the ball

$$\mathcal{B}_{r_0} := \{u\in\mathcal{C}(J,\mathbb{R}) : \|u\|\leq r_0\}.$$

Now we subdivide the operator $\mathcal{G}$ into two operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on $\mathcal{B}_{r_0}$ defined by

$$\begin{aligned}\mathcal{G}_1u(t) =, & \frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^\eta \psi'(s)\,(\psi(\eta)-\psi(s))^{\alpha-1}\,\mathcal{K}_u(s)ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\,\mathcal{K}_u(s)ds\right\}\end{aligned}$$

and

$$\mathcal{G}_2u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} \mathcal{K}_u(s)ds.$$

Taking into account that $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined on $\mathcal{B}_{r_0}$, and for any $u \in \mathcal{C}(J, \mathbb{R})$,

$$\mathcal{G}u(t) = \mathcal{G}_1u(t) + \mathcal{G}_2u(t), \quad t \in J.$$

The proof will be divided into several steps:

Step 1: $\mathcal{G}_1u_1 + \mathcal{G}_2u_2 \in \mathcal{B}_{r_0}$ for every $u_1, u_2 \in \mathcal{B}_{r_0}$. For $u_1 \in \mathcal{B}_{r_0}$ and using the same arguments in (8), we get

$$|\mathcal{K}_{u_1}(\tau)| \leq \frac{(L_1r_0 + \mu)}{1 - L_2}$$

Similarly, for $u_2 \in \mathcal{B}_{r_0}$, we obain

$$|\mathcal{K}_{u_2}(\tau)| \leq \frac{(L_1r_0 + \mu)}{1 - L_2}$$

Now, for $u_1, u_2 \in \mathcal{B}_{r_0}$ and $t \in J$, we have

$$\begin{aligned}
|\mathcal{G}_1u_1(t) + \mathcal{G}_2u_2(t)| \leq& |\mathcal{G}_1u_1(t)| + |\mathcal{G}_2u_2(t)| \\
\leq& \left\{ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} |\mathcal{K}_{u_2}(s)| \, ds \right. \\
&+ \frac{1}{\Lambda} \left\{ \frac{\lambda}{\Gamma(\alpha)} \int_a^\eta \psi'(s) \left(\psi(\eta) - \psi(s)\right)^{\alpha-1} |\mathcal{K}_{u_1}(s)| \, ds \right. \\
&\left.\left. + \frac{1}{\Gamma(\alpha)} \int_a^T \psi'(s) \left(\psi(T) - \psi(s)\right)^{\alpha-1} |\mathcal{K}_{u_1}(s)| \, ds \right\} \right\} \\
=& \Lambda_1 r_0 + \Lambda_2 < r_0
\end{aligned}$$

which gives

$$\|\mathcal{G}_1u_1 + \mathcal{G}_2u_2\| \leq r_0.$$

This proves that $\mathcal{G}_1u_1 + \mathcal{G}_2u_2 \in \mathcal{B}_{r_0}$ for every $u_1, u_2 \in \mathcal{B}_{r_0}$.

Step 2: $\mathcal{G}_1$ is a contration mapping on $\mathcal{B}_n$ since $\mathcal{G}$ is contraction mapping as in Theorem 1, then $\mathcal{G}_1$ is a contraction map too.

Step 3: The operator $\mathcal{G}_2$ is completely continuous on $\mathcal{B}_{r_0}$ First, from the continuity of $\mathcal{K}_u(\cdot)$, we conclude that the operator $\mathcal{G}_2$ is continuous. Next, It is easy to verify that

$$\|\mathcal{G}_2 u\| \leq \frac{L_1 r_0 + \mu}{1 - L_2} \frac{(\psi(t) - \psi(s))^{\alpha}}{\Gamma(\alpha+1)} < r_0$$

due to definitions of Λ and r_0. This proves that $\mathcal{G}_2$ is uniformly bounded on $\mathcal{B}_{r_0}$ Finally, we prove that $\mathcal{G}_2$ maps bounded sets into equicontinuous sets of $\mathcal{C}(J, \mathbb{R})$, i.e., $(\mathcal{G}\mathcal{B}_{r_0})$ is equicontinuous. We estimate the derivative of $\mathcal{G}_2 u(t)$

$$\begin{aligned}
\left|(\mathcal{G}_2 u)'(t)\right| &= \left|\frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-2}\,\mathcal{K}_u(s)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-2}\,|\mathcal{K}_u(s)|\,ds \\
&\leq \frac{L_1 r_0 + \mu}{1 - L_2} \frac{(\psi(T) - \psi(s))^{\alpha-1}}{\Gamma(\alpha+1)} := K.
\end{aligned}$$

Now, Let $t_1, t_2 \in J$, with $t_1 < t_2$ and for any $u \in \mathcal{B}_n$.
Thus, we get

$$|\mathcal{G}_2 u\,(t_1) - \mathcal{G}_2 u\,(t_2)| = \int_{t_1}^{t_2} \left|(\mathcal{G}_2 u)'(s)\right| ds \leq K\,(t_2 - t_1)\,.$$

From the last estimate, we deduce that

$$|\mathcal{G}_2 u\,(t_1) - \mathcal{G}_2 u\,(t_2)| \to 0 \text{ when } \quad t_2 \to t_1, u \in \mathcal{B}_{r_0}.$$

This proves that $\mathcal{G}_2$ is equicontinuous on $\mathcal{B}_{r_0}$. In view of the foregoing arguments, the Arzelá-Ascoli theorem applies and hence $\mathcal{G}_2$ is compact on $\mathcal{B}_{r_0}$. Thus, the hypothesis of Krasnoselskii fixed point theorem is fulfilled, which leads to the conclusion that there exists at least one solution on J. □

4. Ulam-Hyers Stability

In the recent section, we interested to studied UH and GUH of ψ–Caputo-type for the problem (1) The following observations are taken from [24, 27, 9]

Definition 4. The problem (1) is UH stable, if there exists a real number $L_f > 0$, such that for each $\varepsilon > 0$ and for every solution $\tilde{u} \in \mathcal{C}(J, \mathbb{R})$ of the inequality

$$\left|{}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(t) - f\left(t, \tilde{u}(t), {}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(t)\right)\right| \leq \varepsilon, \quad t \in J \tag{11}$$

there exists a unique solution $u \in \mathcal{C}(J, \mathbb{R})$ of (1) with

$$|\tilde{u}(t) - u(t)| \leq L_f \varepsilon, \quad t \in J.$$

Definition 5. The problem (1) is GUH stable if there exists $\varphi \in \mathcal{C}([0, \infty), [0, \infty))$ with $\varphi(0) = 0$, such that for each solution $\bar{u} \in \mathcal{C}(J, \mathbb{R})$ of the inequality

$$\left|{}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(t) - f\left(t, \tilde{u}(t), {}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(t)\right)\right| \leq \varepsilon, \quad t \in J \tag{12}$$

there exists a unique solution $u \in \mathcal{C}(J, \mathbb{R})$ of (1) with

$$|\tilde{u}(t) - u(t)| \leq \varphi(\varepsilon), t \in J.$$

Remark 1. Let $\alpha > 0$. A function $\tilde{u} \in \mathcal{C}(J, \mathbb{R})$ is a solution of the inequality (11) defined by

$$\left|{}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(t) - f\left(t, \tilde{u}(t), {}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(t)\right)\right| \leq \varepsilon, \quad t \in J.$$

If and only if there exist a function $h_{\tilde{u}} \in \mathcal{C}(J, \mathbb{R})$ such that

(1) $|h_{\tilde{u}}(t)| \leq \varepsilon$ for all $t \in J$.
(2) $\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(t) = f\left(t, \tilde{u}(t), {}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(t)\right) + h_{\tilde{u}}(t), t \in J.$

Lemma 6. *Let $\tilde{u} \in \mathcal{C}(J, \mathbb{R})$ is a solution of the inequality* (11)*. Then $\tilde{u}$ is a solution of the following integral inequality:*

$$\left|\tilde{u}(t) - w_{\tilde{u}} - \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t) - \psi(s)\right)^{\alpha-1} f\left(s, \tilde{u}(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi}\bar{u}(s)\right) ds\right|$$

$$\leq\left\{\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\Lambda}\left\{\frac{\lambda\left(\psi(\eta)-\psi(s)\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}\varepsilon$$

where

$$\begin{aligned} w_{\tilde{u}} = &\left\{\frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^{\eta}\psi'(s)\left(\psi(\eta)-\psi(s)\right)^{\alpha-1}\mathcal{K}_u(s)ds\right.\right.\\ &\left.\left.+\frac{1}{\Gamma(\alpha)}\int_a^{T}\psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}\mathcal{K}_u(s)ds\right\}\right\}. \end{aligned} \tag{13}$$

Proof. In view of Remark 1 and Theorem 1, we obtain

$$\begin{aligned} \widetilde{u}(t) =& \frac{1}{\Gamma(\alpha)}\int_a^{t}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\left[f\left(s,\widetilde{u}(s),{}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}\widetilde{u}(s)\right)+h_{\widetilde{u}}(s)\right]ds\\ &+\left\{\frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^{\eta}\psi'(s)\left(\psi(\eta)-\psi(s)\right)^{\alpha-1}\left[f\left(s,\widetilde{u}(s),{}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}\widetilde{u}(s)\right)+h_{\widetilde{u}}(s)\right]ds\right.\right.\\ &\left.\left.-\frac{1}{\Gamma(\alpha)}\int_a^{T}\psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}\left[f\left(s,\widetilde{u}(s),{}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}\widetilde{u}(s)\right)+h_{\widetilde{u}}(s)\right]ds\right\}\right\}. \end{aligned} \tag{14}$$

It follows that

$$\begin{aligned} &\left|\tilde{u}(t)-w_{\tilde{u}}-\frac{1}{\Gamma(\alpha)}\int_a^{t}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}f\left(s,\tilde{u}(s),{}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(s)\right)ds\right|\\ &\leq\frac{1}{\Gamma(\alpha)}\int_a^{t}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\left|h_{\widetilde{u}}(s)\right|ds\\ &\quad+\left\{\frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^{\eta}\psi'(s)\left(\psi(\eta)-\psi(s)\right)^{\alpha-1}\left|h_{\widetilde{u}}(s)\right|ds\right.\right.\\ &\quad\left.\left.+\frac{1}{\Gamma(\alpha)}\int_a^{T}\psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}\left|h_{\widetilde{u}}(s)\right|ds\right\}\right\}\\ &\leq\frac{\varepsilon}{\Gamma(\alpha)}\int_a^{t}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}ds\\ &\quad+\left\{\frac{\varepsilon}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^{\eta}\psi'(s)\left(\psi(\eta)-\psi(s)\right)^{\alpha-1}ds\right.\right.\\ &\quad\left.\left.+\frac{1}{\Gamma(\alpha)}\int_a^{T}\psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}ds\right\}\right\} \end{aligned}$$

$$\leq\left\{\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\Lambda}\left\{\frac{\lambda\left(\psi(\eta)-\psi(s)\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(s))^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}\varepsilon. \tag{15}$$

□

Theorem 3. *If hypotheses of Theorem* 1 *are fulfilled. Then the problem* (1) *is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$, and $\tilde{u} \in \mathcal{C}(J, \mathbb{R})$ be a function which satisfies the inequality (11), and let $u \in \mathcal{C}(J, \mathbb{R})$ be the unique solution of the following ψ–Caputo fractional differential equation

$$ {}^C\mathcal{D}_{a^+}^{\alpha;\rho} u(t) = f\left(t, u(t), {}^C\mathcal{D}_{a^+}^{\alpha\rho} u(t)\right), \quad t \in J \tag{16} $$

with

$$ u(\eta) = \tilde{u}(\eta), u(T) = \tilde{u}(T) \tag{17} $$

where $0 < \alpha < 1$. Using Lemma 4, It is easily seen that $u(\cdot)$ satisfies the integral equation

$$ u(t) = w_u + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi} u(s)\right) ds $$

where

$$ \begin{aligned} w_u = \Bigg\{ \frac{1}{\Lambda} \Bigg\{ \frac{\lambda}{\Gamma(\alpha)} & \int_a^\eta \psi'(s) \left(\psi(\eta) - \psi(s)\right)^{\alpha-1} f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi} u(s)\right) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_a^T \psi'(s) \left(\psi(T) - \psi(s)\right)^{\alpha-1} f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi} u(s)\right) ds \Bigg\}\Bigg\}. \end{aligned} $$

Applying Lemma 6, we obtain

$$ \left| \tilde{u}(t) - w_{\tilde{u}} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} f\left(s, \tilde{u}(s), {}^C\mathcal{D}_{a^+}^{\alpha;\psi} \tilde{u}(s)\right) ds \right| \leq V\varepsilon \tag{18} $$

where

$$ V := \left\{ \frac{(\psi(T) - \psi(s))^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Lambda} \left\{ \frac{\lambda (\psi(\eta) - \psi(s))^\alpha}{\Gamma(\alpha+1)} + \frac{(\psi(T) - \psi(s))^\alpha}{\Gamma(\alpha+1)} \right\}\right\}. $$

From (17) we can easily get that $|w_{\tilde{u}} - w_u| \to 0$. Indeed, from (H1)and (17), we obtain that

$$ |w_{\tilde{u}} - w_u| = \left| \left\{ \frac{1}{\Lambda} \left\{ \frac{\lambda}{\Gamma(\alpha)} \int_a^\eta \psi'(s) \left(\psi(\eta) - \psi(s)\right)^{\alpha-1} f\left(s, \tilde{u}(\tau), {}^C\mathcal{D}_{a^+}^{\alpha;\psi} \tilde{u}(s)\right) ds \right.\right.\right. $$

$$-\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1} f\left(s,\tilde{u}(\tau),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(s)\right)ds\Big\}\Big\}$$
$$-\left\{\frac{1}{\Lambda}\left\{\frac{\lambda}{\Gamma(\alpha)}\int_a^\eta \psi'(s)\left(\psi(\eta)-\psi(s)\right)^{\alpha-1} f\left(s,u(s),{}^C\mathcal{D}_{a+}^{\alpha}u(s)\right)ds\right.\right.$$
$$\left.\left.+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1} f\left(s,u(s),{}^C\mathcal{D}_{a+}^{\alpha}u(s)\right)ds\right\}\right\}\Big|$$
$$\leq\frac{1}{\Lambda}\left\{\lambda\mathcal{I}_{a^+}^{\alpha;\psi}\left|f\left(\eta,\widetilde{u}(\eta),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}\widetilde{u}(\eta)\right)-f\left(\eta,u(\eta),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}u(\eta)\right)\right|\right.$$
$$\left.+\mathcal{I}_{a^+}^{\alpha;\psi}\left|f\left(T,\widetilde{u}(T),{}^C\mathcal{D}_{a^+}^{\alpha,\rho}\widetilde{u}(T)\right)-f\left(T,u(T),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}u(T)\right)\right|\right\}$$

since,

$$\begin{aligned}&\left|f\left(T,\widetilde{u}(T),{}^C\mathcal{D}_{a^+}^{\alpha\rho}\widetilde{u}(T)\right)-f\left(T,u(T),{}^C\mathcal{D}_{a^+}^{\alpha\rho}u(T)\right)\right|\\&\leq L_1|u(T)-u(T)|+L_2|{}^C\mathcal{D}_{a^+}^{\alpha,\rho}\tilde{u}(T)-{}^C\mathcal{D}_{a^+}^{\alpha\rho}u(T)\\&\leq\frac{L_1}{1-L_2}|\widetilde{u}(T)-u(T)|.\end{aligned}$$

Similarly, we obtain

$$\left|f\left(\eta,\widetilde{u}(\eta),{}^C\mathcal{D}_{a^+}^{\alpha\rho}\widetilde{u}(\eta)\right)-f\left(\eta,u(\eta),{}^C\mathcal{D}_{a^+}^{\alpha\rho}u(T)\right)\right|\leq\frac{L_1}{1-L_2}|\widetilde{u}(\eta)-u(\eta)| \quad (19)$$

which implies

$$|w_{\tilde{u}}-w_u|\leq\frac{L_1}{(1-L_2)}\frac{1}{\Lambda}\left\{\lambda\mathcal{I}_{a^+}^{\alpha\rho}|\tilde{u}(\eta)-u(\eta)|+\mathcal{I}_{a^+}^{\alpha\rho}|\tilde{u}(\eta)-u(\eta)|\right\}\to 0.$$

Hence,

$$u(t)=w_u+\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} f\left(s,u(s),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}u(s)\right)ds.$$

According to (18), (H1) and (19), we obtain

$$\begin{aligned}&|\tilde{u}(t)-u(t)|\\&\leq\left|\tilde{u}(t)-w_{\tilde{u}}-\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} f\left(s,\tilde{u}(s),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(s)\right)ds\right|\\&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\Big|f\left(s,\tilde{u}(s),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}\tilde{u}(s)\right)\\&\quad-f\left(s,u(s),{}^C\mathcal{D}_{a^+}^{\alpha;\psi}u(s)\right)\Big|\,ds\end{aligned}$$

$$\leq V\varepsilon + \frac{L_1}{1-L_2}\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\widetilde{u}(s)-u(s)|ds.$$

By Lemma 2, there exists a constant $L_f > 0$ independent of ε such that

$$|\tilde{u}(t)-u(t)| \leq L_f\varepsilon \tag{20}$$

Therefore the problem (1) is Ulam-Hyers stable. □

Corollary 1. *Under assumptions of Theorem 3, Assume that $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$. Then tne problem (4) is generalized Ulam-Hyers stable.*

Proof. One can repeat the same processes in Theorem 3 with putting $L_f\varepsilon = \varphi(\varepsilon)$, and $\varphi(0) = 0$, we conclude that

$$|\tilde{u}(t)-u(t)| \leq \varphi(\varepsilon).$$ □

5. Example

This section is devoted to the illustration of the results derived in the last section.

Example 1. Consider the following problem of implicit fractional differential equations involving ψ–Caputo type:

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{4};t}u(t) = \left[\dfrac{1}{3}e^{\sqrt{t}+1} + \dfrac{2+|u(t)|+\left|\mathcal{D}_{0^+}^{\frac{1}{2};t}u(t)\right|}{8e^{2-t}\left(1+|u(t)|+\left|\mathcal{D}_{0^+}^{\frac{1}{2};t}u(t)\right|\right)}\right], t\in[0,1] \\ u(T) = \lambda u(\eta). \end{cases} \tag{21}$$

Where

$$\alpha = \frac{1}{4},\ \lambda = \frac{3}{4},\ \eta = \frac{1}{2},\ a = 0,\ T = 1,\ \psi(t) = t,$$

Set:

$$f(t,u,v) = \left[\frac{1}{3}e^{\sqrt{t+1}} + \frac{2+u+v}{8e^{2-t}(1+u+v)}\right], t\in[0,1], u,v\in\mathbb{R}^+.$$

Clearly, the function $f \in C([0,1])$. For each $u_1, v_1, u_2, v_2 \in \mathbb{R}^+$ and $t \in [0,1]$

$$\begin{aligned}|f(t,u,v) - f(t,u_1,v_1)| &= \left|\frac{2+u_1+v_1}{8e^{2-t}(1+u_1+v_1)} - \frac{2+u_2+v_2}{8e^{2-t}(1+u_2+v_2)}\right| \\ &\leq \frac{1}{8e^{2-t}}\left(|u_1-u_2| + |v_1-v_2|\right) \\ &\leq \frac{1}{8e}\left(|u_1-u_2| + |v_1-v_2|\right). \end{aligned} \tag{22}$$

Hence, the condition (H1) is satisfied with $L_1 = L_2 = \frac{1}{8e}$. It is easy to verify that

$$\Lambda_1 = \frac{L_1}{1-L_2}\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Lambda}\left\{\frac{\lambda\eta^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right\}\right\} < 1.$$

Clearly, the hypothesis of Theorem 1 are fulfilled and hence its conclusion implies the existence of a unique solution of the problem in Equation (21) on $[0,1]$.

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