Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **16** (2021), No. 4, pp. 303-319
DOI: 10.21915/BIMAS.2021402

DETERMINATION OF TIME DEPENDENT DIFFUSION COEFFICIENT IN TIME FRACTIONAL DIFFUSION EQUATIONS BY FRACTIONAL SCALING TRANSFORMATIONS METHOD

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Abstract

This study is devoted to investigation of inverse problem of identifying unknown time-dependent diffusion coefficient in time fractional diffusion equation in the sense of the modified Riemann-Liouville fractional derivative, by employing fractional scaling transformations method. By means of this method fractional order derivatives turns into integer order derivatives which allows us to deal with the easier problem. After establishing the solution and unknown coefficient of integer order diffusion problem, by utilizing the inverse transformation, we construct the solution and unknown coefficient of time fractional diffusion problem. Presented examples illustrate that identified unknown coefficient and the solution of the problem are in a high agreement with the exact solution of the corresponding the inverse problem.

1. Introduction

Last couple of decades fractional differential equations play a significant role in modelling of various processes. As a result, they attract growing attention of many scientists in diverse branches of sciences such as engineering, mathematics, chemistry and physics [1, 2, 3, 4, 5]. Consequently, numerous analytical and numerical methods have been utilized to construct

Received September 8, 2021.

AMS Subject Classification: 35R11.

Key words and phrases: Time fractional diffusion equation, fractional scaling transformations method, modified Riemann-Liouville fractional derivative, Inverse problem.

solutions of mathematical problems including fractional differential equations [6, 7, 8, 9, 10, 11, 12].

Therefore, identification of unknown coefficients in fractional differential equations with or without additional measured data becomes one of the trend challenges in inverse problems [13, 14, 15]. Hence, many researchers in various research areas have been developing new methods to tackle with this kind of inverse problems including fractional derivatives [13, 14, 15, 16, 17].

In this research, our focus is on establishing time dependent diffusivity coefficient and the solution of the mathematical problem including time fractional diffusion equation by means of fractional scaling transformation methods. The main advantage of this method is that it turns fractional order differential equations into integer order differential equations which makes the problem easier to tackle with. We remark that this method works out for the fractional differential equations in the sense of modified Riemann-Liouville fractional derivative. The main goal in this article is to reveal the unknown coefficient of the following governing time fractional diffusion equation:

$$D_t^\alpha u(x,t) = a(t)u_{xx}(x,t), 0 < x < x_1, 0 < t < t_1, 0 < \alpha \leqslant 1, \tag{1}$$

where $u(x,t)$ and $a(t) > 0$ represent the temperature and thermal diffusivity, respectively. Associated to (1) the prescribed initial condition is

$$u(x,0) = \varphi(x), 0 \leqslant x \leqslant x_1, \tag{2}$$

and the prescribed Dirichlet boundary conditions are

$$u(0,t) = \mu_1(t), 0 < t \leqslant t_1, \tag{3}$$

$$u(x_1,t) = \mu_2(t), 0 < t \leqslant t_1, \tag{4}$$

with additional condition

$$\int_0^{x_1} u(x,t)dx = E(t), 0 < t \leqslant t_1. \tag{5}$$

Having the condition $a(t) > 0$ makes the problem (1)-(5) well-posed.

2. Preliminaries

The modified Riemann-Liouville fractional calculus [18] is defined as

$$D_t^\alpha F(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha-1}[F(\xi)-F(0)]d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha}[F(\xi)-F(0)]d\xi, & 0 < \alpha < 1, \\ \left[F^{\alpha-n}(t)\right]^{(n)}, & n \le \alpha \le n+1, n \ge 1 \end{cases} \tag{6}$$

where α is a decimal, Γ denotes the gamma function. According to the definitions of the fractional calculus Eq.(6), the following properties are held:

$$D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \tag{7}$$

$$D_t^\alpha [F(t)G(t)] = G(t)[D_t^\alpha F(t)] + F(t)[D_t^\alpha G(t)], \tag{8}$$

$$D_t^\alpha F[G(t)] = F'_G[G(t)] D_t^\alpha G(t) = (D_G^\alpha F[G(t)])[G'(t)]^\alpha \tag{9}$$

where $F(t)$ and $G(t)$ are arbitrary functions, $\gamma > -1$ is a constant.

3. Analysis of the New Fractional Derivative

By means of the following fractional scaling transformations

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)}, u(x,t) = V(x,T), \tag{10}$$

the problem (1)-(5) is converted to into the following integer order problem

$$V_T = a(T)V_{xx}, 0 < x < x_1, 0 < T < \frac{t_1^\alpha}{\Gamma(1+\alpha)}, \tag{11}$$

with initial conditions

$$V(x,0) = \overline{\varphi}(x), 0 < x \leqslant x_1, \tag{12}$$

and the prescribed Dirichlet boundary conditions are

$$V(0,T) = \overline{\mu_1}(T), 0 < T \leqslant \frac{t_1^\alpha}{\Gamma(1+\alpha)}, \tag{13}$$

$$V(x_1,T)=\overline{\mu_2}(T), 0<T\leqslant \frac{t_1^\alpha}{\Gamma(1+\alpha)} \tag{14}$$

and additional condition

$$\int_0^{x_1} V(x,T)dx=\overline{E}(T), 0<T\leqslant \frac{t_1^\alpha}{\Gamma(1+\alpha)}. \tag{15}$$

After establishing the solution and unknown coefficient of problem (11)-(15), by employing inverse transformation we obtain the solution $u(x,t)$ and an unknown diffusivity coefficient $a(t)$.

4. Illustrative Examples

In this section, we illustrate three examples of inverse problems about determination of unknown time dependent coefficient.

Example 1. Consider the inverse coefficient problem involving time fractional differential equations:

$$D_t^\alpha u(x,t)=a(t)u_{xx}(x,t), \qquad 0<x<1, 0<t<(\Gamma(1+\alpha))^{\frac{1}{\alpha}}, \tag{16}$$

$$u(x,0)=\exp(x), \qquad 0\leqslant x\leqslant 1, \tag{17}$$

$$u(0,t)=\exp\Big(\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}\Big), \quad 0<t\leqslant (\Gamma(1+\alpha))^{\frac{1}{\alpha}}, \tag{18}$$

$$u(1,t)=\exp\Big(1+\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}\Big), \ 0<t\leqslant (\Gamma(1+\alpha))^{\frac{1}{\alpha}}, \tag{19}$$

$$\int_0^1 u(x,t)dx=\exp\Big(\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}\Big)(\exp(1)-1), \quad 0<t\leqslant (\Gamma(1+\alpha))^{\frac{1}{\alpha}}. \tag{20}$$

By taking fractional scaling transformation methods into account the problem (16)-(20) turns into following integer order problem:

$$V_T=a(T)V_{xx}, 0<x<1, 0<T<1, \tag{21}$$

with initial conditions

$$V(x,0)=\exp(x), 0<x\leqslant 1, \tag{22}$$

and the prescribed Dirichlet boundary conditions are

$$V(0,T) = \exp(T+T^2), 0 < T \leqslant 1 \tag{23}$$

$$V(1,T) = \exp(1+T+T^2), 0 < T \leqslant 1 \tag{24}$$

and additional condition

$$\int_0^1 V(x,T)dx = \exp(T+T^2)(\exp(1)-1), 0 < T \leqslant 1. \tag{25}$$

This inverse problem have the solution $V(x,T) = \exp(x+T+T^2)$ and unknown diffusivity coefficient becomes $a(T) = 2T+1$ [13]. As seen from Figs.1-4, by means of inverse transformation the solution of problem (21)-(25) and unknown diffusivity coefficient are obtained in the following form respectively.

$$u(x,t) = \exp\left(x + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}\right) \tag{26}$$

and

$$a(t) = b + c\frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{27}$$

where $b = 1$ and $c = 2$.

Moreover, the values of exact and approximate solutions of problem (16)-(20) at $t=0.8$ for different values of orders of α are presented in Table 1.

Table 1: The table of exact and approximate solution of Ex. 1 at $t = 0.8$.

x	Exact	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
0	4.22070	4.22070	4.82614	5.50024	6.21906
0.2	5.15517	5.15517	5.89466	6.71801	7.59598
0.4	6.29654	6.29654	7.19976	8.20539	9.27775
0.6	7.69061	7.69061	8.79381	10.02209	11.33186
0.8	9.39333	9.39333	10.74078	12.24100	13.84077
1	11.47304	11.47304	13.11882	14.95120	16.90516

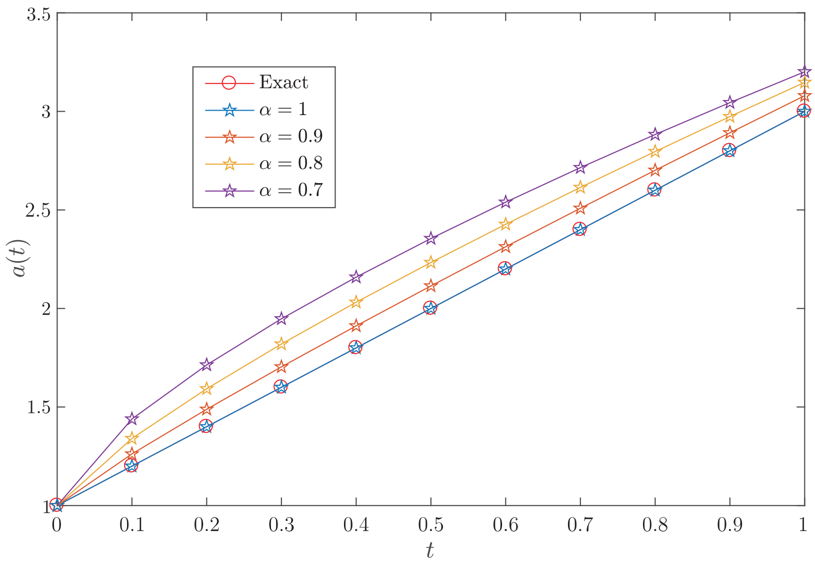

Figure 1: The graphics of approximate solution for $a(t)$ in Ex. 1.

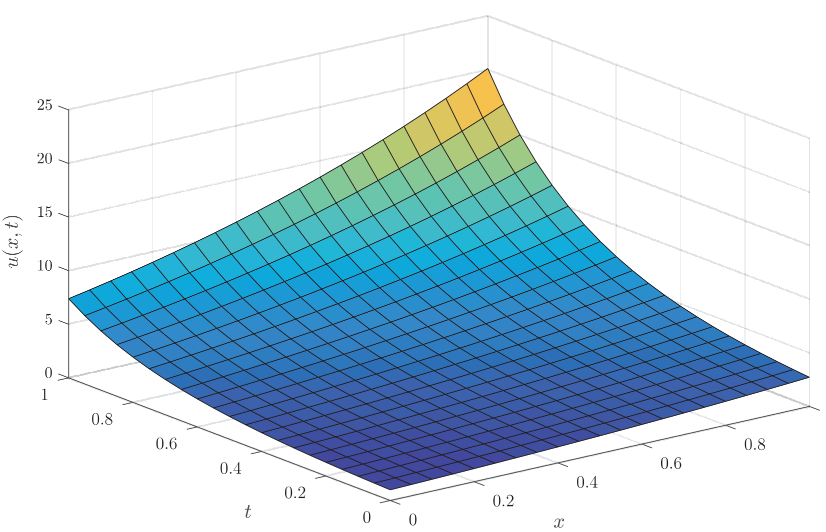

Figure 2: The graphics of exact solution for $u(x,t)$ in Ex. 1.

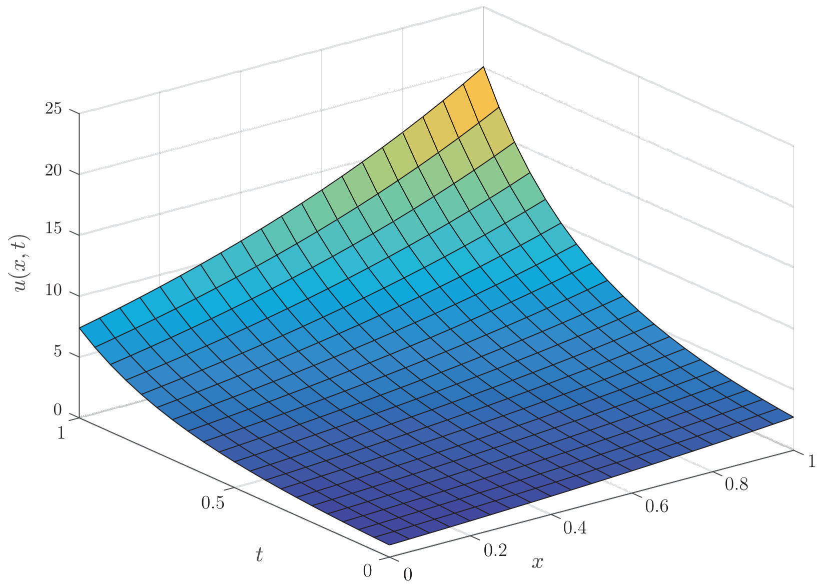

Figure 3: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ in Ex. 1.

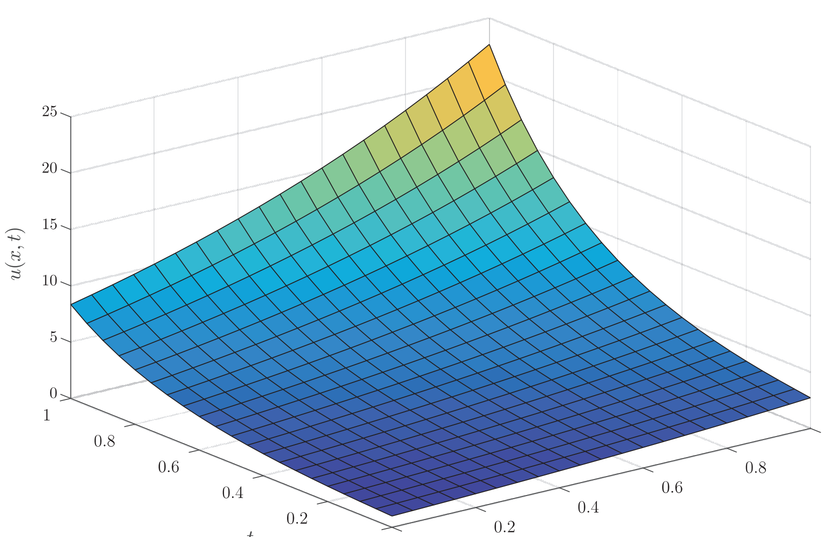

Figure 4: The graphics of approximate solution for $u(x,t)$ with $\alpha = 0.9$ in Ex. 1.

Example 2. Consider the inverse coefficient problem involving time fractional differential equations:

$$D_t^{\alpha} u(x,t) = a(t) u_{xx}(x,t), \qquad 0 < x < 1, 0 < t < (\Gamma(1+\alpha))^{\frac{1}{\alpha}}, \tag{28}$$

$$u(x,0) = \exp(x), \qquad 0 \leqslant x \leqslant 1, \tag{29}$$

$$u(0,t) = \exp\Big((4+\pi)\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\pi^3}{6}\frac{t^{3\alpha}}{(\Gamma(1+\alpha))^3} + \frac{\pi^5}{120}\frac{t^{5\alpha}}{(\Gamma(1+\alpha))^5}\Big), \quad 0 < t \leqslant (\Gamma(1+\alpha))^{\frac{1}{\alpha}}, \tag{30}$$

$$u(1,t) = \exp\Big(1 + (4+\pi)\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\pi^3}{6}\frac{t^{3\alpha}}{(\Gamma(1+\alpha))^3} + \frac{\pi^5}{120}\frac{t^{5\alpha}}{(\Gamma(1+\alpha))^5}\Big), \quad 0 < t \leqslant (\Gamma(1+\alpha))^{\frac{1}{\alpha}}, \tag{31}$$

$$\int_0^1 u(x,t)dx = \exp\Big((4+\pi)\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\pi^3}{6}\frac{t^{3\alpha}}{(\Gamma(1+\alpha))^3} + \frac{\pi^5}{120}\frac{t^{5\alpha}}{(\Gamma(1+\alpha))^5}\Big)(\exp(1)-1), \quad 0 < t \leqslant (\Gamma(1+\alpha))^{\frac{1}{\alpha}}. \tag{32}$$

By taking fractional scaling transformation methods into account the problem (28)-(32) turns into following integer order problem:

$$V_T = a(T)V_{xx}, 0 < x < 1, 0 < T < 1, \tag{33}$$

with initial conditions

$$V(x,0) = \exp(x), 0 < x \leqslant 1, \tag{34}$$

and the prescribed Dirichlet boundary conditions are

$$V(0,T) = \exp((4+\pi)T - T^3 + T^5), \qquad 0 < T \leqslant 1, \tag{35}$$

$$V(1,T) = \exp(1 + (4+\pi)T - T^3 + T^5), \quad 0 < T \leqslant 1, \tag{36}$$

and additional condition

$$\int_0^1 V(x,T)dX = \exp((4+\pi)T - T^3 + T^5)(\exp(1)-1), 0 < T \leqslant 1. \tag{37}$$

This inverse problem have the solution $V(x,T) = \exp(x + (4+\pi)T - T^3 + T^5)$ and unknown diffusivity coefficient becomes $a(T) = (4+\pi) - \frac{\pi^3 T^2}{2} + \frac{\pi^5 T^4}{24}$

[13]. As seen from Figs.5-8, by means of inverse transformation the solution of problem (33)-(37) and unknown diffusivity coefficient are obtained in the following form respectively.

$$u(x,t) = \exp\left(x+(4+\pi)\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\pi^3}{6}\frac{t^{3\alpha}}{(\Gamma(1+\alpha))^3} + \frac{\pi^5}{120}\frac{t^{5\alpha}}{(\Gamma(1+\alpha))^5}\right) \quad (38)$$

and

$$a(t) = b + c\frac{t^{\alpha}}{\Gamma(1+\alpha)} + d\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2} + e\frac{t^{3\alpha}}{(\Gamma(1+\alpha))^3} + f\frac{t^{4\alpha}}{(\Gamma(1+\alpha))^4} \quad (39)$$

where $b = 4+\pi$,$c = 0$,$d = -\frac{\pi^3}{2}$, $e = 0$ and $f = \frac{\pi^5}{24}$.
Moreover, the values of exact and approximate solutions of problem (16)-(20) at $t{=}0.8$ for different values of orders of α are presented in Table 2.

Table 2: The table of exact and approximate solution of Ex. 1 at $t = 0.8$.

x	Exact	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
0	49.55301	49.55301	56.24642	64.10626	73.49413
0.2	60.52418	60.52418	68.69953	78.29956	89.76593
0.4	73.92440	73.92440	83.90980	95.63530	109.64036
0.6	90.29146	90.29146	102.48766	116.80922	133.91504
0.8	110.28224	110.28224	125.17871	142.67110	163.56420
1	134.69904	134.69904	152.89362	174.25888	199.77776

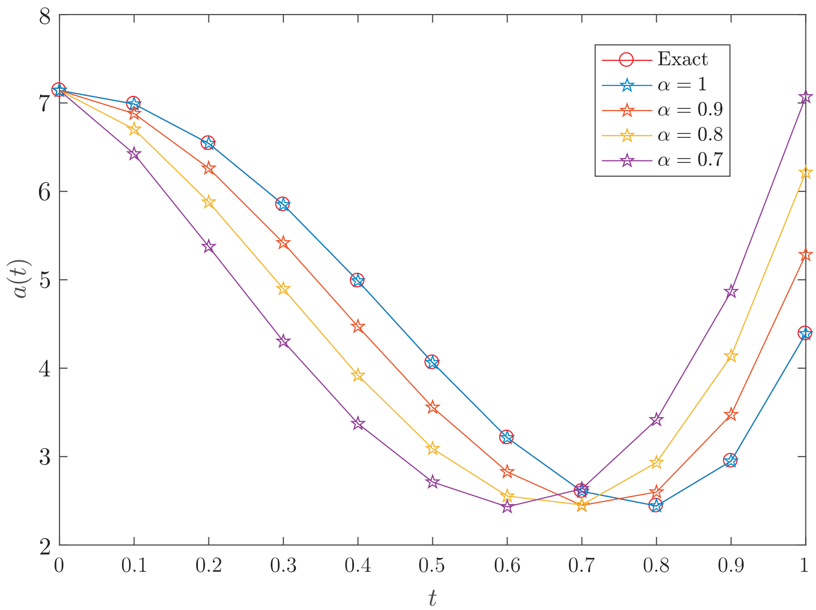

Figure 5: The graphics of approximate solution for $a(t)$ in Ex. 2.

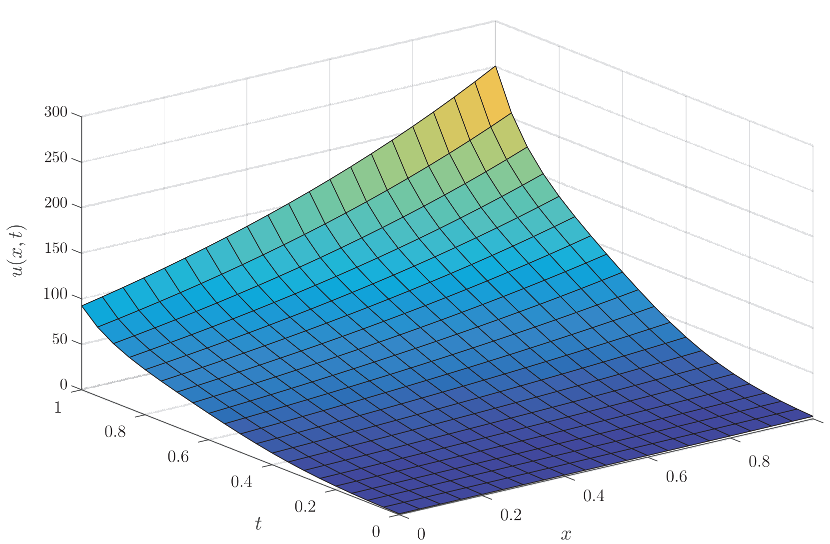

Figure 6: The graphics of exact solution for $u(x,t)$ in Ex. 2.

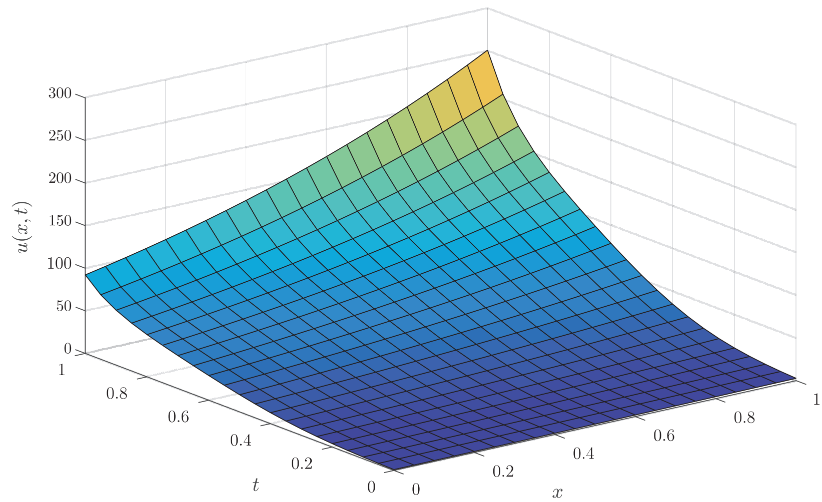

Figure 7: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ in Ex. 2.

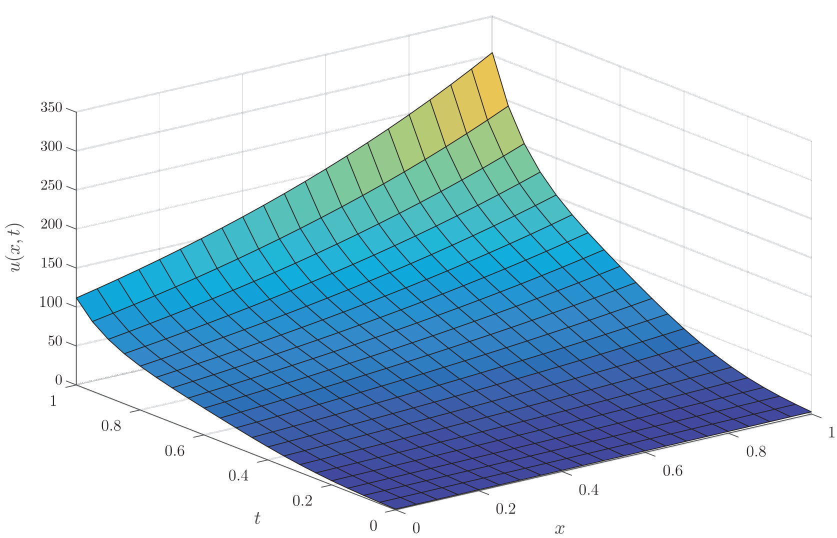

Figure 8: The graphics of approximate solution for $u(x,t)$ with $\alpha = 0.9$ in Ex. 2.

Example 3. Consider the inverse coefficient problem involving time fractional differential equations:

$$D_t^\alpha u(x,t) = a(t)u_{xx}(x,t), \quad 0 < x < 1, 0 < t < \Big(\Gamma(1+\alpha)\Big)^{\frac{1}{\alpha}}, \tag{40}$$

$$u(x,0) = \begin{cases} \exp(x+\frac{1}{4}), & 0 \leqslant x \leqslant \frac{1}{2}, \\ \exp(x+\frac{1}{2}), & \frac{1}{2} \leqslant x \leqslant 1, \end{cases} \tag{41}$$

$$u(0,t) = \begin{cases} \exp(\frac{1}{4} + \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{1}{2}\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}), 0 \leqslant t \leqslant (\frac{\Gamma(1+\alpha)}{2})^{\frac{1}{\alpha}}, \\ \exp(\frac{1}{2} + \frac{1}{2}\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}), (\frac{\Gamma(1+\alpha)}{2})^{\frac{1}{\alpha}} \leqslant t \leqslant \Gamma(1+\alpha)^{\frac{1}{\alpha}}, \end{cases} \tag{42}$$

$$u(1,t) = \begin{cases} \exp(1+\frac{1}{4} + \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{1}{2}\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}), 0 \leqslant t \leqslant (\frac{\Gamma(1+\alpha)}{2})^{\frac{1}{\alpha}}, \\ \exp(1+\frac{1}{2} + \frac{1}{2}\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}), (\frac{\Gamma(1+\alpha)}{2})^{\frac{1}{\alpha}} \leqslant t \leqslant \Gamma(1+\alpha)^{\frac{1}{\alpha}}, \end{cases} \tag{43}$$

$$\int_0^1 u(x,t)dx = \begin{cases} \exp(\frac{1}{4} + \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{1}{2}\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2})(\exp(1)-1), 0 \leqslant t \leqslant (\frac{\Gamma(1+\alpha)}{2})^{\frac{1}{\alpha}}, \\ \exp(\frac{1}{2} + \frac{1}{2}\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2})(\exp(1)-1), (\frac{\Gamma(1+\alpha)}{2})^{\frac{1}{\alpha}} \leqslant t \leqslant \Gamma(1+\alpha)^{\frac{1}{\alpha}}, \end{cases} \tag{44}$$

By taking fractional scaling transformation methods into account the problem (40)-(44) turns into following integer order problem:

$$V_T = a(T)V_{xx}, 0 < x < 1, 0 < T < 1, \tag{45}$$

with initial conditions

$$V(x,0) = \begin{cases} \exp(x+\frac{1}{4}), 0 \leqslant x \leqslant \frac{1}{2}, \\ \exp(x+\frac{1}{2}), \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \tag{46}$$

and the prescribed Dirichlet boundary conditions are

$$V(0,T) = \begin{cases} \exp(\frac{1}{4} + T - T^2), & 0 \leqslant T \leqslant \frac{1}{2}, \\ \exp(\frac{1}{2} + \frac{T^2}{2}), & \frac{1}{2} \leqslant T \leqslant 1, \end{cases} \tag{47}$$

$$V(1,T) = \begin{cases} \exp(\frac{5}{4} + T - T^2), & 0 \leqslant T \leqslant \frac{1}{2}, \\ \exp(\frac{3}{2} + \frac{T^2}{2}), & \frac{1}{2} \leqslant T \leqslant 1, \end{cases} \tag{48}$$

$$\int_0^1 V(x,T)dx = \begin{cases} \exp(\frac{1}{4} + T - T^2)(\exp(1)-1), & 0 \leqslant T \leqslant \frac{1}{2}, \\ \exp(\frac{1}{2} + \frac{T^2}{2})(\exp(1)-1), & \frac{1}{2} \leqslant T \leqslant 1. \end{cases} \tag{49}$$

This inverse problem have the solution

$$V(x,T)=\begin{cases}\exp(x+\frac{1}{4}+T-T^2), & 0<x,T\leqslant\frac{1}{2},\\ \exp(x+\frac{1}{2}+\frac{T^2}{2}), & \frac{1}{2}\leqslant x,T\leqslant 1\end{cases}$$

and unknown diffusivity coefficient becomes

$$a(T)=\begin{cases}1-T, & 0<T\leqslant\frac{1}{2},\\ T, & \frac{1}{2}\leqslant T\leqslant 1\end{cases}$$

[13]. As seen from Figs.9-12, by means of inverse transformation the solution of problem (33)-(37) and unknown diffusivity coefficient are obtained in the following form respectively

$$u(x,t)=\begin{cases}\exp(x+\frac{1}{4}+\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{1}{2}\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}), & 0\leqslant x,t\leqslant\frac{1}{2},\\ \exp(x+\frac{1}{2}+\frac{1}{2}\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}), & \frac{1}{2}\leqslant x,t\leqslant 1\end{cases} \quad (50)$$

and

$$a(t)=b+c\frac{t^\alpha}{\Gamma(1+\alpha)} \quad (51)$$

where $b=1$ and $c=-1$ for $0\leqslant t\leqslant\frac{1}{2}$, $b=0$ and $c=1$ for $\frac{1}{2}\leqslant t\leqslant 1$. Moreover, the values of exact and approximate solutions of problem (16)-(20) at $t=0.8$ for different values of orders of α are presented in Table 3.

Table 3: The table of exact and approximate solution of Ex. 1 at $t=0.8$.

x	Exact	$\alpha=1$	$\alpha=0.9$	$\alpha=0.8$	$\alpha=0.7$
0	2.07508	2.07508	2.09350	2.10605	2.11337
0.2	2.53451	2.53451	2.55700	2.57233	2.58127
0.4	3.09566	3.09566	3.12313	3.14185	3.15277
0.6	4.13712	4.13712	4.31345	4.49662	4.67913
0.8	5.05309	5.05309	5.26846	5.49219	5.71510
1	6.17186	6.17186	6.43491	6.70817	6.98044

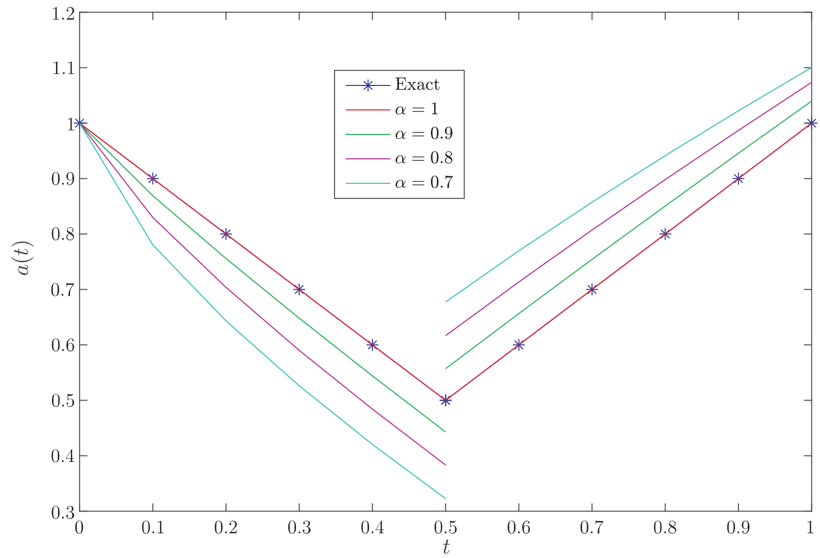

Figure 9: The graphics of approximate solution for $a(t)$ in Ex. 3.

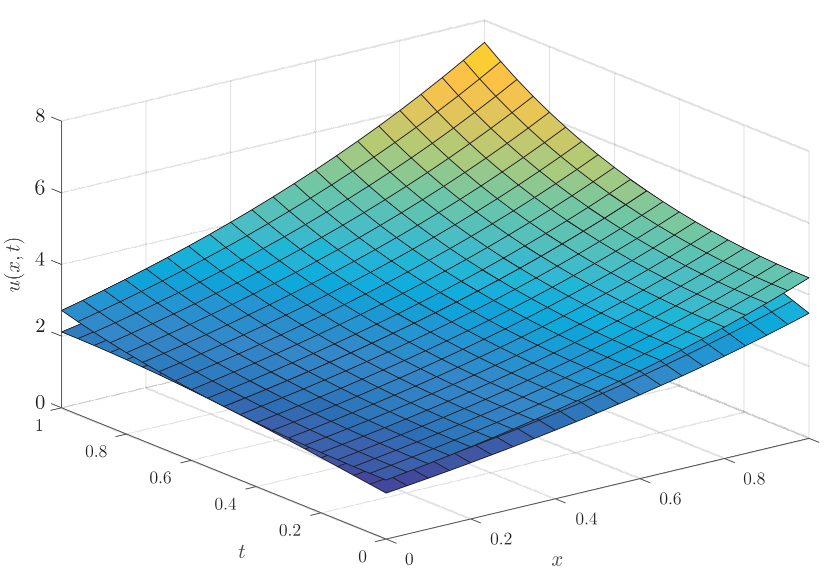

Figure 10: The graphics of exact solution for $u(x,t)$ in Ex. 3.

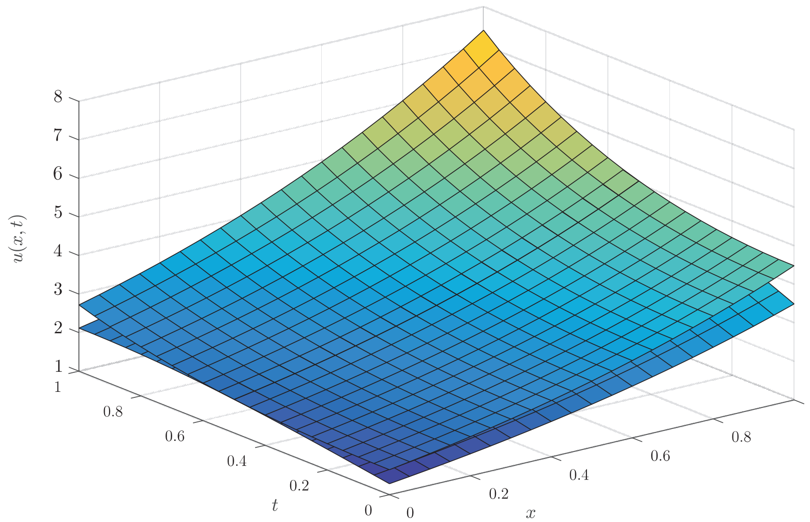

Figure 11: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ in Ex. 3.

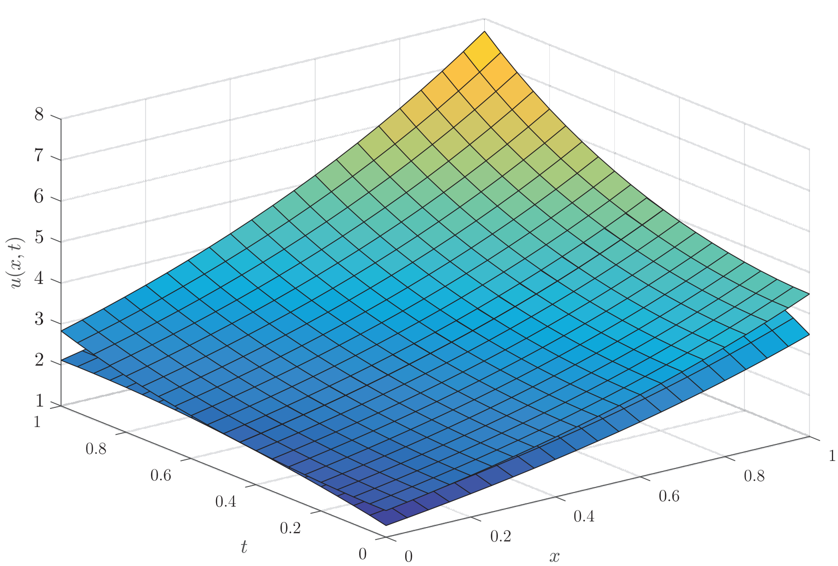

Figure 12: The graphics of approximate solution for $u(x,t)$ with $\alpha = 0.9$ in Ex. 3.

5. Conclusion

In this research, the inverse problem of revealing unknown time-dependent diffusivity coefficient in mathematical problem including time fractional diffusion equation is taken into consideration. Fractional scaling transformation methods are implemented successfully to turn the fractional problem into integer order problem to determine the solution and to identify unknown time-dependent diffusion coefficient. The considerable advantage of this method is that we get rid of difficulties of fractional derivative which makes the problem easier to deal with. Future work will be concerned with the construction of the unknown parameter in time fractional differential equations with various boundary conditions.

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