

ON THE \mathcal{D} -ISOMETRIC DEFORMATION OF KENMOTSU MANIFOLDS AND BIHARMONIC MAPS

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Abstract

In this paper, we introduce a \mathcal{D} -isometric deformation of almost contact metric manifold where we give some new results of this type of deformation, the harmonicity and the biharmonicity of the identity map are characterized by using this deformation.

1. Introduction

1.1. Harmonic and biharmonic maps

Harmonic maps $\phi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds are the critical points of the energy functional

$$E(\phi) = \frac{1}{2} \int_D |d\phi|^2 dv_g,$$

for every compact domain $D \subset M$. The Euler-Lagrange equation associated to $E(\phi)$ is

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of ϕ , one can refer to [2] for background on harmonic maps. As the generalizations of harmonic maps, biharmonic maps

Received April 21, 2022.

AMS Subject Classification: 53C25, 53D15, 58E20.

Key words and phrases: Harmonic and biharmonic maps, Almost contact metric manifolds, Kenmotsu manifolds, \mathcal{D} -isometric deformation.

are defined as follows. The map $\phi : (M^m, g) \rightarrow (N^n, h)$ is biharmonic if it is a critical point of the bi-energy functional defined by

$$E_2(\phi) = \frac{1}{2} \int_D |\tau(\phi)|^2 dv_g.$$

The first variation formula for the bi-energy which is derived in [8] shows that the Euler-Lagrange equation for bi-energy is

$$\tau_2(\phi) = -Tr_g \left(\nabla^\phi \right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi) d\phi = 0.$$

We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ . The methods of construction of biharmonic maps are the subject of several papers, see for example [3-5] and [14-16].

1.2. Almost contact metric manifold

An odd-dimensional Riemannian manifold (M^{2m+1}, g) is said to be an almost contact metric manifold if there exists on M , a $(1, 1)$ -tensor field φ , a vector field ξ (the structure vector field) and 1-form η such that

$$\begin{aligned} \varphi^2 X &= -X + \eta(X) \xi, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X) \eta(Y). \end{aligned}$$

Moreover, if the almost contact metric manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$ satisfies the condition

$$(\nabla_X \varphi)(Y) = g(\varphi X, Y) \xi - \eta(X) \eta(Y)$$

or equivalently

$$\nabla_X \xi = X - \eta(X) \xi,$$

for all $X, Y \in \Gamma(TM)$, then $(M^{2m+1}, \varphi, \xi, \eta, g)$ is known as a Kenmotsu manifold (see [8]). It is well known that a Kenmotsu manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$ satisfies the following relations :

$$\begin{aligned} R(X, Y) \xi &= \eta(X) Y - \eta(Y) X, \\ R(\xi, X) Y &= \eta(X) Y - g(X, Y) \xi, \end{aligned}$$

and

$$Ricci(\xi) = Tr_g R(\xi, \cdot) \cdot = -2m\xi.$$

The notion of a \mathcal{D} -homothetic deformation on a contact metric manifold is studied in [19] and [17], the authors investigated the generalized \mathcal{D} -conformal deformations of nearly K -cosymplectic, quasi-Sasakian and β -Kenmotsu manifolds. In [1], the authors prove some new results on the generalized \mathcal{D} -conformal deformation where they characterize the harmonicity of the identity map relative to this deformation. The authors in [7] have introduced a new deformation of almost contact metric structures using a function and a 1-form and prove some basic properties and they constructed some examples based on the three types (Sasakian, Kenmotsu, cosymplectic). The harmonicity and biharmonicity on the Kenmotsu manifolds are studied in [13], [18] and [20] where the authors present some construction methods to characterize them. In this paper, we introduce the \mathcal{D} -isometric deformation on the almost contact metric manifold, this paper is organized in the following way : In Section 2, we prove some new properties of a \mathcal{D} -isometric deformation on a almost contact metric manifold, where we treat the case of Kenmotsu manifolds. The results obtained are used in Section 3 in order to study the harmonicity and the biharmonicity of the identity map relatively to this type of deformation, which led us to construct some new examples of harmonic and biharmonic maps.

2. The Main Results

In this section, we consider $(M^{2m+1}, \varphi, \xi, \eta, g)$ an almost contact metric manifold. A \mathcal{D} -isometric deformation is defined as change of structure tensors of the form (see [9])

$$\bar{\varphi} = \varphi, \quad \bar{\eta} = \alpha\eta, \quad \bar{\xi} = \frac{1}{\alpha}\xi, \quad \bar{g} = g + (\alpha^2 - 1)\eta \otimes \eta,$$

where α is a positive function on M ; one can easily check that $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is an almost contact metric manifold, too. Denoted by ∇ and $\bar{\nabla}$ the Levi-Civita connections on $(M^{2m+1}, \varphi, \xi, \eta, g)$ and $(M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ respectively.

2.1. Some properties of the \mathcal{D} -isometric deformation

If we consider an orthonormal frame $\{e_i, \varphi e_i, \xi\}_{i=1}^m$ on the almost contact metric manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$, then an orthonormal frame on

$(M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is given by

$$\left\{ \bar{e}_i = e_i, \bar{\varphi}e_i = \varphi e_i, \bar{\xi} = \frac{1}{\alpha}\xi \right\}_{i=1}^m .$$

Proposition 1 ([1]). *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and let $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a \mathcal{D} -isometric deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$. Then, we have*

$$\begin{aligned} \bar{g}(\bar{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) + \alpha\eta(Y)\eta(Z)X(\alpha) + \alpha\eta(X)\eta(Z)Y(\alpha) \\ &\quad - \alpha\eta(X)\eta(Y)Z(\alpha) + (\alpha^2 - 1)\eta(Z)\eta(\nabla_X Y) \\ &\quad + \frac{1}{2}(\alpha^2 - 1)\eta(X)\{g(\nabla_Y \xi, Z) - g(\nabla_Z \xi, Y)\} \\ &\quad + \frac{1}{2}(\alpha^2 - 1)\eta(Y)\{g(\nabla_X \xi, Z) - g(\nabla_Z \xi, X)\} \\ &\quad + \frac{1}{2}(\alpha^2 - 1)\eta(Z)\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\} . \end{aligned}$$

By applying Proposition 1, we obtain the following theorem :

Theorem 1 ([1]). *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and let $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a \mathcal{D} -isometric deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$. For any $X, Y \in \Gamma(TM)$, the relation between $\bar{\nabla}_X Y$ and $\nabla_X Y$ is given by*

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y - \alpha\eta(X)\eta(Y)grad\alpha + \frac{\alpha^2 - 1}{\alpha}\eta(X)\eta(Y)\xi(\alpha)\xi \\ &\quad - \frac{\alpha^2 - 1}{2}\{\eta(X)Tr_g g(\nabla \cdot \xi, Y) \cdot + \eta(Y)Tr_g g(\nabla \cdot \xi, X) \cdot\} \\ &\quad + \frac{(\alpha^2 - 1)^2}{2\alpha^2}\{\eta(X)g(\nabla_\xi \xi, Y) + \eta(Y)g(\nabla_\xi \xi, X)\}\xi \\ &\quad + \frac{\alpha^2 - 1}{2\alpha^2}\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\}\xi \\ &\quad + \frac{1}{\alpha}\{\eta(Y)X(\alpha) + \eta(X)Y(\alpha)\}\xi \\ &\quad + \frac{\alpha^2 - 1}{2}\{\eta(Y)\nabla_X \xi + \eta(X)\nabla_Y \xi\} , \end{aligned} \tag{1}$$

where

$$Tr_g g(X, \nabla \cdot \xi) \cdot = g(X, \nabla_{e_i} \xi) e_i + g(X, \nabla_{\varphi e_i} \xi) \varphi e_i + g(X, \nabla_\xi \xi) \xi .$$

In particular, if $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold, we obtain the following result.

Corollary 1. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold, then*

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y - \alpha \eta(X) \eta(Y) \operatorname{grad} \alpha \\ &\quad + \frac{\alpha^2 - 1}{\alpha} \eta(X) \eta(Y) \xi(\alpha) \xi \\ &\quad + \frac{\alpha^2 - 1}{\alpha^2} \{g(X, Y) - \eta(X) \eta(Y)\} \xi \\ &\quad + \frac{1}{\alpha} \{\eta(Y) X(\alpha) + \eta(X) Y(\alpha)\} \xi. \end{aligned} \tag{2}$$

In particular, if the function α depends only on the direction of ξ , the equation (2) becomes

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y - \frac{1}{\alpha} \eta(X) \eta(Y) \xi(\alpha) \xi \\ &\quad + \frac{1}{\alpha} \{\eta(Y) X(\alpha) + \eta(X) Y(\alpha)\} \xi \\ &\quad + \frac{\alpha^2 - 1}{\alpha^2} \{g(X, Y) - \eta(X) \eta(Y)\} \xi. \end{aligned} \tag{3}$$

Proof of Corollary 1. As $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold, we have

$$\nabla_X \xi = X - \eta(X) \xi, \quad \nabla_Y \xi = Y - \eta(Y) \xi, \quad \nabla_\xi \xi = 0.$$

It follows that

$$\nabla_{e_i} \xi = e_i, \quad \nabla_{\varphi e_i} \xi = \varphi e_i,$$

which gives us

$$\operatorname{Tr}_g g(\nabla \cdot \xi, X) \cdot = X - \eta(X) \xi, \quad \operatorname{Tr}_g g(\nabla \cdot \xi, Y) \cdot = Y - \eta(Y) \xi.$$

By returning to equation (1), we deduce that

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y - \alpha \eta(X) \eta(Y) \operatorname{grad} \alpha \\ &\quad + \frac{\alpha^2 - 1}{\alpha} \eta(X) \eta(Y) \xi(\alpha) \xi \end{aligned}$$

$$\begin{aligned}
& + \frac{\alpha^2 - 1}{\alpha^2} \{g(X, Y) - \eta(X)\eta(Y)\} \xi \\
& + \frac{1}{\alpha} \{\eta(Y)X(\alpha) + \eta(X)Y(\alpha)\} \xi.
\end{aligned}$$

In particular, if the function α depends only on the direction of ξ , then $\text{grad}\alpha = \xi(\alpha)\xi$ and we obtain

$$\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X Y - \frac{1}{\alpha} \eta(X)\eta(Y)\xi(\alpha)\xi \\
& + \frac{1}{\alpha} \{\eta(Y)X(\alpha) + \eta(X)Y(\alpha)\} \xi \\
& + \frac{\alpha^2 - 1}{\alpha^2} \{g(X, Y) - \eta(X)\eta(Y)\} \xi.
\end{aligned}$$

By considering the elements of the orthonormal frames of $(M^{2m+1}, \varphi, \xi, \eta, g)$ and $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$, by Corollary 1 we deduce the following remarks.

Remark 1. Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold and let $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a \mathcal{D} -isometric deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$ where the function α depends only on the direction of ξ . If we consider an orthonormal frame $\{e_i, \varphi e_i, \xi\}_{i=1}^m$ on $(M^{2m+1}, \varphi, \xi, \eta, g)$, the Corollary 1 gives the following formulas

$$\begin{aligned}
\bar{\nabla}_{e_i} e_i &= \nabla_{e_i} e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi, \\
\bar{\nabla}_{\varphi e_i} \varphi e_i &= \nabla_{\varphi e_i} \varphi e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi
\end{aligned}$$

and

$$\bar{\nabla}_{\xi} \xi = \frac{1}{\alpha} \xi(\alpha) \xi.$$

Similarly, for an orthonormal frame $\{\bar{e}_i = e_i, \bar{\varphi} e_i = \varphi e_i, \bar{\xi} = \frac{1}{\alpha} \xi\}_{i=1}^m$ on $(M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$, we obtain

$$\begin{aligned}
\bar{\nabla}_{\bar{e}_i} \bar{e}_i &= \bar{\nabla}_{e_i} e_i = \nabla_{e_i} e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi, \\
\bar{\nabla}_{\bar{\varphi} e_i} \bar{\varphi} e_i &= \bar{\nabla}_{\varphi e_i} \varphi e_i = \nabla_{\varphi e_i} \varphi e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi
\end{aligned}$$

and

$$\bar{\nabla}_{\bar{\xi}} \bar{\xi} = 0.$$

Remark 2. The fact that the function α depends only on the direction of ξ gives us

$$X(\alpha) = g(X, e_i) e_i(\alpha) + g(X, \varphi e_i) (\varphi e_i)(\alpha) + g(X, \xi) \xi(\alpha) = \eta(X) \xi(\alpha)$$

and

$$Y(\alpha) = g(Y, e_i) e_i(\alpha) + g(Y, \varphi e_i) (\varphi e_i)(\alpha) + g(Y, \xi) \xi(\alpha) = \eta(Y) \xi(\alpha).$$

As a result, the equation (3) will take the following form

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + \frac{1}{\alpha} \eta(X) \eta(Y) \xi(\alpha) \xi \\ &+ \frac{\alpha^2 - 1}{\alpha^2} \{g(X, Y) - \eta(X) \eta(Y)\} \xi. \end{aligned}$$

Proposition 2. Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold. For the orthonormal frame $\{e_i, \varphi e_i, \xi\}_{i=1}^m$ of M , we have

$$Tr_g \nabla^2 \xi = -2m\xi$$

and

$$Tr_g \nabla^2 f \xi = \xi^{(2)}(f) \xi + 2m\xi(f) \xi - 2mf\xi$$

where the function $f \in C^\infty(M)$ depends only on the direction of ξ and $\xi^{(2)}(f) = \xi(\xi(f))$.

Proof of Proposition 2. Using the fact that $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold, we have $\nabla_\xi \xi = 0$. Then by definition we obtain

$$Tr_g \nabla^2 \xi = \nabla_{e_i} \nabla_{e_i} \xi - \nabla_{\nabla_{e_i} e_i} \xi + \nabla_{\varphi e_i} \nabla_{\varphi e_i} \xi - \nabla_{\nabla_{\varphi e_i} \varphi e_i} \xi.$$

A simple calculation gives

$$\nabla_{e_i} \xi = e_i$$

and

$$\begin{aligned} \nabla_{\nabla_{e_i} e_i} \xi &= \nabla_{e_i} e_i - \eta(\nabla_{e_i} e_i) \xi \\ &= \nabla_{e_i} e_i - g(\nabla_{e_i} e_i, \xi) \xi \\ &= \nabla_{e_i} e_i + g(e_i, \nabla_{e_i} \xi) \xi \\ &= \nabla_{e_i} e_i + g(e_i, e_i) \xi \\ &= \nabla_{e_i} e_i + m\xi. \end{aligned}$$

Then

$$\nabla_{e_i} \nabla_{e_i} \xi - \nabla_{\nabla_{e_i} e_i} \xi = \nabla_{e_i} e_i - \nabla_{e_i} e_i - m\xi = -m\xi.$$

The same method gives

$$\nabla_{\varphi e_i} \nabla_{\varphi e_i} \xi - \nabla_{\nabla_{\varphi e_i} \varphi e_i} \xi = -m\xi.$$

It follows that

$$Tr_g \nabla^2 \xi = -2m\xi$$

If we suppose that the function $f \in C^\infty(M)$ depends only on the direction of ξ , we have

$$Tr_g \nabla^2 f\xi = \nabla_{e_i} \nabla_{e_i} f\xi - \nabla_{\nabla_{e_i} e_i} f\xi + \nabla_{\varphi e_i} \nabla_{\varphi e_i} f\xi - \nabla_{\nabla_{\varphi e_i} \varphi e_i} f\xi + \nabla_\xi \nabla_\xi f\xi.$$

Term by term, we obtain

$$\begin{aligned} \nabla_{e_i} \nabla_{e_i} f\xi - \nabla_{\nabla_{e_i} e_i} f\xi &= f \nabla_{e_i} \nabla_{e_i} \xi - f \nabla_{\nabla_{e_i} e_i} \xi - (\nabla_{e_i} e_i)(f)\xi \\ &= f \nabla_{e_i} \nabla_{e_i} \xi - f \nabla_{\nabla_{e_i} e_i} \xi - (\nabla_{e_i} e_i)(f)\xi \\ &= f \nabla_{e_i} \nabla_{e_i} \xi - f \nabla_{\nabla_{e_i} e_i} \xi - g(\nabla_{e_i} e_i, \xi)(f)\xi \\ &= f \nabla_{e_i} \nabla_{e_i} \xi - f \nabla_{\nabla_{e_i} e_i} \xi + m\xi(f)\xi, \end{aligned}$$

$$\nabla_{\varphi e_i} \nabla_{\varphi e_i} f\xi - \nabla_{\nabla_{\varphi e_i} \varphi e_i} f\xi = f \nabla_{\varphi e_i} \nabla_{\varphi e_i} \xi - f \nabla_{\nabla_{\varphi e_i} \varphi e_i} \xi + m\xi(f)\xi$$

and

$$\nabla_\xi \nabla_\xi f\xi = \xi^{(2)}(f)\xi.$$

Which gives

$$Tr_g \nabla^2 f\xi = \xi^{(2)}(f)\xi + 2m\xi(f)\xi + fTr_g \nabla^2 \xi,$$

it follows that

$$Tr_g \nabla^2 f\xi = \xi^{(2)}(f)\xi + 2m\xi(f)\xi - 2mf\xi.$$

Proposition 3. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold and let $(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be a \mathcal{D} -isometric deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$. Then*

$$Tr_{\overline{g}} \nabla^2 \xi = -2m\xi$$

and

$$Tr_{\bar{g}}\nabla^2 f\xi = \frac{1}{\alpha^2}\xi^{(2)}(f)\xi - \frac{1}{\alpha^2}\xi(\alpha)\xi(f)\xi + \frac{2m}{\alpha^2}\xi(f)\xi - 2mf\xi.$$

where the function $f \in C^\infty(M)$ depends only on the direction of ξ .

Proof of Proposition 3. By definition, we have

$$Tr_{\bar{g}}\nabla^2\xi = \nabla_{\bar{e}_i}\nabla_{\bar{e}_i}\xi - \nabla_{\nabla_{\bar{e}_i}\bar{e}_i}\xi + \nabla_{\bar{\varphi}e_i}\nabla_{\bar{\varphi}e_i}\xi - \nabla_{\nabla_{\bar{\varphi}e_i}\bar{\varphi}e_i}\xi + \nabla_{\bar{\xi}}\nabla_{\bar{\xi}}\xi - \nabla_{\nabla_{\bar{\xi}}\bar{\xi}}\xi.$$

Using the fact that

$$\bar{e}_i = e_i, \quad \bar{\varphi}e_i = \varphi e_i, \quad \bar{\xi} = \frac{1}{\alpha}\xi, \quad \nabla_{\bar{\xi}}\bar{\xi} = \nabla_{\xi}\xi = 0,$$

we obtain

$$\begin{aligned} Tr_{\bar{g}}\nabla^2\xi &= \nabla_{e_i}\nabla_{e_i}\xi - \nabla_{\nabla_{e_i}e_i}\xi + \nabla_{\varphi e_i}\nabla_{\varphi e_i}\xi - \nabla_{\nabla_{\varphi e_i}\varphi e_i}\xi \\ &= \nabla_{e_i}\nabla_{e_i}\xi - \nabla_{\nabla_{e_i}e_i}\xi + \nabla_{\varphi e_i}\nabla_{\varphi e_i}\xi - \nabla_{\nabla_{\varphi e_i}\varphi e_i}\xi \\ &= Tr_g\nabla^2\xi, \end{aligned}$$

then

$$Tr_{\bar{g}}\nabla^2\xi = -2m\xi.$$

For the term $Tr_{\bar{g}}\nabla^2 f\xi$, if we suppose that the function $f \in C^\infty(M)$ depends only on the direction of ξ , we obtain

$$\begin{aligned} Tr_{\bar{g}}\nabla^2 f\xi &= \nabla_{\bar{e}_i}\nabla_{\bar{e}_i}f\xi - \nabla_{\nabla_{\bar{e}_i}\bar{e}_i}f\xi + \nabla_{\bar{\varphi}e_i}\nabla_{\bar{\varphi}e_i}f\xi - \nabla_{\nabla_{\bar{\varphi}e_i}\bar{\varphi}e_i}f\xi + \nabla_{\bar{\xi}}\nabla_{\bar{\xi}}f\xi \\ &\quad - \nabla_{\nabla_{\bar{\xi}}\bar{\xi}}f\xi \\ &= \nabla_{e_i}\nabla_{e_i}f\xi - \nabla_{\nabla_{e_i}e_i}f\xi + \nabla_{\varphi e_i}\nabla_{\varphi e_i}f\xi - \nabla_{\nabla_{\varphi e_i}\varphi e_i}f\xi + \frac{1}{\alpha}\nabla_{\xi}\frac{1}{\alpha}\nabla_{\xi}f\xi \\ &= f\nabla_{e_i}\nabla_{e_i}\xi - \nabla_{\nabla_{e_i}e_i}f\xi - \frac{m(\alpha^2-1)}{\alpha^2}\nabla_{\xi}f\xi + f\nabla_{\varphi e_i}\nabla_{\varphi e_i}\xi \\ &\quad - \nabla_{\nabla_{\varphi e_i}\varphi e_i}f\xi - \frac{m(\alpha^2-1)}{\alpha^2}\nabla_{\xi}f\xi + \frac{1}{\alpha}\xi\left(\frac{1}{\alpha}\xi(f)\right)\xi \\ &= f\nabla_{e_i}\nabla_{e_i}\xi - f\nabla_{\nabla_{e_i}e_i}\xi + f\nabla_{\varphi e_i}\nabla_{\varphi e_i}\xi - f\nabla_{\nabla_{\varphi e_i}\varphi e_i}\xi \\ &\quad - (\nabla_{e_i}e_i)(f)\xi - (\nabla_{\varphi e_i}\varphi e_i)(f)\xi \end{aligned}$$

$$\begin{aligned}
& -\frac{2m(\alpha^2-1)}{\alpha^2}\xi(f)\xi + \frac{1}{\alpha}\xi\left(\frac{1}{\alpha}\xi(f)\right)\xi \\
& = \frac{1}{\alpha^2}\xi^{(2)}(f)\xi - \frac{1}{\alpha^3}\xi(\alpha)\xi(f)\xi - 2mf\xi \\
& \quad + 2m\xi(f)\xi - \frac{2m(\alpha^2-1)}{\alpha^2}\xi(f)\xi.
\end{aligned}$$

We deduce that

$$Tr_{\bar{g}}\nabla^2 f\xi = \frac{1}{\alpha^2}\xi^{(2)}(f)\xi - \frac{1}{\alpha^3}\xi(\alpha)\xi(f)\xi + \frac{2m}{\alpha^2}\xi(f)\xi - 2mf\xi.$$

Proposition 4. Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold and let $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a \mathcal{D} -isometric deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$. Then

$$Tr_g\bar{\nabla}^2\xi = \frac{1}{\alpha}\xi^{(2)}(\alpha)\xi + \frac{2m}{\alpha}\xi(\alpha)\xi - \frac{2m}{\alpha^2}\xi$$

and

$$\begin{aligned}
Tr_g\bar{\nabla}^2 f\xi & = \frac{f}{\alpha}\xi^{(2)}(\alpha)\xi + \xi^{(2)}(f)\xi + \frac{2}{\alpha}\xi(\alpha)\xi(f)\xi + \frac{2mf}{\alpha}\xi(\alpha)\xi \\
& \quad + 2m\xi(f)\xi - \frac{2mf}{\alpha^2}\xi.
\end{aligned}$$

where the function $f \in C^\infty(M)$ depends only on the direction of ξ .

Proof of Proposition 4. For the term $Tr_g\bar{\nabla}^2\xi$, we have

$$Tr_g\bar{\nabla}^2\xi = \bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\xi - \bar{\nabla}_{\nabla_{e_i}e_i}\xi + \bar{\nabla}_{\varphi e_i}\bar{\nabla}_{\varphi e_i}\xi - \bar{\nabla}_{\nabla_{\varphi e_i}\varphi e_i}\xi + \bar{\nabla}_\xi\bar{\nabla}_\xi\xi$$

A simple calculation gives

$$\begin{aligned}
\bar{\nabla}_{e_i}\xi & = \nabla_{e_i}\xi = e_i, \\
\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\xi & = \nabla_{e_i}e_i + \frac{m(\alpha^2-1)}{\alpha^2}\xi
\end{aligned}$$

and

$$\bar{\nabla}_{\nabla_{e_i}e_i}\xi = \nabla_{e_i}e_i - \frac{m}{\alpha}\xi(\alpha)\xi + m\xi,$$

then

$$\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\xi - \bar{\nabla}_{\nabla_{e_i}e_i}\xi = \frac{m}{\alpha}\xi(\alpha)\xi - \frac{m}{\alpha^2}\xi.$$

Similarly, we obtain

$$\bar{\nabla}_{\varphi e_i} \bar{\nabla}_{\varphi e_i} \xi - \bar{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} \xi = \frac{m}{\alpha} \xi(\alpha) \xi - \frac{m}{\alpha^2} \xi$$

and

$$\bar{\nabla}_{\xi} \bar{\nabla}_{\xi} \xi = \frac{1}{\alpha} \xi^{(2)}(\alpha) \xi.$$

Finally, we deduce that

$$Tr_g \bar{\nabla}^2 \xi = \frac{1}{\alpha} \xi^{(2)}(\alpha) \xi + \frac{2m}{\alpha} \xi(\alpha) \xi - \frac{2m}{\alpha^2} \xi.$$

To complete the proof, we will simplify $Tr_g \bar{\nabla}^2 f \xi$ where the function $f \in C^\infty(M)$ depends only on the direction of ξ . We have

$$\begin{aligned} Tr_g \bar{\nabla}^2 f \xi &= \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} f \xi - \bar{\nabla}_{\nabla_{e_i} e_i} f \xi + \bar{\nabla}_{\varphi e_i} \bar{\nabla}_{\varphi e_i} f \xi - \bar{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} f \xi + \bar{\nabla}_{\xi} \bar{\nabla}_{\xi} f \xi \\ &= f \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} \xi - f \bar{\nabla}_{\nabla_{e_i} e_i} \xi + f \bar{\nabla}_{\varphi e_i} \bar{\nabla}_{\varphi e_i} \xi - f \bar{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} \xi + f \bar{\nabla}_{\xi} \bar{\nabla}_{\xi} \xi \\ &\quad + \xi^{(2)}(f) \xi + 2\xi(f) \bar{\nabla}_{\xi} \xi - (\nabla_{e_i} e_i)(f) \xi - (\nabla_{\varphi e_i} \varphi e_i)(f) \xi \\ &= f Tr_g \bar{\nabla}^2 \xi + \xi^{(2)}(f) \xi + \frac{2}{\alpha} \xi(\alpha) \xi(f) \xi + 2m \xi(f) \xi. \end{aligned}$$

It follows that

$$\begin{aligned} Tr_g \bar{\nabla}^2 f \xi &= \frac{f}{\alpha} \xi^{(2)}(\alpha) \xi + \xi^{(2)}(f) \xi + \frac{2}{\alpha} \xi(\alpha) \xi(f) \xi + \frac{2mf}{\alpha} \xi(\alpha) \xi \\ &\quad + 2m \xi(f) \xi - \frac{2mf}{\alpha^2} \xi. \end{aligned}$$

Denoted by R and \bar{R} the curvature tensors on $(M^{2m+1}, \varphi, \xi, \eta, g)$ and $(M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ respectively.

Proposition 5. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold and let $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a \mathcal{D} -isometric deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$. Then*

$$Tr_g \bar{R}(\xi, \cdot) \cdot = \frac{2m}{\alpha^3} \xi(\alpha) \xi - \frac{2m}{\alpha^2} \xi.$$

Proof of Proposition 5. By definition, we have

$$Tr_g \bar{R}(\xi, \cdot) \cdot = \bar{R}(\xi, e_i) e_i + \bar{R}(\xi, \varphi e_i) \varphi e_i + \bar{R}(\xi, \xi) \xi,$$

then

$$Tr_g \bar{R}(\xi, \cdot) \cdot = \bar{R}(\xi, e_i) e_i + \bar{R}(\xi, \varphi e_i) \varphi e_i. \quad (4)$$

For the first term $\bar{R}(\xi, e_i) e_i$, we have

$$\bar{R}(\xi, e_i) e_i = \bar{\nabla}_\xi \bar{\nabla}_{e_i} e_i - \bar{\nabla}_{e_i} \bar{\nabla}_\xi e_i - \bar{\nabla}_{[\xi, e_i]} e_i.$$

Using remark 1, we have

$$\bar{\nabla}_{e_i} e_i = \nabla_{e_i} e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi,$$

which gives us

$$\bar{\nabla}_\xi \bar{\nabla}_{e_i} e_i = \bar{\nabla}_\xi \nabla_{e_i} e_i + \bar{\nabla}_\xi \frac{m(\alpha^2 - 1)}{\alpha^2} \xi.$$

A simple calculation gives

$$\bar{\nabla}_\xi \nabla_{e_i} e_i = \nabla_\xi \nabla_{e_i} e_i - \frac{m}{\alpha} \xi(\alpha) \xi$$

and

$$\bar{\nabla}_\xi \frac{m(\alpha^2 - 1)}{\alpha^2} \xi = \frac{m(\alpha^2 + 1)}{\alpha^3} \xi(\alpha) \xi.$$

It follows that

$$\bar{\nabla}_\xi \bar{\nabla}_{e_i} e_i = \nabla_\xi \nabla_{e_i} e_i + \frac{m}{\alpha^3} \xi(\alpha) \xi.$$

The same method gives

$$\bar{\nabla}_{e_i} \bar{\nabla}_\xi e_i = \nabla_{e_i} \nabla_\xi e_i$$

and

$$\bar{\nabla}_{[\xi, e_i]} e_i = \nabla_{[\xi, e_i]} e_i - \frac{m(\alpha^2 - 1)}{\alpha^2} \xi.$$

Then

$$\bar{R}(\xi, e_i) e_i = R(\xi, e_i) e_i + \frac{m}{\alpha^3} \xi(\alpha) \xi + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi.$$

Finally, using the fact that

$$R(\xi, e_i) e_i = -m\xi,$$

we conclude that

$$\overline{R}(\xi, e_i) e_i = \frac{m}{\alpha^3} \xi(\alpha) \xi - \frac{m}{\alpha^2} \xi. \tag{5}$$

A similar calculation gives

$$\overline{R}(\xi, \varphi e_i) \varphi e_i = \frac{m}{\alpha^3} \xi(\alpha) \xi - \frac{m}{\alpha^2} \xi. \tag{6}$$

If we replace (5) and (6) in (4), we deduce that

$$Tr_g \overline{R}(\xi, \cdot) \cdot = \frac{2m}{\alpha^3} \xi(\alpha) \xi - \frac{2m}{\alpha^2} \xi.$$

As the first result, we will study the hamonicity and the biharmonicity of the identity map $Id : (M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$.

2.2. The biharmonicity of $Id : (M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$

Theorem 2. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold and let $(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be a \mathcal{D} -isometric deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$. Then the identity map $Id : (M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ is harmonic if and only if*

$$\xi(\alpha) + 2m\alpha(\alpha^2 - 1) = 0,$$

and it is biharmonic if and only if

$$\begin{aligned} &\alpha^2 \xi^{(3)}(\alpha) - \alpha(\alpha + 9) \xi(\alpha) \xi^{(2)}(\alpha) + 6m\alpha^2 \xi^{(2)}(\alpha) \\ &+ 3(\alpha + 4) (\xi(\alpha))^3 - 2m\alpha(2\alpha + 9) (\xi(\alpha))^2 \\ &- 4m\alpha^2(\alpha^2 - 2m) \xi(\alpha) - 8m^2\alpha^5(\alpha^2 - 1) = 0. \end{aligned}$$

Proof of Theorem 2. The tension field of $Id : (M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ is given by

$$\overline{\tau}(Id) = \nabla_{\overline{e}_i} \overline{e}_i - \overline{\nabla}_{\overline{e}_i} \overline{e}_i + \nabla_{\overline{\varphi e}_i} \overline{\varphi e}_i - \overline{\nabla}_{\overline{\varphi e}_i} \overline{\varphi e}_i + \nabla_{\overline{\xi}} \overline{\xi} - \overline{\nabla}_{\overline{\xi}} \overline{\xi}.$$

Using the following equations

$$\nabla_{\overline{e}_i} \overline{e}_i = \nabla_{e_i} e_i, \quad \nabla_{\overline{\varphi e}_i} \overline{\varphi e}_i = \nabla_{\varphi e_i} \varphi e_i, \quad \nabla_{\overline{\xi}} \overline{\xi} = -\frac{1}{\alpha^3} \xi(\alpha) \xi,$$

$$\bar{\nabla}_{e_i} e_i = \nabla_{e_i} e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi,$$

$$\bar{\nabla}_{\varphi e_i} \varphi e_i = \nabla_{\varphi e_i} \varphi e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi$$

and

$$\bar{\nabla}_\xi \xi = \frac{1}{\alpha} \xi(\alpha) \xi, \quad \bar{\nabla}_{\bar{\xi}} \bar{\xi} = 0,$$

we obtain

$$\bar{\tau}(Id) = -\frac{1}{\alpha^3} \xi(\alpha) \xi - \frac{2m(\alpha^2 - 1)}{\alpha^2} \xi.$$

Then $Id : (M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ is harmonic if and only if

$$\xi(\alpha) + 2m\alpha(\alpha^2 - 1) = 0.$$

The biharmonicity of $Id : (M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ is given by the following equation

$$Tr_{\bar{g}} \nabla^2 \bar{\tau}(Id) + Tr_{\bar{g}} R(\bar{\tau}(Id), \cdot) \cdot = 0. \quad (7)$$

We start with the expansion of the first term, we have

$$Tr_{\bar{g}} \nabla^2 \bar{\tau}(Id) = -Tr_{\bar{g}} \nabla^2 \left(\frac{1}{\alpha^3} \xi(\alpha) \xi \right) - 2m Tr_{\bar{g}} \nabla^2 \left(\frac{\alpha^2 - 1}{\alpha^2} \xi \right).$$

The second equation obtained in Proposition 3 gives us

$$\begin{aligned} Tr_{\bar{g}} \nabla^2 \left(\frac{1}{\alpha^3} \xi(\alpha) \xi \right) &= \frac{1}{\alpha^2} \xi^{(2)} \left(\frac{1}{\alpha^3} \xi(\alpha) \right) \xi - \frac{1}{\alpha^2} \xi(\alpha) \xi \left(\frac{1}{\alpha^3} \xi(\alpha) \right) \xi \\ &\quad + \frac{2m}{\alpha^2} \xi \left(\frac{1}{\alpha^3} \xi(\alpha) \right) \xi - \frac{2m}{\alpha^3} \xi(\alpha) \xi. \end{aligned}$$

It is easy to see that

$$\xi \left(\frac{1}{\alpha^3} \xi(\alpha) \right) = \frac{1}{\alpha^3} \xi^{(2)}(\alpha) - \frac{3}{\alpha^4} (\xi(\alpha))^2$$

and

$$\begin{aligned}
 \xi^{(2)} \left(\frac{1}{\alpha^3} \xi(\alpha) \right) &= \xi \left(\frac{1}{\alpha^3} \xi^{(2)}(\alpha) - \frac{3}{\alpha^4} (\xi(\alpha))^2 \right) \\
 &= \xi \left(\frac{1}{\alpha^3} \xi^{(2)}(\alpha) \right) - \xi \left(\frac{3}{\alpha^4} (\xi(\alpha))^2 \right) \\
 &= \frac{1}{\alpha^3} \xi^{(3)}(\alpha) - \frac{3}{\alpha^4} \xi(\alpha) \xi^{(2)}(\alpha) \\
 &\quad - \frac{6}{\alpha^4} \xi(\alpha) \xi^{(2)}(\alpha) + \frac{12}{\alpha^5} (\xi(\alpha))^3 \\
 &= \frac{1}{\alpha^3} \xi^{(3)}(\alpha) - \frac{9}{\alpha^4} \xi(\alpha) \xi^{(2)}(\alpha) + \frac{12}{\alpha^5} (\xi(\alpha))^3.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 Tr_{\bar{g}} \nabla^2 \left(\frac{1}{\alpha^3} \xi(\alpha) \xi \right) &= \frac{1}{\alpha^5} \xi^{(3)}(\alpha) \xi - \frac{1}{\alpha^5} \xi(\alpha) \xi^{(2)}(\alpha) \xi - \frac{9}{\alpha^6} \xi(\alpha) \xi^{(2)}(\alpha) \xi \\
 &\quad + \frac{2m}{\alpha^5} \xi^{(2)}(\alpha) \xi + \frac{3}{\alpha^6} (\xi(\alpha))^3 \xi + \frac{12}{\alpha^7} (\xi(\alpha))^3 \xi \\
 &\quad - \frac{6m}{\alpha^6} (\xi(\alpha))^2 \xi - \frac{2m}{\alpha^3} \xi(\alpha) \xi.
 \end{aligned}$$

A similar calculation gives

$$\begin{aligned}
 Tr_{\bar{g}} \nabla^2 \left(\frac{\alpha^2 - 1}{\alpha^2} \xi \right) &= \frac{2}{\alpha^5} \xi^{(2)}(\alpha) \xi - \frac{2}{\alpha^5} (\xi(\alpha))^2 \xi - \frac{6}{\alpha^6} (\xi(\alpha))^2 \xi \\
 &\quad + \frac{4m}{\alpha^5} \xi(\alpha) \xi - 2m\xi + \frac{2m}{\alpha^2} \xi.
 \end{aligned}$$

We deduce that

$$\begin{aligned}
 Tr_{\bar{g}} \nabla^2 \bar{\tau}(Id) &= -\frac{1}{\alpha^5} \xi^{(3)}(\alpha) \xi + \frac{1}{\alpha^5} \xi(\alpha) \xi^{(2)}(\alpha) \xi + \frac{9}{\alpha^6} \xi(\alpha) \xi^{(2)}(\alpha) \xi \\
 &\quad - \frac{6m}{\alpha^5} \xi^{(2)}(\alpha) \xi - \frac{3}{\alpha^6} (\xi(\alpha))^3 \xi - \frac{12}{\alpha^7} (\xi(\alpha))^3 \xi \\
 &\quad + \frac{18m}{\alpha^6} (\xi(\alpha))^2 \xi + \frac{4m}{\alpha^5} (\xi(\alpha))^2 \xi + \frac{2m}{\alpha^3} \xi(\alpha) \xi \\
 &\quad - \frac{8m^2}{\alpha^5} \xi(\alpha) \xi + 4m^2 \xi - \frac{4m^2}{\alpha^2} \xi.
 \end{aligned}$$

To complete the proof, it remains to investigate the term $Tr_{\bar{g}} R(\bar{\tau}(Id), \cdot) \cdot$, we have

$$Tr_{\bar{g}} R(\bar{\tau}(Id), \cdot) \cdot = \frac{2m}{\alpha^3} \xi(\alpha) \xi + \frac{4m^2(\alpha^2 - 1)}{\alpha^2} \xi.$$

Going back to equation (7), we conclude that $Id : (M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ is biharmonic if and only if

$$\begin{aligned} &\alpha^2 \xi^{(3)}(\alpha) - \alpha(\alpha + 9)\xi(\alpha)\xi^{(2)}(\alpha) + 6m\alpha^2 \xi^{(2)}(\alpha) \\ &+ 3(\alpha + 4)(\xi(\alpha))^3 - 2m\alpha(2\alpha + 9)(\xi(\alpha))^2 \\ &- 4m\alpha^2(\alpha^2 - 2m)\xi(\alpha) - 8m^2\alpha^5(\alpha^2 - 1) = 0. \end{aligned}$$

As a consequence of the Theorem 2, we will present some examples.

Example 1. We consider the manifold $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold with a φ -basis

$$e_1 = e^{-z} \frac{\partial}{\partial x}, \quad e_2 = e^{-z} \frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z}.$$

The Riemannian metric on M is defined by

$$g = e^{2z} dx^2 + e^{2z} dy^2 + dz^2,$$

The vector fields e_1, e_2 and e_3 satisfy

$$\begin{aligned} \nabla_{e_1} e_1 &= -\xi, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= e_1, \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= -\xi, & \nabla_{e_2} e_3 &= e_2, \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

If we suppose that the function α depends only on z , we deduce that identity map $Id : (M^3, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^3, \varphi, \xi, \eta, g)$ is harmonic if and only if α is a solution of the following differential equation

$$\alpha' + 2\alpha(\alpha^2 - 1) = 0.$$

then $Id : (M, \varphi, \xi, \eta, g) \longrightarrow (M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is harmonic if and only if

$$\alpha = \pm \frac{e^{2z}}{\sqrt{e^{4z} - C}},$$

and we obtain

$$\overline{g} = g + \left(\frac{C}{e^{4z} - C} \right) \eta \otimes \eta.$$

By Theorem 2, the identity map $Id : (M^3, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^3, \varphi, \xi, \eta, g)$ is

biharmonic if and only if α is a solution of the following differential equation

$$\begin{aligned} & \alpha^2 \alpha^{(3)} - \alpha(\alpha + 9) \alpha' \alpha'' + 6\alpha^2 \alpha'' \\ & + 3(\alpha + 4) (\alpha')^3 - 2\alpha(2\alpha + 9) (\alpha')^2 \\ & - 4\alpha^2 (\alpha^2 - 2) \alpha' - 8\alpha^5 (\alpha^2 - 1) = 0. \end{aligned}$$

Example 2. We consider the Kenmotsu manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinate in \mathbb{R}^5 . An orthonormal frame is given by $e_1 = e^{-v} \frac{\partial}{\partial x}$, $e_2 = e^{-v} \frac{\partial}{\partial y}$ and $e_3 = e^{-v} \frac{\partial}{\partial z}$, $e_4 = e^{-v} \frac{\partial}{\partial u}$ and $e_5 = e^{-v} \frac{\partial}{\partial v}$. Taking $e_5 = \xi$ and using Koszul's formula we get the following

$$\begin{array}{ccccc} \nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1 \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2 \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3 \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -e_5, & \nabla_{e_4} e_5 = e_4 \\ \nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0. \end{array}$$

If we suppose that the function α depends only on v , we deduce that the identity map Id is harmonic if and only if the function α is a solution of the following differential equation

$$e^{-v} \alpha' + 4\alpha (\alpha^2 - 1) = 0.$$

The general solution of this equation is

$$\alpha = \pm \frac{e^{4e^v}}{\sqrt{e^{8e^v} - C}}.$$

Then we obtain

$$\bar{g} = g + \left(\frac{C}{e^{8e^v} - C} \right) \eta \otimes \eta.$$

By Theorem 2, the identity map $Id : (M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g}) \rightarrow (M, \varphi, \xi, \eta, g)$ is biharmonic if and only if α is a solution of the following differential equation

$$\begin{aligned} & \alpha^2 \xi^{(3)}(\alpha) - \alpha(\alpha + 9) \xi(\alpha) \xi^{(2)}(\alpha) + 12\alpha^2 \xi^{(2)}(\alpha) + 3(\alpha + 4) (\xi(\alpha))^3 \\ & - 4\alpha(2\alpha + 9) (\xi(\alpha))^2 - 8\alpha^2 (\alpha^2 - 2m) \xi(\alpha) - 32\alpha^5 (\alpha^2 - 1) = 0. \end{aligned}$$

2.2. The biharmonicity of $Id : (M^{2m+1}, \varphi, \xi, \eta, g) \longrightarrow (M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$

As a last result of this paper, we will characterize the harmonicity and the biharmonicity of the identity map $Id : (M^{2m+1}, \varphi, \xi, \eta, g) \longrightarrow (M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$. We obtain the following theorem.

Theorem 3. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold and let $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a \mathcal{D} -isometric deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$. Then $Id : (M^{2m+1}, \varphi, \xi, \eta, g) \longrightarrow (M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is harmonic if and only if*

$$\alpha \xi(\alpha) + 2m(\alpha^2 - 1) = 0,$$

and it is biharmonic if and only if

$$\begin{aligned} &\alpha^4 \xi^{(3)}(\alpha) + 2m\alpha^2(2\alpha^2 + 1)\xi^{(2)}(\alpha) - 2m\alpha(\xi(\alpha))^2 \\ &+ 4m(m\alpha^4 + (2m - 1)\alpha^2 - m)\xi(\alpha) - 8m^2\alpha(\alpha^2 - 1) = 0. \end{aligned}$$

Proof of Theorem 3. By definition, the tension field of the identity map $Id : (M^{2m+1}, \varphi, \xi, \eta, g) \longrightarrow (M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is given by

$$\bar{\tau}(Id) = \bar{\nabla}_{e_i} e_i - \nabla_{e_i} e_i + \bar{\nabla}_{\varphi e_i} \varphi e_i - \nabla_{\varphi e_i} \varphi e_i + \bar{\nabla}_{\xi} \xi - \nabla_{\xi} \xi.$$

Using the following equations

$$\begin{aligned} \bar{\nabla}_{e_i} e_i &= \nabla_{e_i} e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi, \\ \bar{\nabla}_{\varphi e_i} \varphi e_i &= \nabla_{\varphi e_i} \varphi e_i + \frac{m(\alpha^2 - 1)}{\alpha^2} \xi, \\ \nabla_{\xi} \xi &= 0 \end{aligned}$$

and

$$\bar{\nabla}_{\xi} \xi = \frac{1}{\alpha} \xi(\alpha) \xi,$$

we deduce that

$$\bar{\tau}(Id) = \frac{1}{\alpha} \xi(\alpha) \xi + \frac{2m(\alpha^2 - 1)}{\alpha^2} \xi.$$

Then $Id : (M^{2m+1}, \varphi, \xi, \eta, g) \longrightarrow (M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is harmonic if and only if

$$\alpha \xi(\alpha) + 2m(\alpha^2 - 1) = 0.$$

The identity map $Id : (M^{2m+1}, \varphi, \xi, \eta, g) \longrightarrow (M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is biharmonic if and only if

$$Tr_g \bar{\nabla}^2 \bar{\tau}(Id) + Tr_g \bar{R}(\bar{\tau}(Id), \cdot) \cdot = 0.$$

For the term $Tr_g \bar{\nabla}^2 \bar{\tau}(Id)$, we have

$$Tr_g \bar{\nabla}^2 \bar{\tau}(Id) = Tr_g \bar{\nabla}^2 \frac{1}{\alpha} \xi(\alpha) \xi + 2m Tr_g \bar{\nabla}^2 \frac{\alpha^2 - 1}{\alpha^2} \xi.$$

Using Proposition 4, we get

$$\begin{aligned} Tr_g \bar{\nabla}^2 \frac{1}{\alpha} \xi(\alpha) \xi &= \frac{1}{\alpha^2} \xi(\alpha) \xi^{(2)}(\alpha) \xi + \xi^{(2)} \left(\frac{1}{\alpha} \xi(\alpha) \right) \xi + \frac{2}{\alpha} \xi(\alpha) \xi \left(\frac{1}{\alpha} \xi(\alpha) \right) \xi \\ &\quad + \frac{2m}{\alpha^2} (\xi(\alpha))^2 \xi + 2m \xi \left(\frac{1}{\alpha} \xi(\alpha) \right) \xi - \frac{2m}{\alpha^3} \xi(\alpha) \xi. \end{aligned}$$

It is very easy to see that

$$\xi \left(\frac{1}{\alpha} \xi(\alpha) \right) = \frac{1}{\alpha} \xi^{(2)}(\alpha) - \frac{1}{\alpha^2} (\xi(\alpha))^2$$

and

$$\begin{aligned} \xi^{(2)} \left(\frac{1}{\alpha} \xi(\alpha) \right) &= \xi \left(\frac{1}{\alpha} \xi^{(2)}(\alpha) \right) - \xi \left(\frac{1}{\alpha^2} (\xi(\alpha))^2 \right) \\ &= \frac{1}{\alpha} \xi^{(3)}(\alpha) - \frac{3}{\alpha^2} \xi(\alpha) \xi^{(2)}(\alpha) + \frac{2}{\alpha^3} (\xi(\alpha))^3. \end{aligned}$$

It follows that

$$Tr_g \bar{\nabla}^2 \frac{1}{\alpha} \xi(\alpha) \xi = \frac{1}{\alpha} \xi^{(3)}(\alpha) \xi + \frac{2m}{\alpha} \xi^{(2)}(\alpha) \xi - \frac{2m}{\alpha^3} \xi(\alpha) \xi.$$

Likewise, we have

$$\begin{aligned} Tr_g \bar{\nabla}^2 \frac{\alpha^2 - 1}{\alpha^2} \xi &= \frac{\alpha^2 - 1}{\alpha^3} \xi^{(2)}(\alpha) \xi + \xi^{(2)} \left(\frac{\alpha^2 - 1}{\alpha^2} \right) \xi + \frac{2}{\alpha} \xi(\alpha) \xi \left(\frac{\alpha^2 - 1}{\alpha^2} \right) \xi \\ &\quad + \frac{2m(\alpha^2 - 1)}{\alpha^3} \xi(\alpha) \xi + 2m \xi \left(\frac{\alpha^2 - 1}{\alpha^2} \right) \xi - \frac{2m(\alpha^2 - 1)}{\alpha^4} \xi. \end{aligned}$$

A similar calculation leads us to the following results

$$\xi \left(\frac{\alpha^2 - 1}{\alpha^2} \right) = \frac{2}{\alpha^3} \xi(\alpha)$$

and

$$\xi^{(2)} \left(\frac{\alpha^2 - 1}{\alpha^2} \right) = \frac{2}{\alpha^3} \xi^{(2)}(\alpha) - \frac{6}{\alpha^4} (\xi(\alpha))^2$$

which gives us

$$\begin{aligned} Tr_g \bar{\nabla}^2 \frac{\alpha^2 - 1}{\alpha^2} \xi &= \frac{\alpha^2 + 1}{\alpha^3} \xi^{(2)}(\alpha) \xi - \frac{2}{\alpha^4} (\xi(\alpha))^2 \xi \\ &\quad + \frac{2m(\alpha^2 + 1)}{\alpha^3} \xi(\alpha) \xi - \frac{2m(\alpha^2 - 1)}{\alpha^4} \xi. \end{aligned}$$

We conclude that

$$\begin{aligned} Tr_g \bar{\nabla}^2 \bar{\tau}(Id) &= \frac{1}{\alpha} \xi^{(3)}(\alpha) \xi + \frac{2m(2\alpha^2 + 1)}{\alpha^3} \xi^{(2)}(\alpha) \xi - \frac{4m}{\alpha^4} (\xi(\alpha))^2 \xi \\ &\quad + \frac{4m^2(\alpha^2 + 1) - 2m}{\alpha^3} \xi(\alpha) \xi - \frac{4m^2(\alpha^2 - 1)}{\alpha^4} \xi. \end{aligned}$$

Finally, we complete the proof by calculating the term $Tr_g \bar{R}(\bar{\tau}(Id), \cdot) \cdot$. By proposition 5, we obtain

$$\begin{aligned} Tr_g \bar{R}(\bar{\tau}(Id), \cdot) \cdot &= Tr_g \bar{R} \left(\frac{1}{\alpha} \xi(\alpha) \xi + \frac{2m(\alpha^2 - 1)}{\alpha^2} \xi, \cdot \right) \\ &= \left(\frac{1}{\alpha} \xi(\alpha) + \frac{2m(\alpha^2 - 1)}{\alpha^2} \right) Tr_g \bar{R}(\xi, \cdot) \\ &= \left(\frac{1}{\alpha} \xi(\alpha) + \frac{2m(\alpha^2 - 1)}{\alpha^2} \right) \left(\frac{2m}{\alpha^3} \xi(\alpha) \xi - \frac{2m}{\alpha^2} \xi \right) \\ &= + \frac{2m}{\alpha^4} (\xi(\alpha))^2 \xi + \frac{2m((2m - 1)\alpha^2 - 2m)}{\alpha^5} \xi(\alpha) \xi \\ &\quad - \frac{4m^2(\alpha^2 - 1)}{\alpha^4} \xi. \end{aligned}$$

The biharmonicity of the identity map $Id : (M^{2m+1}, \varphi, \xi, \eta, g) \longrightarrow$

$(M^{2m+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is characterized by the following differential equation

$$\begin{aligned} \alpha^4 \xi^{(3)}(\alpha) + 2m\alpha^2(2\alpha^2 + 1) \xi^{(2)}(\alpha) - 2m\alpha(\xi(\alpha))^2 \\ + 4m(m\alpha^4 + (2m - 1)\alpha^2 - m) \xi(\alpha) - 8m^2\alpha(\alpha^2 - 1) = 0. \end{aligned}$$

Example 3. We consider the Kenmotsu manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$,

where (x, y, z, u, v) are the standard coordinate in \mathbb{R}^5 . An orthonormal frame is given by $e_1 = e^{-v} \frac{\partial}{\partial x}$, $e_2 = e^{-v} \frac{\partial}{\partial y}$ and $e_3 = e^{-v} \frac{\partial}{\partial z}$, $e_4 = e^{-v} \frac{\partial}{\partial u}$ and $e_5 = e^{-v} \frac{\partial}{\partial v}$. Taking $e_5 = \xi$ and using Koszul's formula we get the following

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_5, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= 0, & \nabla_{e_1} e_4 &= 0, & \nabla_{e_1} e_5 &= e_1 \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= -e_5, & \nabla_{e_2} e_3 &= 0, & \nabla_{e_2} e_4 &= 0, & \nabla_{e_2} e_5 &= e_2 \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= -e_5, & \nabla_{e_3} e_4 &= 0, & \nabla_{e_3} e_5 &= e_3 \\ \nabla_{e_4} e_1 &= 0, & \nabla_{e_4} e_2 &= 0, & \nabla_{e_4} e_3 &= 0, & \nabla_{e_4} e_4 &= -e_5, & \nabla_{e_4} e_5 &= e_4 \\ \nabla_{e_5} e_1 &= 0, & \nabla_{e_5} e_2 &= 0, & \nabla_{e_5} e_3 &= 0, & \nabla_{e_5} e_4 &= 0, & \nabla_{e_5} e_5 &= 0. \end{aligned}$$

If we suppose that the function α depends only on v , we deduce that the identity map Id is harmonic if and only if the function α is a solution of the following differential equation

$$e^{-v} \alpha \alpha' + 4(\alpha^2 - 1) = 0.$$

The general solution of this equation is

$$\alpha = \pm e^{-4e^v} \sqrt{A + e^{8e^v}}, \quad A \in \mathbb{R}_+.$$

Then $Id : (M, \varphi, \xi, \eta, g) \longrightarrow (M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g} = g + Ae^{-8e^v} \eta \otimes \eta)$ is harmonic. By Theorem 3, we deduce that $Id : (M, \varphi, \xi, \eta, g) \longrightarrow (M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is biharmonic if and only if

$$\begin{aligned} e^v \alpha^4 \alpha^{(3)} + 2e^v \alpha^2 (3\alpha^2 + 2) \alpha'' + e^v (9\alpha^4 + 20\alpha^2 - 16) \alpha' \\ - 4\alpha (\alpha')^2 - 32e^{2v} \alpha (\alpha^2 - 1) = 0. \end{aligned}$$

Example 4. We consider the Kenmotsu manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$. The Riemannian metric on M is defined by

$$g = e^{2z} dx^2 + e^{2z} dy^2 + dz^2,$$

and the orthonormal frame is given by $e_1 = e^{-z} \frac{\partial}{\partial x}$, $e_2 = \varphi e_1 = e^{-z} \frac{\partial}{\partial y}$ and $e_3 = \xi = \frac{\partial}{\partial z}$. The vector fields e_1, e_2 and e_3 satisfy

$$\begin{aligned} \nabla_{e_1} e_1 &= -\xi, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} \xi &= e_1, \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= -\xi, & \nabla_{e_2} \xi &= e_2, \\ \nabla_{\xi} e_1 &= 0, & \nabla_{\xi} e_2 &= 0, & \nabla_{\xi} \xi &= 0. \end{aligned}$$

If we suppose that the function α depends only on v , we deduce that the identity map Id is harmonic if and only if the function α is a solution of the following differential equation

$$\alpha\alpha' + 2(\alpha^2 - 1) = 0.$$

The general solution of this equation is

$$\alpha = \pm e^{-2z} \sqrt{A + e^{4z}}, \quad A \in \mathbb{R}_+.$$

Then $Id : (M, \varphi, \xi, \eta, g) \longrightarrow (M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g} = g + Ae^{-4z}\eta \otimes \eta)$ is harmonic. By Theorem 3, we deduce that $Id : (M, \varphi, \xi, \eta, g) \longrightarrow (M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is biharmonic if and only if

$$\alpha^4\alpha^{(3)} + 2\alpha^2(2\alpha^2 + 1)\alpha'' - 2\alpha(\alpha')^2 + 4(\alpha^4 + \alpha^2 - 1)\alpha' - 8\alpha(\alpha^2 - 1) = 0.$$

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