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SOLVABILITY OF NONLINEAR HYPERBOLIC EQUATION WITH NONLINEAR INTEGRAL NEUMANN CONDITIONS

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Abstract

In this paper, we examine a nonlinear hyperbolic equation with a nonlinear integral condition, where we prove the existence and the uniqueness of the linear problem by the Faedo-Galerkin method. By applying an iterative process to some significant results obtained for the linear problem, the existence and the uniqueness of the weak solution for the nonlinear problem are additionally examined.

1. Introduction and Position of the Problem

Nonlinear hyperbolic partial differential equations on a finite interval rather than on the whole real line arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium or polymer rheology. Most applications of partial differential equations involve domains with boundaries, and it is important to specify data correctly at these locations. The conditions relating the solution to the differential equation to data at boundary are called boundary conditions. A more complete discussion of the theory of boundary conditions for time

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dependent partial differential equations. The problem of determining a solution to a differential equation when both initial data and boundary data are presented is called an initial-boundary value problem. Boundary conditions that determine a unique solution are said to be well-posed. Because of that, we focus on the nonlinear integral conditions which is more complicated for the integral condition [3], [4] and [5], where it's hard to directly measure the minimum and maximum values on the border, however, the overall value or average is known. This method might be utilized for modeling where we can model more complicated domain with nonlinear integral condition.

Motivated by the above works, in this paper we investigate the existence and the uniqueness of the weak solution of nonlinear hyperbolic equation with nonlinear integral condition where we based our demonstration into two parts. The first one we prove the existence and the uniqueness of the linear problem. In the second section, by applying an iterative process to the results obtained for the linear problem, the existence and the uniqueness of the weak solution for the nonlinear problem are additionally examined.

2. Solvability of the Solution of Linear Hyperbolic Problem with Integral Condition of Second Type by the Method of Faedo-Galerkin

Let $Q = \left\{(x,t) \in \mathbb{R}^2,\ x \in \Omega =]0,l[\text{ and } 0 < t < T\right\}$. Consider the following nonlinear problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a\dfrac{\partial^2 u}{\partial x^2} + bu = f(x,t,u) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \\ \dfrac{\partial u}{\partial x}(0,t) = \displaystyle\int_0^l k(x,t)g(u)(x,t)dx \\ \dfrac{\partial u}{\partial x}(l,t) = \displaystyle\int_0^l k(x,t)h(u)(x,t)dx \end{cases}, \tag{P_1}$$

Assume that $f, \varphi \in L^2(Q)$.

3. Position of the linear problem (P_1)

In the rectangular area $Q = \Omega \times (0, T)$, and $T < \infty$. Consider the following linear problem (P_2)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a\dfrac{\partial^2 u}{\partial x^2} + bu = f(x,t) & \forall\, (x,t) \in Q \\ u\,(x,0) = \varphi\,(x) & \forall x \in (0,l) \\ u_t\,(x,0) = \psi\,(x) & \forall x \in (0,l) \\ \dfrac{\partial u}{\partial x}(0,t) = \displaystyle\int_0^l k(x,t)g(u)(x,t)dx & \forall t \in (0,T) \\ \dfrac{\partial u}{\partial x}(l,t) = \displaystyle\int_0^l k(x,t)g(u)(x,t)dx & \forall t \in (0,T) \end{cases}, \qquad (P_2)$$

whose hyperbolic equation is given as follows

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} + bu = f(x,t), \tag{1}$$

with the initial conditions

$$\ell u = \begin{cases} u(x,0) = \varphi\,(x) \\ u_t\,(x,0) = \psi\,(x) \end{cases}, \qquad x \in (0,l)\,,$$

and the integral condition of the second type

$$\begin{cases} \dfrac{\partial u}{\partial x}(0,t) &= \int_0^l k(x,t)g(u)(x,t)dx, & t \in (0,T) \\ \dfrac{\partial u}{\partial x}(l,t) &= \int_0^l k(x,t)h(u)(x,t)dx, & t \in (0,T) \end{cases},$$

where

$$k(x,t) \geq 0, \;\; \forall\; (x,t) \in Q, \text{ and } g(u)(x,t) \leq h(u)(x,t)\; \forall\; (x,t) \in Q.$$

We define a space V by:

$$V = \left\{ u \in H^1\,(\Omega) \right\}.$$

The space V provided with the norm $\|v\|_V = \|v\|_{H^1(\Omega)}$ is a Hilbert space. We are now able to formulate the problem (P_2), precisely to study it, we will need the following hypothesis :

$$(H):\begin{cases} f\in L^2\left(0,T;L^2\left(\Omega\right)\right) & (H.1)\\ \varphi\in H^1\left(\Omega\right) & (H.2)\end{cases}.$$

Definition 1. The weak solution to the problem (P_2) is a function that checks:

(i) $u\in L^2\left(0,T;H^1\left(\Omega\right)\right)\cap L^\infty\left(0,T;H^1\left(\Omega\right)\right)$.

(ii) u admits a strong derivative $\frac{\partial u}{\partial t}\in L^2\left(0,T;L^2\left(\Omega\right)\right)$.

(iii) $u\left(0\right)=\varphi, u_t\left(0\right)=\psi$.

(iv) Identity

$$\left(u_{tt},v\right)+a\left(u_x,v_x\right)+b\left(u,v\right)=\left(f,v\right)+u_x(l,t)v(l)-u_x(0,t)v(0)\qquad \forall v\in V,\ \forall t\in\left[0,T\right].$$

3.1. Variational formulation

By multiplying the equation

$$\frac{\partial^2 u}{\partial t^2}-a\frac{\partial^2 u}{\partial x^2}+bu=f\left(x,t\right). \tag{2}$$

By an element $v\in V$, and integrating it over Ω, we obtain:

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\cdot vdx-a\int_\Omega \frac{\partial^2 u}{\partial x^2}\cdot vdx+b\int_\Omega u\cdot vdx=\int_\Omega f\cdot vdx. \tag{3}$$

By using the boundary conditions and using Green's formula, (3) becomes

$$\left(u_{tt},v\right)+a\left(u_x,v_x\right)+b\left(u,v\right)=\left(f,v\right)+u_x(l,t)v(l)-u_x(0,t)v(0)\quad \forall v\in V, \tag{4}$$

where $(\cdot,\cdot)$ denotes the scalar product on $L^2\left(\Omega\right)$.

3.2. Study of the existence of weak solution of the problem ($\mathbf{P}_2$)

The demonstration of the existence of the solution of the problem (P_2) is based on the Faedo-Galerkin method which consists of carrying out the following three steps:

3.2.1. Step 1: Construction of the approximate solutions

The space V is separable, then there exists a sequence $w_1, w_2, \ldots, w_m$, having the following properties:

$$\begin{cases} w_i \in V, & \forall i, \\ \forall m, w_1, w_2, \ldots, w_m & \text{are linearly independent}, \\ V_m = \langle \{w_1, w_2, \ldots, w_m\} \rangle & \text{is dense in } V. \end{cases} \tag{5}$$

In particular :

$$\forall \varphi \in V \implies \exists (\alpha_{km})_m \in IN^*,\ \varphi_m = \sum_{k=1}^{m} \alpha_{km} w_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \tag{6}$$

$$\forall \psi \in V \implies \exists (\beta_{km})_m \in IN^*,\ \psi_m = \sum_{k=1}^{m} \beta_{km} w_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \tag{7}$$

Faedo-Galerkin's approximation consists of searching for any integer $m \geq 1$, a function

$$t \mapsto u_m(x,t) = \sum_{i=1}^{m} g_{im}(t)\, w_i(x)$$

verifies

$$\begin{cases} u_m(t) \in V_m, & \forall t \in [0,T] \\ ((u_m(t))_{tt}, w_k) + A(u_m(t), w_k) = (f(t), w_k) & \forall k = \overline{1,m} \end{cases}. \tag{P_3}$$

We have

$$\begin{aligned} ((u_m(t))_{tt}, w_k) &= \left(\left(\sum_{i=1}^{m} g_{im}(t)\, w_i \right)_{tt}, w_k \right) \\ &= \left(\sum_{i=1}^{m} \frac{\partial^2 g_{im}}{\partial t}(t)\, w_i(x), w_k \right) \\ &= \sum_{i=1}^{m} (w_i, w_k) \frac{\partial^2 g_{im}}{\partial t^2}(t), \end{aligned} \tag{8}$$

and

$$\begin{aligned} A\left(u_m(t), w_k\right) =& A\left(\sum_{i=1}^{m} g_{im}\left(t\right) w_i, w_k\right) \\ =& a\sum_{i=1}^{m} g_{im}\left(t\right)\left[\int_{\Omega} \frac{\partial w_i}{\partial x}\frac{\partial w_k}{\partial x}dx - \frac{\partial w_i}{\partial x}\left(l\right) w_k(l) + \frac{\partial w_i}{\partial x}\left(0\right) w_k(0)\right] \\ =& a\sum_{i=1}^{m} g_{im}\left(t\right)\int_{\Omega} \frac{\partial w_i\left(x\right)}{\partial x}\frac{\partial w_k(x)}{\partial x}dx - a\sum_{i=1}^{m} g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(l\right) w_k(l) \\ & + a\sum_{i=1}^{m} g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(0\right) w_k(0) \\ =& \sum_{i=1}^{m} A\left(w_i, w_k\right) g_{im}\left(t\right). \end{aligned} \tag{9}$$

Also, we have

$$\begin{aligned} u_m(0) &= \sum_{i=1}^{m} g_{im}\left(0\right) w_i\left(x\right) \\ &= \varphi_m \\ &= \sum_{i=1}^{m} \alpha_{im} w_i(x). \end{aligned}$$

and

$$\begin{aligned} u_m'(0) &= \sum_{i=1}^{m} g_{im}'\left(0\right) w_i\left(x\right) \\ &= \beta_m \\ &= \sum_{i=1}^{m} \beta_{im} w_i(x). \end{aligned}$$

We obtain a system of second order nonlinear differential equations :

$$\begin{cases} \sum\limits_{i=1}^{m}\left(w_i, w_k\right)\frac{\partial^2}{\partial t^2} g_{im}\left(t\right) + a\sum\limits_{i=1}^{m}\left(\frac{\partial w_i}{\partial x}, \frac{\partial w_k}{\partial x}\right) g_{im}\left(t\right) \\ = \left(f(t), w_k\right) + a\sum\limits_{i=1}^{m} g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(l\right) w_k(l) - a\sum\limits_{i=1}^{m} g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(0\right) w_k(0) \\ g_{im}\left(0\right) = \alpha_{im} \qquad \forall i = \overline{1,m}. \\ g_{im}'\left(0\right) = \beta_{im} \qquad \forall i = \overline{1,m}. \end{cases} \tag{P_4}$$

We consider the vector

$$g_m = (g_{1m}(t), \dots, g_{mm}(t)), f_m = ((f, w_1), \dots, (f, w_m))$$

and the matrix

$$B_m = ((w_i, w_j))_{\substack{1\le i\le m\\ 1\le j\le m}}, A_m = \left(\left(\frac{\partial w_i}{\partial x}, \frac{\partial w_j}{\partial x}\right)\right)_{\substack{1\le i\le m\\ 1\le j\le m}}$$

and

$$C_m = \left(\frac{\partial w_i}{\partial x}(l)\cdot w_j(l)\right)_{\substack{1\le i\le m\\ 1\le j\le m}}, D_m = \left(\frac{\partial w_i}{\partial x}(0)\cdot w_j(0)\right)_{\substack{1\le i\le m\\ 1\le j\le m}}.$$

We write the problem (P_4) in the matrix form, we obtain :

$$\begin{cases} B_m \frac{\partial g_m}{\partial t}(t) + aA_m g_m + aD_m g_m = f_m + aC_m g_m \\ g_m(0) = (\alpha_{im})_{1\le i\le m} \\ g'_m(0) = (\beta_{im})_{1\le i\le m} \end{cases}$$

As the matrix entries B_m are linearly independent (because it is a diagonal matrix), $\det B_m \neq 0$, so it is invertible, then g_m is the solution to

$$\begin{cases} \frac{\partial^2 g_m}{\partial t^2}(t) + \left(aB_m^{-1}A_m + bB_m^{-1}D_m - aB_m^{-1}C_m\right) g_m = B_m^{-1} f_m \\ g_m(0) = (\alpha_{im})_{1\le i\le m}. \\ g'_m(0) = (\beta_{im})_{1\le i\le m}. \end{cases} \qquad (P_5)$$

It is easy that we can verify that this ordinary differential systems have a solution, where the matrices $\left(aB_m^{-1}A_m + bB_m^{-1}D_m - aB_m^{-1}C_m\right)$ with constant coefficients and the vector $B_m^{-1} f_m$ with continuous function are majorized by integrable fuctions on $(0, T)$. Then we can conclude that there exists a t_m depends only on $|\alpha_{im}|, |\beta_{im}|$.

3.2.2. Step 2: A priori estimate

Lemma 1. *For all $m \in \mathbb{N}\backslash\{0\}$ and if*

$$\frac{1}{8a} > k^2,$$

the solution $u_m \in L^2(0,T\ ;V_m)$ to the problem (P_2) checks

$$\begin{aligned} \|u_m\|_{L^2(0,T;H^1(\Omega))} &\leq c_1 \\ \left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(0,T;\ L^2(\Omega))} &\leq c_2 \end{aligned}$$

where $c_{1,}c_2$ are two positive constants independent of m.

Proof. Multiply the equation of (P_3) by $g'_{km}(t)$ and we sum over k, we find

$$\begin{aligned} &\sum_{k=1}^{m}\left((u_m(t))_{tt}\,,w_k\right)\cdot g'_{km}(t)+a\sum_{k=1}^{m}\left(\frac{\partial^2 u_m}{\partial x^2}(t),w_k\right)\cdot g'_{km}(t) \\ &\quad= \sum_{k=1}^{m}\left(f(t),w_k\right)\cdot g'_{km}(t)+a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(l\right)\sum_{k=1}^{m}g'_{km}(t)w_k(l) \\ &\qquad -a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(0\right)\sum_{k=1}^{m}g'_{km}(t)w_k(0). \end{aligned}$$

So, we obtain

$$\begin{aligned} &\left((u_m(t))_{tt}\,,(u_m(t))_t\right)-a\left(\frac{\partial^2 u_m}{\partial x^2}(t),\frac{u_m}{\partial t}(t)\right) \\ &\quad= \left(f(t),(u_m(t))_t\right)+a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(l\right)\sum_{k=1}^{m}g'_{km}(t)w_k(l) \\ &\qquad -a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(0\right)\sum_{k=1}^{m}g'_{km}(t)w_k(0). \end{aligned}$$

Thus, we get

$$\begin{aligned} &\left((u_m(t))_{tt}\,,(u_m(t))_t\right)+\frac{a}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} \\ &\quad= \left(f(t),u_m(t)\right)+a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(l\right)\sum_{k=1}^{m}g'_{km}(t)w_k(l) \\ &\qquad -a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(0\right)\sum_{k=1}^{m}g'_{km}(t)w_k(0). \end{aligned}$$

Integrating over 0 to t, by using the Cauchy inequality with ε, $(|ab| \leq \frac{a^2}{2\varepsilon}+$

$\frac{\varepsilon b^2}{2}$), we get

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla u_m\right\|^2_{L^2(\Omega)} \\
\leq &\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + a\sum_{i=1}^{m} g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(l\right)\sum_{k=1}^{m} g'_{km}(t)w_k(l) \\
&- a\sum_{i=1}^{m} g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(0\right)\sum_{k=1}^{m} g'_{km}(t)w_k(0) + \frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla \psi_m\right\|^2_{L^2(\Omega)}, \\
\leq &\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right|^2_{L^2(Q)} + a\frac{\partial u_m}{\partial x}(l,t)\frac{\partial u_m}{\partial t}(l,t) - a\frac{\partial u_m}{\partial x}(0,t)\frac{\partial u_m}{\partial t}(0,t), \\
\leq &\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla \psi_m\right\|^2_{L^2(\Omega)} \\
&+ a\int_0^{\tau}\left[\left(\int_{\Omega} k(x,t)g\left(u_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(l,t)\right. \\
&\left.- \left(\int_{\Omega} k(x,t)h\left(u_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(0,t)\right] \\
\leq &\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla \psi_m\right\|^2_{L^2(\Omega)} \\
&+ a\int_0^{\tau}\left[\left(\int_{\Omega} k(x,t)g\left(u_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(l,t)\right. \\
&\left.- \left(\int_{\Omega} k(x,t)h\left(u_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(0,t)\right] \\
\leq &\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla \psi_m\right\|^2_{L^2(\Omega)} \\
&+ a\int_0^{\tau}\left[\left(\int_{\Omega} k(x,t)g\left(u_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(l,t)\right. \\
&\left.- \left(\int_{\Omega} k(x,t)g\left(u_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(0,t)\right] \\
\leq &\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right|^2_{L^2(Q)} + ak\left\|\frac{\partial u_m}{\partial t}\right\|_{L^{\infty}(0,T;L^2(\Omega))}\left[\int_0^{\tau}\int_{\Omega}\frac{\partial^2 u_m}{\partial t\partial x}dxdt\right] \\
&+ \frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla \psi_m\right\|^2_{L^2(\Omega)} \\
\leq &\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}
\end{aligned}
$$

$$
\begin{aligned}
&+ak\left\|\frac{\partial u_m}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))}\left[\int_\Omega \frac{\partial u_m}{\partial x}dx-\int_\Omega \frac{\partial \psi_m}{\partial x}dx\right]\\
&+\frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)}+\frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)}\\
\leq&\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}+\frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)}+\frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)}\\
&+\left[\frac{(ak)^2}{2\delta}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\delta\left[\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\left\|\nabla\varphi_m\right\|^2_{L^\infty(0,T;L^2(\Omega))}\right]\right]\\
\leq&\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}+\frac{(ak)^2}{2\delta}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&+\delta\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)}+\max(\frac{1}{2},\delta)\left\|\varphi_m\right\|^2_{H^1(\Omega)},\\
\leq&\frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)}+\left(\frac{\varepsilon C_T}{2}+\frac{(ak)^2}{2\delta}\right)\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\delta\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&+\frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)}+\max(\frac{1}{2},ak\delta)\left\|\varphi_m\right\|^2_{H^1(\Omega)},
\end{aligned}
\tag{10}
$$

where the constant $K=\max\int_Q k^2(x,t)\,dxdt$. Then, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)}+\frac{a}{2}\left\|\nabla u_m\right\|^2_{L^2(\Omega)}\\
&\leq\ \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)}+\left(\frac{\varepsilon C_T}{2}+\frac{(ak)^2}{2\delta}\right)\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&\quad+\delta\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)}+\max(\frac{1}{2},\delta)\left\|\varphi_m\right\|^2_{H^1(\Omega)},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\left(\frac{1}{2}-\left(\frac{\varepsilon C_T}{2}+\frac{(ak)^2}{2\delta}\right)\right)\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\left(\frac{a}{2}-\delta\right)\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&\leq\ \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)}+\frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)}+\max(\frac{1}{2},\delta)\left\|\varphi_m\right\|^2_{H^1(\Omega)}.
\end{aligned}
$$

By putting $\varepsilon=\frac{1}{4C_T},\delta=4\left(ak\right)^2$. So, we get

$$
\begin{aligned}
&\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&\leq C_1\left[\left\|f\right\|^2_{L^2(Q)}+\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)}+\left\|\varphi_m\right\|^2_{H^1(\Omega)}\right],
\end{aligned}
\tag{12}
$$

or

$$C_1 = \frac{\max\left(2C_T, \frac{a}{2}, \max(\frac{1}{2}, (ak)^2)\right)}{\min\left\{\frac{1}{4}, \left(\frac{a}{2} - 4\,(ak)^2\right)\right\}}.$$

From (12) we also get this estimate

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega)} \leq \sqrt{C_1}\left[\|f\|^2_{L^2(Q)} + \|\nabla\psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)}\right]^{\frac{1}{2}} \tag{13}$$

then we integrate (13)it over $[0,T]$, we find

$$\begin{aligned}
\left\|\int_0^T \frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega)} \leq \int_0^T \left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega)} &\leq T\sqrt{C_1}\left[\|f\|^2_{L^2(Q)} + \|\nabla\psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)}\right]^{\frac{1}{2}} \\
\|u_m - \varphi_m\|_{L^2(\Omega)} &\leq T\sqrt{C_1}\left[\|f\|^2_{L^2(Q)} + \|\nabla\psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)}\right]^{\frac{1}{2}} \\
\left|\|u_m\|_{L^2(\Omega)} - \|\varphi_m\|_{L^2(\Omega)}\right| &\leq T\sqrt{C_1}\left[\|f\|^2_{L^2(Q)} + \|\nabla\psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)}\right]^{\frac{1}{2}} \\
\left(\|u_m\|_{L^2(\Omega)} - \|\varphi_m\|_{L^2(\Omega)}\right)^2 &\leq TC_1\left[\|f\|^2_{L^2(Q)} + \|\nabla\psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)}\right] \\
\|u_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{L^2(\Omega)} &\leq TC_1\left[\|f\|^2_{L^2(Q)} + \|\nabla\psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)}\right] \\
&\quad + 2\,\|u_m\|_{L^2(\Omega)}\,\|\varphi_m\|_{L^2(\Omega)}
\end{aligned}$$

By applying the Cauchy inequality with γ, ($|ab| \leq \frac{a^2}{2\gamma} + \frac{\gamma b^2}{2}$)

$$\begin{aligned}
\|u_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{L^2(\Omega)} \;&\leq\; TC_1\left[\|f\|^2_{L^2(Q)} + \|\nabla\psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)}\right] \\
&\quad + \frac{1}{\gamma}\,\|u_m\|^2_{L^2(\Omega)} + \gamma\,\|\varphi_m\|^2_{L^2(\Omega)}
\end{aligned} \tag{14}$$

By putting $\gamma = 2$, finally we get

$$\|u_m\|^2_{L^2(\Omega)} \leq 4TC_1\left[\|f\|^2_{L^2(Q)} + \|\nabla\psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)}\right]. \tag{15}$$

It follows from (12) and (15) that the solution to the initial value problem

for the system of ODE (P_4) can be extended to $[0,T]$. This confirms what we have demonstrated in the first step. We obtained

$$\begin{cases} u_m \text{ uniformly bounded in } L^2(0,T;\ H^1(\Omega)) \\ (u_m)_t \text{ uniformly bounded in } L^2\left(0,T;\ L^2(\Omega)\right) \end{cases}. \tag{16}$$

□

3.2.3. Step 3: Convergence and result of existence

Theorem 1. *There is a function* $u \in L^2(0,T;\ H^1(\Omega)) \cap L^\infty(0,T;\ L^2(\Omega))$ *with* $\frac{\partial u}{\partial t} \in L^2\left(0,T;\, L^2(\Omega)\right)$ *and a subsequence denoted by* $(u_{m_k})_k \subseteq (u_m)_m$, *such that*

$$\begin{cases} u_{m_k} \rightharpoonup u & \text{in } L^2(0,T;\ H^1(\Omega)) \\ \frac{\partial u_{m_k}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \text{in } L^2\left(0,T;\ L^2(\Omega)\right) \end{cases},$$

when $m \longrightarrow +\infty$.

Proof. We deduce from Lemma (1.2) there are subsequences denoted by (u_{m_k}), $\left(\frac{\partial u_{m_k}}{\partial t}\right)$ of (u_m) and $(u_m)_t$ respectively, such that

$$u_{m_k} \rightharpoonup u \quad \text{in} \quad L^2(0,T;\ H^1(\Omega)). \tag{17}$$

$$\frac{\partial u_{m_k}}{\partial t} \rightharpoonup w \quad \text{in} \quad L^2\left(0,T;\ L^2(\Omega)\right). \tag{18}$$

We know that according to Relikh-Kondrachoff's theorem that the injection of $H^1(Q)$ into $L^2(Q)$ is compact. And like the results of Rellich's theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence which converges strongly in $L^2(Q)$. So

$$u_{m_k} \longrightarrow u \qquad \text{in } L^2(Q)\ . \tag{19}$$

On the one hand, from Lemma (1.3) there is a subsequence of $(u_{m_k})_k$ is still denoted by u_{m_k} converges almost everywhere to u, such that

$$u_{m_k} \longrightarrow u \qquad \text{almost everywhere } Q\ . \tag{20}$$

It remains to demonstrate that $w = \frac{\partial u}{\partial t}$, for that it suffices to prove :

$$u(t) = \varphi + \int_0^t w(\tau)d\tau. \tag{21}$$

As

$$u_{m_k} \rightharpoonup u \qquad \text{in } L^2(0,T\ ;\ L^2(\Omega))\ ,$$

then, the proof of (21) is equivalent to demonstrate that

$$u_{m_k} \rightharpoonup \varphi + \chi \qquad \text{in } L^2(0,T\ ;\ L^2(\Omega))\ ,$$

which means

$$\lim\,(u_{m_k} - \varphi - \chi, v)_{L^2(0,T\ ;\ L^2(\Omega))} = 0,\ \forall v \in L^2(0,T\ ;\ L^2(\Omega)),$$

as

$$\chi(t) = \int_0^t w(\tau)d\tau.$$

Using equality

$$u_{m_k} - \varphi_{m_k} = \int_0^t \frac{\partial u_{m_k}}{\partial \tau}d\tau,\ \text{for all } t \in [0,T],$$

which results from $u_{m_k} \in L^2(0,T;V_{m_k})$ and $(u_{m_k})_t \in L^2(0,T;V_{m_k})$ that

$$\begin{aligned}
&\left(u_{m_k} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\ ;\ L^2(\Omega))} \\
=&\left(u_{m_k} - \varphi_{m_k} - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\ ;\ L^2(\Omega))} + (\varphi_{m_k} - \varphi, v)_{L^2(0,T\ ;\ L^2(\Omega))} \\
=&\left(\int_0^t \Big(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau)\Big)d\tau, v\right)_{L^2(0,T\ ;\ L^2(\Omega))} + (\varphi_{m_k} - \varphi, v)_{L^2(0,T\ ;\ L^2(\Omega))}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } t \in [0,T],
\end{aligned}$$

by virtue of **(ii)** of Lemma (1.6), it comes

$$\left(u_{m_k} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\ ;\ L^2(\Omega))}$$

$$= \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v \right)_{L^2(0,T\,;\,L^2(\Omega))} d\tau + (\varphi_{m_k} - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))},$$

$$\text{for all } t \in [0,T].$$

On the other hand, we have

$$\lim_{k \longrightarrow \infty} \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v \right)_{L^2(0,T\,;\,L^2(\Omega))} d\tau = 0, \text{ for } t \in [0,T]. \tag{22}$$

Also, we have

$$\lim_{k \longrightarrow \infty} (\varphi_m - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))} = 0. \tag{23}$$

So we get

$$\lim_{k \longrightarrow \infty} (u_{m_k} - \varphi - \chi, v)_{L^2(0,T\,;\,L^2(\Omega))} = 0, \quad \forall v \in L^2\left(0,T; L^2(\Omega)\right). \qquad \square$$

Theorem 2. *The function u of the theorem* (1) *is the weak solution to the problem* (P_2) *in the sense of the definition* (1).

Proof. From the theorem (1), we have shown that the limit function u satisfies the first two conditions of the definition (1). Now we will demonstrate (**iii**). According to the theorem (1), we have

$$u_{m_k}(0) \rightharpoonup u(0) \quad \text{in} \quad L^2(\Omega).$$

On the other hand, we have

$$u_{m_k}(0) \longrightarrow \varphi \quad \text{in} \quad L^2(\Omega),$$

so

$$u_{m_k}(0) \rightharpoonup \varphi \quad \text{in} \quad L^2(\Omega).$$

From the uniqueness of the limit, we get

$$u(0) = \varphi.$$

By using the same steps we demonstrate $u_t(0) = \psi$. It remains to demonstrate (**iv**) :

$$(u_{tt}, v) + a(u, v) = (f, v) \qquad \forall v \in V, \text{ and } \forall t \in [0, T].$$

Integrating (P_3) over $(0, T)$, we find

$$\int_0^t ((u_m(t))_{tt}, w_k)\, d\tau + \int_0^t a(u_m(t), w_k)\, d\tau = \int_0^t (f(t), w_k)\, d\tau$$
$$\forall k = \overline{1, m}, \text{ and } \forall t \in [0, T]. \qquad (24)$$

We know that V_m dense in V and passing to the limit in (24), we find

$$\int_0^T (u_{tt}, w_k)\, d\tau + \int_0^T a(u, w_k)\, d\tau = \int_0^T (f, w_k)\, d\tau, \qquad \forall t \in [0, T],$$

so (**iv**) is verified. □

Corollary 1. *The uniqueness of the solution of problem* (P_2) *comes straight through the estimate* (12).

4. Weak Solution of the Nonlinear Problem

First, we propose the concept of the studied solution. Let $v = v(x, t)$ be any function of V, such that

$$V = \left\{ v \in C^1(Q),\ v_x(l, t) = v_x(0, t) = 0,\ t \in [0, T] \right\}.$$

Multiplying

$$\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} = f(x, t, u, u_x)$$

by v and integrate it over Q_τ, we find

$$\int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt - a \int_{Q_\tau} \Delta y(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt$$
$$= \int_{Q_\tau} G(x, t, y, y_x) \cdot \frac{\partial v}{\partial t}(x, t) dx dt,$$

and using an integration by parts and the conditions on y, v we get

$$\int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x,t)\cdot\frac{\partial v}{\partial t}(x,t)dxdt + a\int_{Q_\tau}\frac{\partial y}{\partial x}(x,t)\cdot\frac{\partial^2 v}{\partial x\partial t}(x,t)dxdt$$
$$= \int_{Q_\tau} G(x,t,y,y_x)\cdot\frac{\partial v}{\partial t}(x,t)dxdt, \tag{25}$$

it then results from(25) that

$$A(y,v) = \int_{Q_\tau} G(x,t,y,y_x)\cdot\frac{\partial v}{\partial t}(x,t)dxdt, \tag{26}$$

or

$$A(y,v) \ = \int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x,t)\cdot\frac{\partial v}{\partial t}(x,t)dxdt + a\int_{Q_\tau}\frac{\partial y}{\partial x}(x,t)\cdot\frac{\partial^2 v}{\partial x\partial t}(x,t)dxdt.$$

Building a recurrent sequence starting with $y^{(0)} = 0$. The sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined as follows: given the element $y^{(n-1)}$, then for $n = 1, 2, 3, \ldots$ we will solve the following problem

$$\begin{cases} \frac{\partial^2 y^{(n)}}{\partial t^2} - a\Delta y^{(n)} = G\left(x,t,y^{(n-1)},y_x^{(n-1)}\right) \\ y^{(n)}(x,0) = 0 \\ y_t^{(n)}(x,0) = 0 \\ y_x^{(n)}(0,t) = 0 \\ y_x^{(n)}(l,t) = 0 \end{cases}. \tag{P_4}$$

According to the study of the previous linear problem each time we fix the n, the problem (P_4) admits a unique solution $y^{(n)}(x,t)$ which is given by the Faedo-Galerkin method. Now we suppose

$$z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t),$$

so we get a new problem :

$$\begin{cases} \frac{\partial^2 z^{(n)}}{\partial t^2} - a\Delta z^{(n)} = p^{(n-1)}(x,t) \\ z^{(n)}(x.0) = 0 \\ z_t^{(n)}(x.0) = 0 \\ z_x^{(n)}(0,t) = 0 \\ z_x^{(n)}(l,t)\,dx = 0 \end{cases}, \qquad (P_5)$$

or

$$p^{(n-1)}(x,t) = G\left(x,t,y^{(n)},y_x^{(n)}\right) - G\left(x,t,y^{(n-1)},y_x^{(n-1)}\right).$$

Multiplying

$$\frac{\partial^2 z^{(n)}}{\partial t^2} - a\Delta z^{(n)} = p^{(n-1)}(x,t)$$

by $z^{(n)}$, and integrating it on Q_τ, we obtain :

$$\int_{Q_\tau} \frac{\partial^2 z^{(n)}}{\partial t^2}(x,t) \cdot \frac{\partial z^{(n)}}{\partial t}(x,t)dxdt - a\int_{Q_\tau} \Delta z^{(n)}(x,t) \cdot \frac{\partial z^{(n)}}{\partial t}(x,t)\,dxdt$$
$$= \int_{Q_\tau} p^{(n-1)}(x,t) \cdot \frac{\partial z^{(n)}}{\partial t}(x,t)\,dxdt.$$

Let us use an integration by parts for each term by taking account of the initial condition and the boundary conditions, we find :

$$\frac{1}{2}\int_\Omega (\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2 dx + \frac{a}{2}\int_\Omega \left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$= \int_{Q_\tau} p^{(n-1)}(x,t) \cdot \frac{\partial z^{(n)}}{\partial t}(x,t)\,dxdt.$$

We apply the Cauchy Schwarz inequality on the second part of the equation, we obtain :

$$\frac{1}{2}\int_\Omega (\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2 dx + \frac{a}{2}\int_\Omega \left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$

$$\leqslant \frac{1}{2\varepsilon}\int_{Q^\tau} \mid p^{(n-1)}(x,t)\mid^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2 dxdt$$

$$\frac{1}{2}\int_\Omega (\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2 dx + \frac{a}{2}\int_\Omega \left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dx$$

$$\leqslant \frac{1}{2\varepsilon}\int_{Q^\tau} \mid G\left(x,t,y^{(n)},y_x^{(n)}\right) - G\left(x,t,y^{(n-1)},y_x^{(n-1)}\right)\mid^2 dxdt$$

$$+\frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2 dxdt,$$

Like G is Lipschitz function, we find :

$$\leqslant \frac{k^2}{2\varepsilon}\int_{Q^\tau}(\mid y^{(n)}-y^{(n-1)}\mid+\mid y_x^{(n)}-y_x^{(n-1)}\mid)^2 dxdt+\frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2 dxdt,$$

$$\leqslant \frac{k^2}{2\varepsilon}\int_{Q^\tau}(\mid z^{(n-1)}\mid + \mid z_x^{(n-1)}\mid)^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2 dxdt,$$

$$\leqslant \frac{k^2}{\varepsilon}\int_{Q^\tau}(\mid z^{(n-1)}\mid^2 + \mid z_x^{(n-1)}\mid^2) dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2 dxdt,$$

$$\leqslant \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))} + \frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2 dxdt,$$

We multiply by 2 and apply Grenwell's Lemma, we get

$$\int_\Omega (\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2 dx + a\int_\Omega \left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dx$$

$$\leqslant \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))} + \varepsilon\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2 dxdt,$$

$$\leq \frac{k^2}{\varepsilon}\exp(\varepsilon T)\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))}$$

We integrate on t, we obtain :

$$\int_{Q_T} (\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2 dxdt + a\int_{Q_T} \left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$

$$\leqslant \frac{Tk^2}{\varepsilon} \left\| z^{(n-1)} \right\|_{L^2(0,T,H^1(0,l))} \exp(\varepsilon T)$$

$$\int_{Q_T} \left(\frac{\partial z^{(n)}}{\partial t}(x,\tau)\right)^2 dxdt + \int_{Q_T} \left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$

$$\leqslant \frac{Tk^2 \exp(\varepsilon T)}{\varepsilon \min(1,a)} \left\| z^{(n-1)} \right\|_{L^2(0,T,H^1(0,1))}$$

putting

$$c = \frac{Tk^2 \exp(\varepsilon T)}{\varepsilon \min(1,a)}$$

so

$$\| \frac{\partial z^{(n)}}{\partial t} \|^2_{L^2(0,T,H^1(0,1))} + \| \frac{\partial z^{(n)}}{\partial x} \|^2_{L^2(0,T,H^1(\Omega))} \leqslant c \left\| z^{(n-1)} \right\|^2_{L^2(0,T,H^1(\Omega))};$$

Then, we apply Poincaré inequality, we have

$$\| z^{(n)} \|^2_{L^2(0,T,H^1(\Omega))} \leqslant Tc \left\| z^{(n-1)} \right\|^2_{L^2(0,T,H^1(\Omega))}.$$

$$\sum_{i=1}^{n-1} z^{(i)} = y^{(n)}.$$

According to the convergence criterion of the series $\sum_{n=1}^{\infty} z^{(n)}$ converges if $|c| < 1$, which implies :

$$\left| \frac{(Tk)^2 \exp(\varepsilon T)}{\varepsilon \min(1,a)} \right| < 1 \quad \Rightarrow \quad Tk\sqrt{\frac{\exp(\varepsilon T)}{\varepsilon \min(1,a)}} < 1,$$

So, we get

$$k < \sqrt{\frac{\varepsilon \min(1,a) \exp(-\varepsilon T)}{T}}.$$

Then $(y^{(n)})_n$ converges to an element of $L^2(0,T,H^1(\Omega))$, we call y. We will show that

$$\lim_{n \longrightarrow \infty} y^{(n)}(x,t) = y(x,t)$$

is a solution to the problem (P_5) by showing that y satisfies :

$$A\left(y,v\right)=\int\limits_{Q_\tau} G(x,t,y,y_x)\cdot v(x,t)dxdt.$$

We therefore consider the weak formulation of the problem (P_1) following :

$$A\left(y^{(n)},v\right)=\int\limits_{Q_\tau}\frac{\partial^2 y^{(n)}}{\partial t^2}(x,t)\cdot\frac{\partial v}{\partial t}(x,t)dxdt+\frac{a}{2}\int\limits_{Q_\tau}\frac{\partial y^{(n)}}{\partial x}(x,t)\cdot\frac{\partial^2 v}{\partial t\partial x}(x,t)dxdt.$$

From the linearity of A, we have :

$$\begin{aligned}A\left(y^{(n)},v\right)&=A\left(y^{(n)}-y,v\right)+A\left(y,v\right)\\&=\int\limits_{Q_\tau}\frac{\partial^2(y^{(n)}-y)}{\partial t^2}(x,t)\cdot\frac{\partial v}{\partial t}(x,t)dxdt\\&\quad+\frac{a}{2}\int\limits_{Q_\tau}\frac{\partial(y^{(n)}-y)}{\partial x}(x,t)\cdot\frac{\partial^2 v}{\partial t\partial x}(x,t)dxdt\\&\quad-\int\limits_{Q_\tau}\frac{\partial^2 y}{\partial t^2}(x,t)\cdot\frac{\partial v}{\partial t}(x,t)dxdt+\frac{a}{2}\int\limits_{Q_\tau}\frac{\partial y}{\partial x}(x,t)\cdot\frac{\partial^2 v}{\partial t\partial x}(x,t)dxdt,\end{aligned}$$

So

$$\begin{aligned}A\left(y^{(n)}-y,v\right)=&\int\limits_{Q_\tau}\frac{\partial^2(y^{(n)}-y)}{\partial t^2}(x,t)\cdot\frac{\partial v}{\partial t}(x,t)dxdt\\&+\frac{a}{2}\int\limits_{Q_\tau}\frac{\partial(y^{(n)}-y)}{\partial x}(x,t)\cdot\frac{\partial^2 v}{\partial t\partial x}(x,t)dxdt,\end{aligned}$$

We apply the Cauchy Schwartz inequality, and we find :

$$\begin{aligned}A\left(y^{(n)}-y,v\right)\leqslant&\parallel v_t\parallel_{L^2(Q_\tau)}\left\|(y^{(n)}-y)_{tt}\right\|_{L^2(0,T,L^2(\Omega))}\\&+\frac{a}{2}\parallel v_{xt}\parallel_{L^2(Q_\tau)}\left\|(y^{(n)}-y)_x\right\|_{L^2(0,T,L^2(\Omega))}.\end{aligned}$$

Then, we find

$$A\left(y^{(n)}-y,v\right)\leqslant C(\left\|(y^{(n)}-y)_{tt}\right\|_{L^2(0,T,L^2(\Omega))}$$

$$+\left\|(y^{(n)}-y)_x\right\|_{L^2(0,T,L^2(\Omega))})(\| v_t \|_{L^2(Q_\tau)}+\| v_{xt} \|_{L^2(Q_\tau)}),$$

or

$$C=\max(1,\frac{a}{2}).$$

As

$$y^{(n)}\longrightarrow y \qquad \text{in } L^2\left(0,T,H^1\left(0,l\right)\right)\approxeq H^1\left(Q\right),$$

we have

$$\begin{aligned} y^{(n)} &\longrightarrow y && \text{in } L^2\left(Q\right), \\ y_t^{(n)} &\longrightarrow y_t && \text{in } L^2\left(Q\right), \\ y_x^{(n)} &\longrightarrow y_x && \text{in } L^2\left(Q\right), \end{aligned}$$

Let's go to the limit when $n\longrightarrow+\infty$, we find

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)}-y,v\right)=0.$$

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