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MINIMAL SOLUTION OF IRREGULAR BARRIER REFLECTED BDSDEs WITH LEFT CONTINUOUS AND STOCHASTIC LINEAR GROWTH GENERATORS

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Abstract

In this paper, we deal with reflected backward doubly stochastic differential equations (RBDSDEs in short) with one rcll reflecting barrier when the coefficient f satisfies a stochastic Lipschitz condition, via penalization method we prove the existence and uniqueness of solutions. The comparison theorem is also established. Via an inf-convolution approximation and comparison theorem, we show the existence of a minimal solution to the RBDSDE under continuous and stochastic linear growth condition, also we provide a minimal solution to RBDSDE with left continuous and stochastic linear growth condition.

1. Introduction

In 1994 Pardoux and Peng [10] found a class of backward doubly stochastic differential equations (BDSDEs for abbreviation) of the form

$$\begin{cases} -dY_t = f\left(t, Y_t, Z_t\right) dt + g\left(t, Y_t, Z_t\right) d\overleftarrow{B}_t - Z_t dW_t, \qquad 0 \leq t < T, \\ Y_T = \xi. \end{cases}$$

with a backward stochastic integral $d\overleftarrow{B}_t$ and a forward stochastic integral dW_t, where ξ is a square integrable random variable and f is an progressively measurable process, a so-called driver, the authors proved the result of existence and uniqueness solution under uniformly Lipschitz conditions, their

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goal was to give a probabilistic interpretation of a solution to some quasi-linear stochastic partial differential equation. After almost twenty years Shi et al [16] provided a comparison theorem for this kind of equation, they applied the principle of comparison to prove that there is a minimal solution for BDSDE with continuous coefficient. After several years, Bahlali et al [1] prove the existence and uniqueness of solutions for reflected backward doubly stochastic differential equations (RBDSDEs in short) with continuous reflecting barrier and Lipschitz coefficients of the form

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)\, dt + g(t, Y_t, Z_t)\, d\overleftarrow{B}_t - Z_t dW_t, \qquad 0 \le t < T, \\ Y_T = \xi, \\ Y_t \ge L_t \ a.s. \text{ for any} \qquad t \in [0, T]. \end{cases}$$

The role of the continuous and increasing process $(K_t)_{t\in[0,T]}$ is to push upward the process Y in order to keep it above L, it satisfies the Skorokhod condition

$$\int_0^T (Y_s - L_s)\, dK_s = 0,$$

also, by using the inf-convolution approximation the authors showed that there is a minimal solution of backward doubly stochastic differential equation with one continuous reflecting barrier when the driver f is continuous with linear growth see [14] and [9]. If $g(s, Y_s, Z_s) := 0$, the RBDSDEs convert to reflected backward stochastic differential equations (RBSDEs in short), this type of equations has been studied in 1997 by El Karoui et al [4].

In 2002, Hamadène [5] studies the RBSDEs with one right continuous and left limited barrier. By using Snell envelope notion, the author proved the existence and the uniqueness of the solution in the case where the generator f was Lipschitz and presented the comparison theorem between two solutions and he used this last theorem to find an existence result of solutions for RBSDEs under continuous and linear growth assumptions. In this article we mainly deal with the existence and uniqueness result of the solution for RBDSDEs with rcll reflecting barrier. The difficulty of the task is related to the non-positive jumps of process L which express that the solution Y component of the solution may have non-positive jumps and hence no continuous but only rcll. Via a penalization method we prove the existence and uniqueness solution when the generator f is stochastic Lipschitz with respect to (y, z) . Also by inf-convolution approximation and comparison theorem, we show the existence result of this kind of equation when the driver f satisfying the continuous and stochastic linear growth condition.

This paper is organized as follows. In section 2, we present the basic concepts that we will use throughout this work that help us solve our main problem. The third section is devoted to study the BDSDE with one rcll reflecting barrier when the driver f is stochastic Lipschitz with respect to (y, z), we show that this type of equation admits a unique solution, in the proof we use the penalization technique. In section 4, we try to compare two solutions of RBDSDE with rcll barrier if we compare the barriers as well as the generator f in the same sense. The fifth section is devoted to proving the existence solution of RBDSDE with rcll barrier, but in the case where the coefficient f satisfies the continuity with stochastic linear growth, in the proof, we have approximated f by a sequence of functions f_n which are stochastic Lipschitz. Finally in the last section, we studied the existence of the minimal solution for BDSDE with rcll reflecting barrier and left continuous and stochastic linear growth condition.

2. Setting of the Problem

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $T > 0$, let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independent standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively. Let $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)$ and $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T)$, completed with P-null sets. We put, $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, it should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub $\sigma-$fields, and hence it is not a filtration. For each $t \in [0, T]$, we define

$$\mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B,$$

the collection $(\mathcal{G}_t)_{t\in[0,T]}$ is a filtration. Also, we define an non-decreasing process $\{A_t\}_{t\geq 0}$ such that $A_t = \int_0^t a_s^2 ds < \infty$, where $\{a_t^2\}_{t\geq 0}$ is a stochastic process with values in $\mathbb{R}_+$ such that a_t^2 is $\mathcal{F}_t$-measurable for $a.e\ t \geq 0$

Now, for any k, $d \geq 1$ and $\delta > 0$, we consider the following spaces of precesses:

- $\mathcal{A}^2$ is the set of rcll and increasing, $\mathcal{F}_t$-progressively measurable process $K : [0,T] \times \Omega \to [0,+\infty($ with $K_0 = 0$ and $\mathbb{E}\left(K_T\right)^2 < +\infty$.
- $\mathbb{L}^2\left(\delta, \mathcal{F}_T, \mathbb{R}\right)$ is the set of $\mathcal{F}_T$- measurable random variables $\xi : \Omega \to \mathbb{R}$ with $\mathbb{E}e^{\delta A_T}\left|\xi\right|^2 < +\infty$.
- $\mathcal{M}_\delta^2\left(A,T,\mathbb{R}^d\right)$ is the set of $\mathcal{F}_t-$progressively measurable stochastic processes $\{\vartheta_t; t \in [0,T]\}$, such that

$$||\vartheta||_{\mathcal{M}_\delta^2} = \mathbb{E}\int_0^T e^{\delta A_t}\left|\vartheta_t\right|^2 dt < \infty.$$

- $\mathcal{M}_\delta^{2,a}\left(A,T,\mathbb{R}^d\right)$ is the set of jointly measurable processes $\{\vartheta_t; t \in [0,T]\}$, such that ϑ_t is $\mathcal{F}_t-$measurable for a.e.$t \in [0,T]$ with

$$||\vartheta||_{\mathcal{M}_\delta^{2,a}} = \mathbb{E}\int_0^T a_t^2 e^{\delta A_t}\left|\vartheta_t\right|^2 dt < \infty.$$

- We denote by $\mathcal{S}_\delta^2\left(A,T,\mathbb{R}^d\right)$, the set of rcll and $\mathcal{F}_t-$measurable stochastic processes $\{\vartheta_t; t \in [0,T]\}$, which satisfy

$$||\vartheta||_{\mathcal{S}_\delta^2} = \mathbb{E}\left(\sup_{0\le t\le T} e^{\delta A_t}\left|\vartheta_t\right|^2\right) < \infty.$$

- For $\psi = (\psi_t)_{t\le T} \in \mathcal{S}_\delta^2\left(A,T,\mathbb{R}^d\right)$, $\psi_- = (\psi_{t-})_{t\le T}$ is a process such that $\forall t \in]0,T]$, $\psi_- = \lim_{s\nearrow t}\psi_s$, $\psi_{0-} = \psi_0$ and $\Delta_t\psi = \psi_t - \psi_{t-}$.

For simplicity, we use the following notation for spaces of processes:

$$\mathcal{H}_\delta^2\left(a,T\right) = \mathcal{M}_\delta^{2,a}\left(A,T,\mathbb{R}^d\right) \times \mathcal{M}_\delta^2\left(A,T,\mathbb{R}^{k\times d}\right),$$

with the norm

$$||(Y,Z)||_{\mathcal{H}_\delta^2} = ||Y||_{\mathcal{M}_\delta^{2,a}} + ||Z||_{\mathcal{M}_\delta^2}.$$

Also, we denote the space

$$\mathcal{H}_{\delta,c}^2\left(a,T\right) = \left(\mathcal{S}_\delta^2\left(A,T,\mathbb{R}^d\right) \cap \mathcal{M}_\delta^{2,a}\left(A,T,\mathbb{R}^d\right)\right) \times \mathcal{M}_\delta^2\left(A,T,\mathbb{R}^{k\times d}\right),$$

with the norm

$$||(Y,Z)||_{\mathcal{H}_{\delta,c}^2} = ||Y||_{\mathcal{S}_\delta^2} + ||Y||_{\mathcal{M}_\delta^{2,a}} + ||Z||_{\mathcal{M}_\delta^2}.$$

Finally, $\mathcal{H}_{\delta,c}^2\left(a,T\right)$ endowed with the norm $\left|\left|\left(\bar{\vartheta},\tilde{\vartheta}\right)\right|\right|_{\mathcal{H}_{\delta,c}^2}$ is a Banach space.

Remark 1. If $(a(t))_{t\geq 0}$ and $(b(t))_{t\geq 0}$ are two random processes with positive values such that $a(t)$ and $b(t)$are $\mathcal{F}_t-$measurable for $a.e.$ $t\in[0,T]$, with $b>a$, then

$$\mathcal{M}_{\delta}^{2,b}\left(A,T,\mathbb{R}^d\right)\subset\mathcal{M}_{\delta}^{2,a}\left(A,T,\mathbb{R}^d\right),$$

Therefore,

$$\mathcal{H}_{\delta,c}^{2}\left(b,T\right)\subset\mathcal{H}_{\delta,c}^{2}\left(a,T\right).$$

In this work, we will mainly be interested to study the following RBDSDE

$$\begin{cases} Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T g(s,Y_s,Z_s)d\overleftarrow{B}_s+\int_t^T dK_s-\int_t^T Z_s dW_s, \\ 0\leq t\leq T,\ L_t\leq Y_t,\ \forall\, t\in[0,T], \\ \text{the Skorokhod condition:} \\ \text{i) } \int_0^T (Y_t-L_t)\,dK_t^c=0,\ \text{ where } K^c \text{ is the continuous part of } K, \\ \text{ii) if } K^d \text{ is the discontinuous part of } K,\ \text{ then } K^d \text{ is predictable} \\ \quad \text{and } K_t^d=\sum_{0<s<t}(Y_{s-}-L_{s-})^{-}\,\mathbf{1}_{\{\Delta L_s<0\}}. \end{cases} \tag{1}$$

In this situation, the jumping times of the process Y come from the process L. Then the process K is rcll.

Remark 2. If $K=K^c+K^d$ where K^c (resp. K^d) is the non-discontinuous (resp. purely discontinuous) part of K, then K^d is predictable. Therefore, K^d only works when the process Y has a predictable jump that occurs at a predictable positive jump point of L. This means that

$$\Delta_t K=-\Delta_t Y=-\left(Y_{t-}-L_{t-}\right)^{-}\mathbf{1}_{\{L_{t-}=Y_{t-}\}}.$$

Remark 3. The following condition is equivalent to the Skorokhod condition defined in (1)

$$\int_0^T\left(Y_{t-}-L_{t-}\right)dK_t=0.$$

Indeed

$$\begin{aligned}\int_0^T\left(Y_{t-}-L_{t-}\right)dK_t &=\int_0^T\left(Y_t-L_t\right)dK_t^c+\int_0^T\left(Y_{t-}-L_{t-}\right)dK_t^d,\\ &=\int_0^T\left(Y_t-L_t\right)dK_t^c+\sum_{0\leq t\leq T}\left(Y_{t-}-L_{t-}\right)^{-}\mathbf{\Delta}K_t^d\end{aligned}$$

$$= \int_0^T (Y_t - L_t)\, dK_t^c + \sum_{0\le t\le T} \left|(Y_{t-} - L_{t-})^-\right|^2 \mathbf{1}_{\{\Delta L_s<0\}}$$
$$= 0.$$

3. Irregular Barrier Reflected BDSDE with Stochastic Lipschitz Coefficient

In this section, we mainly deal with the existence and uniqueness solution for RBDSDEs with one rcll reflecting barrier (1) under stochastic Lipschitz conditions.

3.1. Assumptions and definition

Assume the coefficient $f\colon \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$, $g\colon \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l}$ and the terminal value $\xi\colon \Omega \to \mathbb{R}^k$.

For $\delta > 0$ and $(k,d) \in (\mathbb{N}^*)^2$, we say that the data (ξ, f, g, L) satisfies assumptions **(H1)** if the following holds:

(H1.1) f and g are jointly measurable and there exist theree non-negative processes $\{r(t)\}_{0\le t\le T}$, $\{\theta(t)\}_{0\le t\le T}$, $\{\nu(t)\}_{0\le t\le T}$ and a constant $0 < \alpha < 1$, such that:

(1) For any $0 \le t \le T$, $r(t)$, $\theta(t)$ and $\nu(t)$ are $\mathcal{F}_t^W$-measurable.

(2) For all $0 \le t \le T$, $(y, \acute{y}) \in \mathbb{R}^k \times \mathbb{R}^k$ and $(z, \acute{z}) \in (\mathbb{R}^{k\times d})^2$, we have

$$\begin{cases} |f(t,y,z) - f(t,\acute{y},\acute{z})| \le r(t)\,|y-\acute{y}| + \theta(t)\,|z-\acute{z}|, \\ |g(t,y,z) - g(t,\acute{y},\acute{z})|^2 \le \nu(t)\,|y-\acute{y}|^2 + \alpha\,|z-\acute{z}|^2. \end{cases}$$

(H1.2) For all $0 \le t \le T$, $a_t^2 = r(t) + \theta^2(t) + \nu(t) > 0$, and $A(t) = \int_0^t a_s^2 ds < \infty$.

(H1.3) The barrier $L\colon = (L_t)_{t\ge 0}$ is a consist of $\mathcal{F}_t-$progressively measurable real valued process and rcll processes satisfying that

$$\begin{cases} \text{(i) } \mathbb{E}\left(\sup_{0\le t\le T} e^{2\delta A_t} \left|L_t^+\right|^2\right) < +\infty, \quad \text{with} \quad L_t^+ := \max(L_t, 0), \\ \text{and} \\ \text{(ii) } L_T \le \xi,\ \mathbb{P} \text{ almost surely.} \end{cases}$$

(H1.4) The integrability condition holds

$$\mathbb{E}\left(\int_0^T e^{\delta A_t}\Big(\frac{|f(t)|^2}{a_t^2}+|g(t)|^2\Big)dt\right)<\infty,$$

where $\gamma(t):=\gamma(t,0,0)$ with $\gamma=\{f,\ g\}$.

(H1.5) $\xi\in\mathbb{L}^2(\delta,\mathcal{F}_T,\mathbb{R})$.

Now, let us give a definition of the solution of this reflected BDSDEs.

Definition 1. A solution of equation (1) is a $(\mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathbb{R}_+)$-valued $\mathcal{F}_t-$progressively measurable process (Y,Z,K) which satisfies (2.1) and such that $(Y,Z,K)\in\mathcal{H}^2_{\delta,c}(a,T)\times\mathcal{A}^2$.

3.2. Uniqueness

In this subsection, we show the uniqueness of the solutions to reflected BDSDEs. For that, let (Y^1,Z^1,K^1) and (Y^2,Z^2,K^2) be two solutions of the RBDSDE (1) with data (ξ,f,g,L).

Proposition 1. *Assume that* **(H1.1)-(H1.3)** *hold. Then there exists at most one triplet* (Y,Z,K) *solution of the RBDSDE* (1).

Proof. If (Y^1,Z^1,K^1) and (Y^2,Z^2,K^2) be two solutions of RBDSDE (1). Using the fact that $\int_t^T e^{\delta A_s}\left(Y^1_{s-}-Y^2_{s-}\right)\left(g\left(s,Y^1_s,Y^1_s\right)-g\left(s,Y^2_s,Y^2_s\right)\right)d\overleftarrow{B}_s$ and $\int_t^T e^{\delta A_s}\left(Y^1_{s-}-Y^2_{s-}\right)\left(Z^1_s-Z^2_s\right)dW_s$ are two martingales with zero expectation, then by Itô's formula for discontinuous semimartingales and **(H1.1)**, we obtain

$$\begin{aligned}
&e^{\delta A_t}\left|Y^1_t-Y^2_t\right|^2+\delta\int_t^T e^{\delta A_s}\left|Y^1_s-Y^2_s\right|^2dA_s+\int_t^T e^{\delta A_s}\left|Z^1_s-Z^2_s\right|^2ds\\
&\quad+\sum_{t<s\le T}e^{\delta A_s}\left(\Delta_sY^1-\Delta_sY^2\right)^2\\
&\le 2\mathbb{E}\int_t^T e^{\delta A_s}\left(Y^1_{s-}-Y^2_{s-}\right)\left(r(s)\left|Y^1_s-Y^2_s\right|+\theta(s)\left|Z^1_s-Z^2_s\right|\right)ds\\
&\quad+\mathbb{E}\int_t^T e^{\delta A_s}\left(\nu(s)\left|Y^1_s-Y^2_s\right|^2+\alpha\left|Z^1_s-Z^2_s\right|\right)ds\\
&\quad+2\mathbb{E}\int_t^T e^{\delta A_s}\left(Y^1_{s-}-Y^2_{s-}\right)\left(dK^1_s-dK^2_s\right).
\end{aligned}\tag{2}$$

Also thanks to Skorokhod condition (see, e.g., [5, Theorem 1.3, p. 587] for more details), we obtain

$$\int_t^T e^{\delta A_s}\left(Y_{s-}^1 - Y_{s-}^2\right)\left(dK_s^1 - dK_s^2\right) \leq 0,$$

then, by Burkholder-Davis-Gundy inequality we get

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\delta A_t}\left|Y_t^1 - Y_t^2\right|^2 + \int_0^T e^{\delta A_s}\left(\sigma_1\left|Y_s^1 - Y_s^2\right|^2 dA_s + \sigma_2\left|Z_s^1 - Z_s^2\right|^2 ds\right)\right) \leq 0,$$

with $\sigma_1 = \delta - 2 - \frac{1}{\epsilon}$ and $\sigma_2 = 1 - \epsilon - \alpha$. Choosing ϵ, $\delta > 0$ such that $\sigma_1 > 0$ and $\sigma_2 > 0$, we have

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\delta A_t}\left|Y_t^1 - Y_t^2\right|^2 + \int_0^T e^{\delta A_s}\left|Z_s^1 - Z_s^2\right|^2 ds\right) = 0.$$

Finally, we get $Y_t^1 = Y_t^2$ and $Z_t^1 = Z_t^2$ $\mathbb{P}$ $a.s$. □

3.3. Existence results via penalization method

In this subsection, we will prove the existence of a solution to reflected BDSDE (1) via the penalization technic and the fixed point.

3.3.1. The independent case

Our main goal in this part of the subsection is to show that the RBDSDE (1) has a solution when the coefficients f, g does not depend on (y, z), i.e., $\mathbb{P} - a.s.$, $f(t, y, z) = f(t)$ and $g(t, y, z) = g(t)$, for any t, y and z.

Theorem 1. *Under* **(H1.3)-(H1.5)**. *There exists a triplet of processes* $(Y, Z, K) \in \mathcal{H}^2_{\delta,c}(a, T) \times \mathcal{A}^2$ *solve the following reflected BDSDEs with one* *rcll reflecting barriers*

$$\begin{cases} Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s + \int_t^T dK_s - \int_t^T Z_s dW_s, \\ L_t \leq Y_t, \ \forall t \in [0,T], \\ \textit{the Skorokhod condition:} \\ \text{i)} \ \int_0^T (Y_t - L_t)\, dK_t^c = 0, \ \textit{where } K^c \textit{ is the continuous part of } K, \\ \text{ii)} \ \textit{if } K^d \textit{ is the discontinuous part of } K, \ \textit{then } K^d \textit{ is predictable} \\ \quad \textit{and } K_t^d = \sum_{0<s<t} (Y_{s-} - L_{s-})^- \mathbf{1}_{\{\Delta L_s < 0\}}. \end{cases} \tag{3}$$

For any $n \geq 1$, we consider the following BDSDE, which is a penalized version of equation (1).

Let (Y_t^n, Z_t^n) be the solution of the following BDSDEs

$$Y_t^n = \xi + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s + \int_t^T dK_s^n - \int_t^T Z_s^n dW_s, \tag{4}$$

where $K_t^n = n \int_0^t (L_s - Y_s^n)^+ ds$, $f : \Omega \times [0,T] \to \mathbb{R}$ and $g : \Omega \times [0,T] \to \mathbb{R}^k$.

According to the result of Pardoux & Peng (1994) [10], the BDSDE (4) has a unique solution $(Y^n, Z^n) \in \mathcal{S}_0^2 (A, T, \mathbb{R}^d) \times \mathcal{M}_0^2 (A, T, \mathbb{R}^{k\times d})$, for any $n \in \mathbb{N}$ and $\forall t \in [0,T]$.

To proof this theorem we need the following three lemmas.

Lemma 1. *Under* **(H1.3)-(H1.5),** *there exists a positive constant C independent of n such that*

$$||(Y^n, Z^n)||_{\mathcal{H}^2_{\delta,c}} + \mathbb{E}\,(K_T^n)^2 \leq C.$$

Proof. Applying Itô's formula for rcll semimartingales to $e^{\delta A_t} |Y_t^n|^2$, we get

$$\begin{aligned} &e^{\delta A_t} |Y_t^n|^2 + \delta \int_t^T e^{\delta A_s} |Y_s^n|^2 dA_s + \sum_{t<s\leq T} e^{\delta A_s} (\Delta_s Y^n)^2 \\ &= e^{\delta A_T} |\xi|^2 + 2 \int_t^T e^{\delta A_s} Y_{s-}^n f(s)ds + 2 \int_t^T e^{\delta A_s} Y_{s-}^n g(s) d\overleftarrow{B}_s \\ &\quad - 2 \int_t^T e^{\delta A_s} Y_{s-}^n Z_s^n dW_s - \int_t^T e^{\delta A_s} |Z_s^n|^2 ds + \int_t^T e^{\delta A_s} |g(s)|^2 ds \\ &\quad - 2 \int_t^T e^{\delta A_s} Y_{s-}^n dK_s^n. \end{aligned} \tag{5}$$

Using the inequalities $2ab \leq \epsilon a^2 + \frac{1}{\epsilon}b^2$, $2ab \leq \frac{2}{\delta}a^2 + \frac{\delta}{2}b^2$ and taking the expectation, then for $t = 0$, we obtain

$$\mathbb{E}\left(|Y_0^n|^2 + \frac{\delta}{2}\int_0^T e^{\delta A_s}|Y_s^n|^2 dA_s + \int_0^T e^{\delta A_s}|Z_s^n|^2 ds\right)$$
$$\leq \mathbb{E}\Bigg(e^{\delta A_T}|\xi|^2 + \frac{2}{\delta}\int_0^T e^{\delta A_s}\left|\frac{f(s)}{a_s}\right|^2 ds + \epsilon\Big(\sup_{0\leq t\leq T} e^{2\delta A_t}\left|L_t^+\right|^2\Big) + \frac{1}{\epsilon}|K_T^n|^2$$
$$+ \int_0^T e^{\delta A_s}|g(s)|^2 ds\Bigg), \tag{6}$$

Furthermore,

$$K_T^n = Y_0^n - \xi - \int_0^T f(s)ds - \int_0^T g(s)d\overleftarrow{B}_s + \int^T Z_s^n dW_s.$$

By the Cauchy–Schwarz inequality and isometry formula, we have

$$\mathbb{E}|K_T^n|^2 \leq C\mathbb{E}\left\{e^{\delta A_T}|\xi|^2 + |Y_0^n|^2 + \int_0^T e^{\delta A_s}\left(\frac{1}{\delta}\left|\frac{f(s)}{a_s}\right|^2 + |g(s)|^2 + |Z_s^n|^2\right)ds\right\}. \tag{7}$$

Combining (6) with (7), we have

$$\mathbb{E}\left(\int_0^T e^{\delta A_s}|Y_s^n|^2 dA_s + \int_0^T e^{\delta A_s}|Z_s^n|^2 ds + |K_T^n|^2\right)$$
$$\leq C\mathbb{E}\left(e^{\delta A_T}|\xi|^2 + \int_0^T e^{\delta A_s}\Big(\left|\frac{f(s)}{a_s}\right|^2 + |g(s)|^2\Big)ds + \sup_{0\leq t\leq T} e^{2\delta A_t}\left|L_t^+\right|^2\right).$$

Using again equation (5), by **(H1.3)-(H1.5)** and the Burkholder-Davis-Gundy inequality we deduce that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\delta A_t}|Y_t^n|^2 + \int_0^T e^{\delta A_s}|Y_s^n|^2 dA_s + \int_0^T e^{\delta A_s}|Z_s^n|^2 ds + |K_T^n|^2\right) \leq C. \quad \square$$

Lemma 2. *For each $n \in \mathbb{N}^*$, we get*

$$\lim_{n\to+\infty}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{2\delta A_t}\left|(L_t - Y_t^n)^+\right|^2\right) = 0.$$

Proof. Using the following transformation

$$\begin{cases} \bar{\xi}_n = \xi + \int_0^T f(s)ds + \int_0^T g(s)d\overleftarrow{B}_s, \\ \bar{L}_t = L_t + \int_0^t f(s)ds + \int_0^t g(s)d\overleftarrow{B}_s, \\ \bar{Y}_t^n = Y_t^n + \int_0^t f(s)ds + \int_0^t g(s)d\overleftarrow{B}_s. \end{cases}$$

We conclude that

$$\bar{Y}_t^n = \bar{\xi}_n + n\int_t^T \left(\bar{L}_s - \bar{Y}_s^n\right)^+ ds - \int_t^T Z_s^n dW_s.$$

Let

$$\begin{cases} d\hat{Y}_t^n = n\left(\bar{L}_s - \hat{Y}_t^n\right) dt + Z_t^n dW_t, \\ \hat{Y}_T^n = \bar{L}_T. \end{cases}$$

Since $\bar{L}_T \leq \bar{\xi}_T$, then the comparison theorem (see, [1, Lemma 3.1]) shows that, for $t \in [0,T]$, $\hat{Y}_t^n \leq \bar{Y}_t^n$ $a.s.$

Now, let σ be a $\mathcal{G}_t-$stopping time, and put $\tau = \sigma \wedge T$. The integration by parts formula implies that

$$e^{-n(t-\tau)}\hat{Y}_t^n = e^{-n(T-\tau)}\hat{Y}_T^n + n\int_t^T e_s^{-n(s-\tau)}\bar{L}_s ds + \int_t^T e^{-n(s-\tau)}Z_s^n dW_s.$$

Taking $t = \tau$ and the conditional expectation, the sequence $\hat{Y}^n$ satisfies the following equality

$$\hat{Y}_\tau^n = \mathbb{E}\left\{e^{-n(T-\tau)}\bar{L}_T + n\int_\tau^T e^{-n(s-\tau)}\bar{L}_s ds|\mathcal{G}_\tau\right\}. \tag{8}$$

In the other hand we obtain that

$$\lim_{n\to+\infty}\left(n\int_\tau^T e^{-n(s-\tau)}\bar{L}_s ds\right) = \lim_{n\to+\infty}\left(I_1^n + I_2^n\right),$$

where

$$\begin{cases} I_1^n = n\int_\tau^T e^{-n(s-\tau)}\left(\bar{L}_s - \bar{L}_\tau\right) ds, \\ \text{and} \\ I_2^n = n\int_\tau^T e^{-n(s-\tau)}\bar{L}_\tau ds. \end{cases}$$

Firstly, we calculate the limit of I_1^n

$$\begin{aligned}|I_1^n| &= \left| n \int_\tau^T e^{-n(s-\tau)} \left(\bar{L}_s - \bar{L}_\tau \right) ds \right| \\ &\le n \int_\tau^T e^{-n(s-\tau)} V_s ds,\end{aligned}$$

where $\sup_{u\in[\tau,s]} \left|\bar{L}_u - \bar{L}_\tau\right| = V_s$, using again the integration by parts formula, we have

$$\lim_{n\to+\infty} \int_\tau^T ne^{-n(s-\tau)} V_s ds = \lim_{n\to+\infty} \left(-e^{-n(T-\tau)} V_T + \int_\tau^T e^{-n(s-\tau)} dV_s \right) = 0,$$

so, we have

$$\lim_{n\to+\infty} I_1^n = 0.$$

On the other hand, it's clear that $\lim_{n\to\infty} I_2^n = \bar{L}_\tau$, which implies that

$$\lim_{n\to+\infty} \left(e^{-n(T-\tau)} \bar{L}_T + n \int_\tau^T e^{-n(s-\tau)} \bar{L}_s ds \right) = \bar{L}_\tau,$$

consequently by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \hat{Y}_\tau^n = \lim_{n\to+\infty} \mathbb{E} \left\{ e^{-n(T-\tau)} \bar{L}_T + n \int_\tau^T e^{-n(s-\tau)} \bar{L}_s ds | \mathcal{G}_\tau \right\} = \bar{L}_\tau \ a.s.$$

From (8), Jensen's inequality and Doob's maximal quadratic inequality (see

[11, Theorem 20, p. 11]), we get

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T} e^{2\delta A_t} \left| \left(\bar{L}_t - \hat{Y}_t^n \right)^+ \right|^2 \right) &\le \mathbb{E}\left(\sup_{0\le t\le T} e^{2\delta A_t} \left| \mathbb{E}\left(\left(\bar{L}_t - \hat{y}_t^n \right)^+ \middle| \mathcal{G}_t \right) \right|^2 \right) \\ &\le 4\mathbb{E}\left(\sup_{0\le t\le T} e^{2\delta A_t} \left| \left(\bar{L}_t - \hat{y}_t^n \right)^+ \right|^2 \right),\end{aligned}$$

where

$$\hat{y}_t^n = e^{-n(T-t)} \bar{L}_T + n \int_t^T e^{-n(s-t)} \bar{L}_s ds.$$

The sequence is defined by

$$(\pi_t^n)_{n\geq 1} : = \left\{ \bar{L}_t - e^{-n(T-t)}\bar{L}_T + n\int_t^T e^{-n(s-t)}\bar{L}_s ds \right\}_{n\geq 1},$$

using the fact that $(\pi_t^n)_{n\geq 1}$ and $\left(e^{2\delta A_t}\pi_t^n\right)_{n\geq 1}$ are uniform convergent in t. Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{2\delta A_t}\left|\left(\bar{L}_t - \hat{y}_t^n\right)^+\right|^2\right) \leq \lim_{n\to\infty}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{2\delta A_t}\left|(\pi_t^n)^+\right|^2\right) = 0.$$

Then

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{2\delta A_t}\left|\left(\bar{L}_t - \hat{Y}_t^n\right)^+\right|^2\right) \underset{n\to\infty}{\to} 0.$$

Using the fact $\hat{Y}_t^n \leq \bar{Y}_t^n$ for all $t \leq T$, we deduce that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{2\delta A_t}\left|\left(\bar{L}_t - \bar{Y}_t^n\right)^+\right|^2\right) \underset{n\to\infty}{\to} 0.$$

Finally, we get

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{2\delta A_t}\left|\left(L_t - Y_t^n\right)^+\right|^2\right) \underset{n\to\infty}{\to} 0. \qquad \square$$

In the following lemma, we prove that the triplet (Y^n, Z^n, K^n) converge to (Y, Z, K).

Lemma 3. *There exists* $(Y, Z, K) \in \mathcal{H}^2_{\delta,c}(a,T) \times \mathcal{A}^2$ *such that,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\delta A_t}\left|Y_t^n - Y_t\right|^2 + \int_0^T e^{\delta A_s}\left|Y_s^n - Y_s\right|^2 dA_s\right)$$
$$+\mathbb{E}\left(\int_0^T e^{\delta A_s}\left|Z_s^n - Z_s\right|^2 ds + \sup_{0\leq t\leq T}\left|K_t^n - K_t\right|^2\right) \underset{n\to\infty}{\to} 0.$$

Proof. For any $n \geq p$, it follows by Itô's formula for discontinuous semimartingale that

$$\mathbb{E}\left(e^{\delta A_t}\left|Y_t^n - Y_t^p\right|^2 + \delta\int_t^T e^{\delta A_s}\left|Y_s^n - Y_s^p\right|^2 dA_s + \int_t^T e^{\delta A_s}\left|Z_s^n - Z_s^p\right|^2 ds\right)$$
$$\leq 2\mathbb{E}\int_t^T \left(Y_{s-}^n - Y_{s-}^p\right)\left(dK_s^n - dK_s^p\right).$$

It is clearly seen that

$$\mathbb{E}\int_t^T \left(Y_{s-}^n - Y_{s-}^p\right)\left(dK_s^n - dK_s^p\right)$$
$$\leq \mathbb{E}\left(\sup_{0\leq t\leq T} e^{\delta A_s}\left(Y_s^n - L_s\right)^- K_T^p + \sup_{0\leq t\leq T} e^{\delta A_s}\left(Y_s^p - L_s\right)^- K_T^n\right).$$

Then by virtue of **Lemma 2**, we obtain

$$\mathbb{E}\left(\int_t^T e^{\delta A_s}\left|Y_s^n - Y_s^p\right|^2 dA_s + \int_t^T e^{\delta A_s}\left|Z_s^n - Z_s^p\right|^2 ds\right)$$
$$\leq 2\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\delta A_s}\left(Y_s^n - L_s\right)^- K_T^p + \sup_{0\leq t\leq T} e^{\delta A_s}\left(Y_s^p - L_s\right)^- K_T^n\right),$$
$$\underset{n,p\to\infty}{\longrightarrow} 0.$$

It follows that (Y^n, Z^n) is a Cauchy sequence in $\mathcal{H}_\delta^2(a,T)$. Then there exists a couple of processes $(Y,Z)\in\mathcal{H}_\delta^2(a,T)$ such that

$$\lim_{n\to\infty}\mathbb{E}\left(\int_t^T e^{\delta A_s}\left|Y_s^n - Y_s\right|^2 dA_s + \int_t^T e^{\delta A_s}\left|Z_s^n - Z_s\right|^2 ds\right) = 0.$$

Using the Burkholder-Davis-Gundy inequality we deduce that

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\delta A_t}\left|Y_t^n - Y_t\right|^2\right) = 0.$$

Now, we show prove that $\lim_{n\to\infty} K_t^n = K_t$ for any $t\in[0,T]$ *a.s.*, where $K\in\mathcal{A}^2$. Since

$$K_t = Y_0 - Y_t - \int_0^t f(s)ds - \int_0^t g(s)d\overleftarrow{B}_s + \int_0^t Z_s dW_s,$$

we have

$$|K_t^n - K_t|^2 \leq 3\left(e^{\delta A_t}|Y_t^n - Y_t|^2 + |Y_0^n - Y_0|^2 + \int_0^t e^{\delta A_s}|Z_s^n - Z_s|^2 ds\right).$$

Then it easy to get that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|K_t^n - K_t|^2\right) \to 0, \text{as } n \to \infty.$$

Finally, we deduce that

$$||Y^n - Y||_{\mathcal{S}_\delta^2} + ||Y^n - Y||_{\mathcal{M}_\delta^{2,a}} + ||Z^n - Z||_{\mathcal{M}_\delta^2} + \mathbb{E}\left(\sup_{0\leq t\leq T}|K_t^n - K_t|^2\right) \underset{n\to\infty}{\to} 0.$$

□

***Proof.* of Theorem 1** It remains to prove that $L_t \leq Y_t, \forall t \in [0,T]$, a.s, and the Skorokhod conditions.

- Using Lemma 2, we have for $n \in \mathbb{N}$, $L_t \leq Y_t^n$, then $L_t \leq Y_t$.
- Now, we prove that, $\int_0^T (Y_t - L_t)\, dK_t^c = \int_0^T (Y_{t-} - L_{t-})\, dK_t^d = 0$. We define the processes

$$\rho_t = L_t 1_{\{t\leq T\}} + \xi 1_{\{t=T\}} + \int_0^t f(s)ds + \int_0^t g(s)d\overleftarrow{B}_s$$

Note that ρ are rcll processes and uniformly square integrable. Using the Snell envelope notion, we have that $S(\rho) : = \underset{\tau\in\mathcal{T}_{0,T}}{ess\sup}\mathbb{E}(\rho_t\,|\mathcal{G}_t)$, where $\mathcal{T}_{0,T}$ is the set of $\mathcal{G}$ stopping time, such that

$$\mathcal{T}_{0,T} : = \{\tau \text{ stopping time with } 0 \leq \tau \leq T \; a.s\}.$$

Now, $S(\rho)$ is a the smallest discontinuous supermartingale which dominates the process ρ. Then, by the Doob–Meyer decomposition theorem, there exists an $\mathcal{G}_t$-uniformly integrable martingale M and a unique $\mathcal{G}_t$-adapted rcll non-decreasing process $K = K_t^c - K_t^d$ with $\mathbb{E}(K_T) < \infty$ and $K_0 = 0$, such that $S(\rho_t) = M_t - K$. Then

$$S(\rho_t) = \mathbb{E}\left(\xi + \int_0^T f(s)ds + \int_0^T g(s)d\overleftarrow{B}_s + K_T\,|\mathcal{G}_t\right) - K_t.$$

Using [3, Theorem 2.34] or [5, Proposition A.4], we obtain

$$\left\{\Delta K^d = K^d - K_-^d \neq 0\right\} \subset \left\{S_-\left(\rho\right) = \rho_-\right\}.$$

Then the following holds

$$\int_0^T \left(S_-\left(\rho\right) - \rho_-\right) dK_t^d = 0.$$

By some property of the Snell envelope (see [8, Lemma A.4]), we get

$$\int_0^T \left(S\left(\rho\right) - \rho\right) dK_t^c = \int_0^T \left(Y_t - L_t\right) dK_t^c = 0.$$

Theorem 3.1 is then proved. □

The main result of this section is the following.

3.3.2. THE GENERAL CASE

We now state the existence and uniqueness result for equation (1).

Theorem 2. *Under* **(H1),** *the reflected BDSDE with one rcll reflecting barriers associated with* (ξ, f, g, L) *has a unique solution* (Y_t, Z_t, K_t).

Proof. We will show the existence of the solution to (1) by applying the fixed point theorem. Let $\mathcal{H}_\delta^2\left(a, T\right)$ be the space of $\mathcal{F}$-measurable processes (Y, Z) endowed with the norm

$$||(Y, Z)||_\delta = \left\{\mathbb{E}\int_0^T e^{\delta A_t}\left(a_t^2\left|Y_t\right|^2 + \left|Z_t\right|^2\right) dt\right\}^{\frac{1}{2}}; \quad \delta > 0.$$

Let Φ be the map from $\mathcal{H}_\delta^2\left(a, T\right)$ into itself, which to (Y, Z) associates $\Phi\left(Y, Z\right) = \left(\tilde{Y}, \tilde{Z}\right)$ where $(\tilde{Y}, \tilde{Z}, \tilde{K})$ is the solution of the following reflected BDSDE

$$\begin{aligned}\tilde{Y}_t = &\xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s + \int_t^T d\tilde{K}_s \\ &- \int_t^T \tilde{K}_s dW_s, \qquad 0 \leq t \leq T.\end{aligned}$$

Let $\left(Y^{'}, Z^{'}\right)$ be another couple of $\mathcal{H}_\delta^2\left(a, T\right)$ and $\Phi\left(\acute{Y}, \acute{Z}\right) = \left(\tilde{Y}^{'}, \tilde{Z}^{'}\right)$.

Then using Itô's formula for discontinuous semimartingales, we obtain, for any $t \in [0,T]$,

$$\begin{aligned}
&e^{\delta A_t}\left|\tilde{Y}_t-\tilde{Y}_t^{'}\right|^2+\delta\int_t^T e^{\delta A_s}\left|\tilde{Y}_s-\tilde{Y}_s^{'}\right|^2 dA_s+\int_t^T e^{\delta A_s}\left|\tilde{Z}_s-\tilde{Z}_s^{'}\right|^2 ds\\
&\quad+\sum_{t<s\leq T} e^{\delta A_s}\left(\Delta_s\tilde{Y}-\Delta_s\tilde{Y}^{'}\right)^2\\
&=2\int_t^T e^{\delta A_s}\left(\tilde{Y}_{s-}-\tilde{Y}_{s-}^{'}\right)\left(f\left(s,Y_s,Z_s\right)-f\left(s,Y_s^{'},Z_s^{'}\right)\right)ds\\
&\quad+2\int_t^T e^{\delta A_s}\left(\tilde{Y}_{s-}-\tilde{Y}_{s-}^{'}\right)\left(d\tilde{K}_s-d\tilde{K}_s^{'}\right)\\
&\quad+2\int_t^T e^{\delta A_s}\left(\tilde{Y}_{s-}-\tilde{Y}_{s-}^{'}\right)\left(g\left(s,Y_s,Z_s\right)-g\left(s,Y_s^{'},Z_s^{'}\right)\right)d\overleftarrow{B}_s\\
&\quad-2\int_t^T e^{\delta A_s}\left(\tilde{Y}_{s-}-\tilde{Y}_{s-}^{'}\right)\left(\tilde{Z}_s-\tilde{Z}_s^{'}\right)dW_s\\
&\quad+\int_t^T e^{\delta A_s}\left|g\left(s,Y_s,Z_s\right)-g\left(s,Y_s^{'},Z_s^{'}\right)\right|^2 ds.
\end{aligned}$$

Using the fact that $\int_t^T e^{\delta A_s}\left(\tilde{Y}_{s-}-\tilde{Y}_{s-}^{'}\right)\left(d\tilde{K}_s-d\tilde{K}_s^{'}\right)\leq 0$ and $2ab\leq\frac{1}{\epsilon}a^2+\epsilon b^2$, then taking expectation, we get for any $\epsilon>0$:

$$\begin{aligned}
&\mathbb{E}\int_t^T e^{\delta A_s}\left(\delta a_s^2\left|\tilde{Y}_s-\tilde{Y}_s^{'}\right|^2+\left|\tilde{Z}_s-\tilde{Z}_s^{'}\right|^2\right)ds\\
&\leq 2\mathbb{E}\int_t^T e^{\delta A_s}\left(\tilde{Y}_{s-}-\tilde{Y}_{s-}^{'}\right)\left(r\left(s\right)\left|Y_s-Y_s^{'}\right|+\theta\left(s\right)\left|Z_s-Z_s^{'}\right|\right)ds\\
&\quad+\mathbb{E}\int_t^T e^{\delta A_s}\left(\nu\left(s\right)\left|Y_s-Y_s^{'}\right|^2+\alpha\left|Z_s-Z_s^{'}\right|^2\right)ds\\
&\leq\left(1+\frac{1}{\epsilon}\right)\mathbb{E}\Big(\int_t^T e^{\delta A_s}a_s^2\left|\tilde{Y}_s-\tilde{Y}_s^{'}\right|^2 ds+\int_t^T e^{\delta A_s}a_s^2\left|Y_s-Y_s^{'}\right|^2 ds\\
&\quad+\epsilon\int_t^T e^{\delta A_s}\left|Z_s-Z_s^{'}\right|^2 ds\Big)\\
&\quad+\mathbb{E}\left(\int_t^T e^{\delta A_s}a_s^2\left|Y_s-Y_s^{'}\right|^2 ds+\alpha\int_t^T e^{\delta A_s}\left|Z_s-Z_s^{'}\right|^2 ds\right).
\end{aligned}$$

This implies that,

$$\begin{aligned}&\mathbb{E}\int_t^T e^{\delta A_s}\left(\left(\delta-1-\epsilon^{-1}\right)a_s^2\left|\tilde{Y}_s-\tilde{Y}_s^{'}\right|^2+\left|\tilde{Z}_s-\tilde{Z}_s^{'}\right|^2\right)ds\\&\quad\leq 2\mathbb{E}\int_t^T e^{\delta A_s}a_s^2\left|Y_s-Y_s^{'}\right|^2 ds+(\epsilon+\alpha)\,\mathbb{E}\int_t^T e^{\delta A_s}\left|Z_s-Z_s^{'}\right|^2 ds.\end{aligned}$$

Choosing $\delta>0$, such that $\delta>1+\epsilon^{-1}+\frac{2}{\epsilon+\alpha}$ and define $\zeta=\frac{2}{\epsilon+\alpha}$, we obtain

$$\begin{aligned}&\mathbb{E}\left(\zeta\int_t^T e^{\delta A_s}a_s^2\left|\tilde{Y}_s-\tilde{Y}_s^{'}\right|^2 ds+\int_t^T e^{\delta A_s}\left|\tilde{Z}_s-\tilde{Z}_s^{'}\right|^2 ds\right)\\&\quad\leq(\epsilon+\alpha)\,\mathbb{E}\left(\zeta\int_t^T e^{\delta A_s}a_s^2\left|Y_s-Y_s^{'}\right|^2 ds+\int_t^T e^{\delta A_s}\left|Z_s-Z_s^{'}\right|^2 ds\right).\end{aligned}$$

Therefore, choosing $\epsilon>0$ such that $(\epsilon+\alpha)<1$, then Φ is a contraction on $\mathcal{H}_\delta^2(a,T)$ and it has a unique fixed point on $\mathcal{H}_\delta^2(a,T)$, which is the unique solution of RBDSDE (1) with data (ξ,f,g,L). □

4. Comparison Theorem

In this section, we show a comparison theorem for the reflected BDSDEs with rcll barrier (1).

Theorem 3. *Suppose that L, $L^{'}$ be two obstacles and f, $f^{\,'}$ be two stochastic Lipschitz drivers. We assume, in addition, the following assumption*

$$(\mathbf{H1.6})\begin{cases}\xi\leq\xi^{'},\\ f(s,y,z)\leq f^{'}(s,y,z),\\ \forall(s,\ y,\ z)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}.\ d\mathbb{P}\times dt\ a.\ s.\end{cases}$$

Let (Y,Z,K) and $(\acute{Y},\acute{Z},\acute{K})$be two solutions to the reflected BDSDE associated with $(\xi,\ f,\ g,\ L)$ and (respectively $(\acute{\xi},\ \acute{f},\ g,\ \acute{L})$), then we have

$$\forall t\in[0,T],\qquad Y_t\leq\acute{Y}_t,\qquad \mathbb{P}-a.s.$$

Proof. Let us show that $Y_t\leq Y_t^{'}$, by Meyer-Itô's formula (see, e.g., [11, Theorem 66 p. 214] or [2, p. 349]) with the convex function $x\to(x^+)^2$

implies that, for $0 \leq t \leq T$,

$$\begin{aligned}
&\left(\left(Y_t - Y_t^{'}\right)^{+}\right)^2 + \sum_{t<s\leq T}\Bigg\{\left(Y_s - Y_s^{'}\right)^2 - \left(\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\right)^2 \\
&\quad - 2\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\Delta_s\left(Y - Y^{'}\right)\Bigg\} \\
&= \left(\left(\xi - \xi^{'}\right)^{+}\right)^2 + 2\int_t^T \left(Y_{s-} - Y_{s-}^{'}\right)^{+}\left(f(s, Y_s, Z_s) - \acute{f}(s, Y_s^{'}, Z_s^{'})\right) ds \\
&\quad + 2\int_t^T \left(Y_{s-} - Y_{s-}^{'}\right)\left(g(s, Y_s, Z_s) - g(s, Y_s^{'}, Z_s^{'})\right) d\overleftarrow{B}_s \\
&\quad + 2\int_t^T \left(Y_{s-} - Y_{s-}^{'}\right)^{+}\left(dK_s - dK_s^{'}\right) - 2\int_t^T \left(Y_{s-} - Y_{s-}^{'}\right)^{+}\left(Z_s - \acute{Z}_s\right) dW_s \\
&\quad + \int_t^T 1_{\{Y_s > Y_s^{'}\}}\left|g(s, Y_s, Z_s) - g(s, Y_s^{'}, Z_s^{'})\right|^2 ds - \int_t^T 1_{\{Y_s > Y_s^{'}\}}\left|Z_s - \acute{Z}_s\right|^2 ds,
\end{aligned}$$

Since $\left(Y_s - Y_s^{'}\right)^2 - \left(\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\right)^2 - 2\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\Delta_s\left(Y - Y^{'}\right) \geq 0$ and $\left(\xi - \xi^{'}\right)^{+} = 0$, then the integration by parts formula give

$$\begin{aligned}
&e^{\delta A_t}\left(\left(Y_t - Y_t^{'}\right)^{+}\right)^2 + \delta\int_t^T e^{\delta A_s}\left(\left(Y_s - Y_s^{'}\right)^{+}\right)^2 dA_s \\
&\quad + \int_t^T 1_{\{Y_s > Y_s^{'}\}} e^{\delta A_s}\left|Z_s - \acute{Z}_s\right|^2 ds \\
&\leq 2\int_t^T e^{\delta A_s}\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\left(f(s, Y_s, Z_s) - \acute{f}(s, Y_s^{'}, Z_s^{'})\right) ds \\
&\quad + 2\int_t^T e^{\delta A_s}\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\left(dK_s - dK_s^{'}\right) \\
&\quad + 2\int_t^T e^{\delta A_s}\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\left(g(s, Y_s, Z_s) - g(s, Y_s^{'}, Z_s^{'})\right) d\overleftarrow{B}_s \\
&\quad - 2\int_t^T e^{\delta A_s}\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\left(Z_s - \acute{Z}_s\right) dW_s \\
&\quad + \int_t^T 1_{\{Y_s > Y_s^{'}\}} e^{\delta A_s}\left|g(s, Y_s, Z_s) - g(s, Y_s^{'}, Z_s^{'})\right|^2 ds.
\end{aligned}$$

Using the Burkholder-Davis-Gundy inequality (see, [6, 12]) we deduce that the process $\int_0^t e^{\delta A_s}\left(Y_{s-} - Y_{s-}^{'}\right)^{+}\left(g(s, Y_s, Z_s) - g(s, Y_s^{'}, Z_s^{'})\right) d\overleftarrow{B}_s$

and $\int_t^T e^{\delta A_s}\left(Y_{s-}-Y_{s-}^{'}\right)^+\left(Z_s-\acute{Z}_s\right)dW_s$ are uniformly integrable martingales, then taking the expectation on both sides, we obtain

$$\begin{aligned}
&\mathbb{E}\Bigg\{e^{\delta A_t}\left(\left(Y_t-Y_t^{'}\right)^+\right)^2+\delta\int_t^T e^{\delta A_s}\left(\left(Y_s-Y_s^{'}\right)^+\right)^2 dA_s \\
&\quad+\int_t^T 1_{\left\{Y_s>Y_s^{'}\right\}}e^{\delta A_s}\left|Z_s-\acute{Z}_s\right|^2 ds\Bigg\} \\
&\le 2\mathbb{E}\int_t^T e^{\delta A_s}\left(Y_{s-}-Y_{s-}^{'}\right)\left(f(s,Y_s,Z_s)-\acute{f}(s,Y_s^{'},Z_s^{'})\right)ds \\
&\quad+2\mathbb{E}\Bigg(\int_t^T e^{\delta A_s}\left(Y_{s-}-Y_{s-}^{'}\right)^+\left(dK_s-dK_s^{'}\right) \\
&\quad+\int_t^T 1_{\left\{Y_s>Y_s^{'}\right\}}e^{\delta A_s}\left|g(s,Y_s,Z_s)-g(s,Y_s^{'},Z_s^{'})\right|^2 ds\Bigg).
\end{aligned}$$

Note that if $Y\geq Y^{'}$, then $Y>L$, which implies that $dK^c=0$ and thus

$$\int_t^T\left(Y_{s-}-Y_{s-}^{'}\right)^+dK_s^c=0.$$

Also, when the purely discontinuous K^d increases at s, we should have $L_{s-}=Y_{s-}$, which implies that

$$\sum_{t<s\le T}\left(Y_{s-}-Y_{s-}^{'}\right)^+\Delta K_s^d=\sum_{t<s\le T}\left(L_{s-}-Y_{s-}^{'}\right)^+\Delta K_s^d=0.$$

In the same way, we obtain

$$\int_t^T\left(Y_{s-}-Y_{s-}^{'}\right)^+dK_s^{'}=0.$$

Since $f\left(t,Y_t^{'},Z_t^{'}\right)-\acute{f}\left(t,Y_t^{'},Z_t^{'}\right)\le 0$, therefore for any $t\le T$

$$\begin{aligned}
&\mathbb{E}\Bigg\{e^{\delta A_t}\left(\left(Y_t-Y_t^{'}\right)^+\right)^2+\delta\int_t^T e^{\delta A_s}\left(\left(Y_s-Y_s^{'}\right)^+\right)^2 dA_s \\
&\quad+\int_t^T 1_{\left\{Y_s>Y_s^{'}\right\}}e^{\delta A_s}\left|Z_s-\acute{Z}_s\right|^2 ds\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\mathbb{E}\int_t^T e^{\delta A_s}\left(Y_{s-}-Y'_{s-}\right)\left(f(s,Y_s,Z_s)-\acute{f}(s,Y'_s,Z'_s)\right)ds\\
&\quad+\mathbb{E}\int_t^T 1_{\left\{Y_s>Y'_s\right\}}e^{\delta A_s}\left|g(s,Y_s,Z_s)-g(s,Y'_s,Z'_s)\right|^2 ds,
\end{aligned}
$$

we obtain from **(H1.1)** and **(H1.2)** that

$$
\begin{aligned}
&\mathbb{E}\Bigg\{e^{\delta A_t}\left(\left(Y_t-Y'_t\right)^+\right)^2+\delta\int_t^T e^{\delta A_s}\left(\left(Y_s-Y'_s\right)^+\right)^2 dA_s\\
&\qquad+\int_t^T e^{\delta A_s}\left|Z_s-\acute{Z}_s\right|^2 ds\Bigg\}\\
&\leq 2\mathbb{E}\int_t^T e^{\delta A_s}\left|Y_{s-}-Y'_{s-}\right|\left(r\left(s\right)\left|Y_s-Y'_s\right|+\theta\left(s\right)\left|Z_s-Z'_s\right|\right)ds\\
&\qquad+\mathbb{E}\left(\int_t^T \nu\left(s\right)e^{\delta A_s}\left|Y_s-Y'_s\right|^2 ds+\alpha\int_t^T e^{\delta A_s}\left|Z_s-\acute{Z}_s\right|^2 ds\right).
\end{aligned}
$$

Using the inequality $2ab\leq\epsilon a^2+\frac{1}{\epsilon}b^2$, we have

$$
\begin{aligned}
\mathbb{E}\Bigg\{e^{\delta A_t}\left(\left(Y_t-Y'_t\right)^+\right)^2+\sigma_1\int_t^T e^{\delta A_s}\left(\left(Y_s-Y'_s\right)^+\right)^2 dA_s&\\
+\sigma_2\int_t^T e^{\delta A_s}\left|Z_s-\acute{Z}_s\right|^2 ds\Bigg\}&\leq 0
\end{aligned}
$$

with $\sigma_1=\delta-\epsilon-2$ and $\sigma_2=1-\frac{1}{\epsilon}-\alpha$. Choosing $\delta\geq 3+\epsilon$ and $\epsilon=\frac{2}{1-\alpha}$, we derive by Gronwall's lemma that

$$
\mathbb{E}\left[e^{\delta A_t}\left(\left(Y_t-Y'_t\right)^+\right)^2\right]=0.
$$

Finally, $\forall t\in[0,T]$, $Y_t\leq Y'_t$, $\mathbb{P}-a.s.$ □

Corollary 1. *Assume that:*

$$
\begin{cases}
\text{i)}\quad f \text{ independent of } z,\\
\text{ii)}\quad f \text{ and } g \text{ satisfy } \textbf{(H1.1)},\ \xi\leq\xi' \text{ and } L_t\leq L'_t, \text{ for any } t\leq T,\\
\text{iii)}\ f\left(t,Y'_t\right)\leq f'\left(t,Y'_t,Z'_t\right).
\end{cases}
$$

Let (Y, Z, K) *and* $\left(\acute{Y}, \acute{Z}, \acute{K}\right)$ *be two solutions to the RBDSDE associated with* $(\xi,\ f,\ g,\ L)$ *and (respectively* $(\acute{\xi},\ \acute{f},\ g,\ \acute{L})$*). Then we have*

$$\mathbb{P}-a.s. \quad Y_t \leq Y_t^{'}.$$

5. Irregular Barrier Reflected BDSDE with Continuous and Stochastic Linear Growth Coefficients

Our goal in this section is to prove an existence theorem for BDSDEs with one rcll reflecting barrier (1) when the coefficient f is continuous with stochastic linear growth.

5.1. Assumptions, approximation lemma and definition

Now, we assume the following assumptions

- **(A1.6)** For all $(t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$, $f(t, \omega, y, z)$ is $\mathcal{F}_t-$measurable.
- **(A1.7)** For fixed ω and t, $f(t, \omega, \cdot, \cdot)$ is continuous.
- **(A1.8)** For all $(t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$\begin{cases} \text{i)}\ \ |f(t, \omega, y, z)| \leq \varphi(t) + r(t)|y| + \theta(t)|z|, \\ \text{ii)}\ \ |g(t, \omega, y, z) - g(t, \omega, \acute{y}, \acute{z})|^2 \leq \nu(t)\left(|y-\acute{y}|^2\right) + \alpha\left(|z-\acute{z}|^2\right), \end{cases}$$

  with $0 < \alpha < 1$. Where φ, r, θ and ν are four nonnegative processes such that for *a.e.* $t \in [0, T]$, $\varphi(t)$, $r(t)$, $\theta(t)$ and $\nu(t)$ $\mathcal{F}_t^W-$measurable.
- **(A1.9)** The integrability condition holds:

$$\mathbb{E}\left(e^{\delta A_t}|\xi|^2 + \int_0^T e^{\delta A_t}\left(\frac{|\varphi(t)|^2}{a_t^2} + |g(t, \omega, 0, 0)|^2\right)dt\right) < \infty.$$

Before to state our main result, we first give the following technical approximation lemma, which generalizes the corresponding result of Lepeltier and San Martin [7].

Lemma 4. *Let* $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ *be a measurable function such that:*

For a.s. every $(t,\omega) \in [0,T] \times \Omega,\ f(t,\omega,y,z)$ *is continuous.*

For every $(t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$|f(t,\omega,y,z)| \le \varphi(t) + r(t)|y| + \theta(t)|z|.$$

where φ, r *and* θ *are three nonnegative processes such that for a.e.* $t \in [0,T]$, $\varphi(t)$, $r(t)$ *and* $\theta(t)$ $\mathcal{F}_t^W-$*measurable.*

Then there exists the sequence of functions f_n

$$f_n(t,\omega,y,z) = \inf_{(\tilde{y},\tilde{z})\in\mathbb{Q}} \{f(t,\omega,\tilde{y},\tilde{z}) + n(r(t)|y-\tilde{y}| + \theta(t)|z-\tilde{z}|)\},$$

are well defined for $n \ge 1$ *and satisfy the following conditions*

(i) *For all* $n \ge 1$, $(t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$|f_n(t,\omega,y,z)| \le \varphi(t) + r(t)|y| + \theta(t)|z|.$$

(ii) *For any* (t,ω,y,z), $f_n(t,\omega,y,z)$ *is non-decreasing in* n.

(iii) *For all* $(t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$, *if* $(t,\omega,y_n,z_n) \to (t,\omega,y,z)$, *then*

$$f_n(t,\omega,y_n,z_n) \to f(t,\omega,y,z).$$

(iv) *For any* $n \ge 1$, $(t,\omega) \in [0,T] \times \Omega$, *for all* $(t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$ *and* $(t,\omega,\tilde{y},\tilde{z}) \in [0,T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$, *we have*

$$|f_n(t,\omega,y,z) - f_n(t,\omega,\tilde{y},\tilde{z})| \le n(r(t)|y-\tilde{y}| + \theta(t)|z-\tilde{z}|).$$

Now, we will introduce the definition of a minimal solution to the BDSDE with one rcll reflecting barrier.

Definition 2. A triplet of processes $(\underline{Y}, \underline{Z}, \underline{K})$ of $\mathcal{H}^2_{\delta,c}(a,T) \times \mathcal{A}^2$ is called a minimal solution of (1) if for any other solution (Y,Z,K), we have $\underline{Y}_t \le Y_t$.

5.2. Existence result

Now, by the inf-convolution approximation of the function f (**Lemma 4**) and the comparison theorem (**Theorem 3**), we establish the following existence theorem.

Theorem 4. *Assume that* **(A1.6)-(A1.9) and (H1.2), (H1.3)** *hold. Then, the RBDSDE* (1) *has a minimal solution* $\left(\underline{Y},\underline{Z},\underline{K}\right)\in\mathcal{H}^2_{\delta,c}(a,T)\times\mathcal{A}^2$.

From **Lemma 4** there exists a sequence of functions f_n associated with f, which is of stochastic Lipschitz-continuous and non-decreasing in n. Now, for every $n\in\mathbb{N}^*$, let a^n and A^n be two stochastic processes with nonnegative values defined by

$$(a^n_t)^2=nr(t)+n^2\theta^2(t)+\nu(t)>0,\qquad \textit{and}\qquad A^n_t=\textstyle\int_0^t(a^n_s)^2\,ds<\infty.$$

Then, from **(A1.6)-(A1.8)** and **(H1.2)**, a^n_t and A^n_t are $\mathcal{F}^W_t-$measurable, for a.e. $t\in[0,T]$ such that $\forall n\in\mathbb{N}^*$, $0<a<a_n$ and $A<A^n<n^2A$. Thus, by **Remark 2.1**, for any $n\geq1$

$$\mathcal{H}^2_{\delta,c}(a^n,T)\subset\mathcal{H}^2_{\delta,c}(a,T)\,. \tag{9}$$

Furthermore, from **(A1.9)**, it is easy to see that the data (ξ,f_n,g) satisfy the following properties $\forall n\in\mathbb{N}^*$

$$\begin{cases}\mathbb{E}\left(e^{\delta A^n_T}|\xi|^2\right)\leq\mathbb{E}\left(e^{\delta n^2A_T}|\xi|^2\right)<\infty,\\ \mathbb{E}\int_0^Te^{\delta A^n_t}\left(\frac{|f_n(t,\omega,0,0)|^2}{(a^n_t)^2}+|g(t,\omega,0,0)|^2\right)dt\\ \qquad\leq\int_0^Te^{\delta n^2A_t}\left(\frac{|\varphi(t)|^2}{a^2_t}+|g(t,\omega,0,0)|^2\right)dt<\infty.\end{cases}$$

Therefore, using section 3 (stochastic Lipschitz case), we get that for every $n\in\mathbb{N}^*$ there exists a unique solution $(Y^n,Z^n,K^n)\in\mathcal{H}^2_{\delta,c}(a^n,T)\times\mathcal{A}^2$ for the following RBDSDE

$$\begin{aligned}Y^n_t=&\,\xi+\int_t^Tf_n(s,Y^n_s,Z^n_s)ds+\int_t^Tg(s,Y^n_s,Z^n_s)d\overleftarrow{B}_s\\&+\int_t^TdK^n_s-\int_t^TZ^n_sdW_s,\qquad 0\leq t\leq T.\end{aligned} \tag{10}$$

Consequently, by (9) $\forall\ n\geq1$, $(Y^n,Z^n,K^n)\in\mathcal{H}^2_{\delta,c}(a,T)\times\mathcal{A}^2$.

Also, the existence and uniqueness result for stochastic Lipschitz case implies that there exists a unique solution $(U,V,K)\in\mathcal{H}^2_{\delta,c}(a,T)\times\mathcal{A}^2$ for following reflected BDSDEs

$$\begin{aligned} U_t =& \xi + \int_t^T H(s, U_s, V_s)ds + \int_t^T g(s, U_s, V_s)d\overleftarrow{B}_s + \int_t^T dK_s \\ & - \int_t^T V_s dW_s, \qquad 0 \leq t \leq T, \end{aligned}$$

because, the function $H(t) = \varphi(t) + r(t)|y| + \theta(t)|z|$ is stochastic Lipschitz.

Secondly, since for fixed (t, ω, y, z) and $\forall n \in \mathbb{N}^*$,

$$f_n(t, \omega, y, z) \leq f_{n+1}(t, \omega, y, z) \leq H(t),$$

it follows from the comparison theorem (**Theorem 3**) that for every $n \geq 1$,

$$Y^n \leq Y^{n+1} \leq U, \quad d\mathbb{P} \otimes dt - a.\ s. \tag{11}$$

The idea of the proof is to establish that the limit of the sequence (Y^n, Z^n, K^n) is a solution of the reflected BDSDE (1). In the next lemma, we prove that the norm $||(Y, Z)||_{\mathcal{H}^2_{\delta,c}}$ is bounded independently of n.

Lemma 5. *Under* **(A1.6)-(A1.9) and (H1.2), (H1.3).** *There exists a constant $C > 0$ independent of n such that*

$$||(Y, Z)||_{\mathcal{H}^2_{\delta,c}} \leq C. \tag{12}$$

Proof. For any $\delta > 0$, Itô's formula for applied to $e^{\delta A_t}|Y_t^n|^2$ provides

$$\begin{aligned} & e^{\delta A_t}|Y_t^n|^2 + \delta \int_t^T e^{\delta A_s}|Y_s^n|^2 dA_s + \int_t^T e^{\delta A_s}|Z_s^n|^2 ds + \sum_{t<s\leq T} e^{\delta A_s}(\Delta_s Y^n)^2 \\ & = e^{\delta A_T}|\xi|^2 + 2\int_t^T e^{\delta A_s} Y_{s-}^n f_n(s, Y_s^n, Z_s^n)ds + 2\int_t^T e^{\delta A_s} Y_{s-}^n g(s, Y_s^n, Z_s^n)d\overleftarrow{B}_s \\ & \quad - 2\int_t^T e^{\delta A_s} Y_{s-}^n Z_s^n dW_s + \int_t^T e^{\delta A_s}|g(s, Y_s^n, Z_s^n)|^2 ds + 2\int_t^T e^{\delta A_s} Y_{s-}^n dK_s^n. \end{aligned} \tag{13}$$

Taking expectation and $t = 0$, we get

$$\mathbb{E}\left(|Y_0^n|^2 + \delta \int_0^T e^{\delta A_s}|Y_s^n|^2 dA_s + \int_0^T e^{\delta A_s}|Z_s^n|^2 ds\right)$$

$$\leq \mathbb{E}\left(e^{\delta A_T}|\xi|^2+2\int_0^T e^{\delta A_s}Y^n_{s-}f_n(s,Y^n_s,Z^n_s)ds\right)$$
$$+\mathbb{E}\left(\int_0^T e^{\delta A_s}|g(s,Y^n_s,Z^n_s)|^2ds+2\int_0^T e^{\delta A_s}Y^n_{s-}dK^n_s\right).$$

Using the inequalities $2ab\leq \frac{1}{\epsilon}a^2+\epsilon b^2$, $(a+b)^2\leq (1+\epsilon)a^2+\frac{b^2}{1+\epsilon}$ and hypothesis **(A1.8), (H1.2),** we get

$$\mathbb{E}\left(|Y^n_0|^2+\delta\int_0^T e^{\delta A_s}|Y^n_s|^2dA_s+\int_0^T e^{\delta A_s}|Z^n_s|^2ds\right)$$
$$\leq \mathbb{E}\left(e^{\delta A_T}|\xi|^2+\left(2+\frac{1}{\epsilon}\right)\int_0^T e^{\delta A_s}|Y^n_s|^2dA_s+\epsilon\int_0^T e^{\delta A_s}|Z^n_s|^2ds+\int_0^T e^{\delta A_s}\frac{|\varphi(s)|^2}{a_s^2}ds\right)$$
$$+\mathbb{E}\Bigg((1+\epsilon)\int_0^T e^{\delta A_s}a_s^2|Y^n_s|^2ds+(1+\epsilon)\alpha\int_0^T e^{\delta A_s}|Z^n_s|^2ds$$
$$+\left(1+\frac{1}{\epsilon}\right)\int_0^T e^{\delta A_s}|g(s,\omega,0,0)|^2ds\Bigg)+2\mathbb{E}\int_0^T e^{\delta A_s}Y^n_{s-}dK^n_s.$$

Therefore, choosing ϵ, δ such that $(1-\epsilon-(1+\epsilon)\alpha)\geq 0$ and $\left(2+\frac{1}{\epsilon}\right)+(1+\epsilon)\leq\delta$, we obtain

$$\mathbb{E}\left(|Y^n_0|^2+\int_0^T e^{\delta A_s}|Y^n_s|^2dA_s+\int_0^T e^{\delta A_s}|Z^n_s|^2ds\right)$$
$$\leq C\mathbb{E}\Bigg(e^{\delta A_T}|\xi|^2+\int_0^T e^{\delta A_s}\frac{|\varphi(s)|^2}{a_s^2}ds$$
$$+\int_0^T e^{\delta A_s}|g(s,0,0)|^2ds+\frac{1}{\epsilon}\left(\sup_{0\leq t\leq T}e^{2\delta A_t}\left|L_t^+\right|^2\right)+\epsilon|K^n_T|^2\Bigg). \tag{14}$$

By equation (10), we have

$$\mathbb{E}|K^n_T|^2=\mathbb{E}\left(Y^n_0-\xi-\int_0^T f_n(s,Y^n_s,Z^n_s)ds-\int_0^T g(s,Y^n_s,Z^n_s)d\overleftarrow{B}_s+\int_0^T Z^n_sdW_s\right)^2,$$

Using the Cauchy–Schwarz inequality, isometry formula and the inequalites $\frac{r^2(t)}{a_t^2}\leq a_t^2$ and $\theta(t)\leq a_t^2$, we provide the existence of a positive constant C

independent of n such that

$$\mathbb{E}\left|K_T^n\right|^2 \leq C\mathbb{E}\Bigg(1+\left|Y_0^n\right|^2+e^{\delta A_T}\left|\xi\right|^2+\int_0^T e^{\delta A_s}\frac{\left|\varphi\left(s\right)\right|^2}{a_s^2}ds + \int_0^T e^{\delta A_s}a_s^2\left|Y_s^n\right|^2 ds+\int_0^T e^{\delta A_s}\left|Z_s^n\right|^2 ds\Bigg). \tag{15}$$

So, combining (14) with (15) and using assumptions **(H1.3)**, **(A1.9),** we obtain

$$\begin{aligned}
&\mathbb{E}\left(\left|Y_0^n\right|^2+\int_0^T e^{\delta A_s}a_s^2\left|Y_s^n\right|^2 ds+\int_0^T e^{\delta A_s}\left|Z_s^n\right|^2 ds\right)\\
&\leq C\mathbb{E}\Bigg(1+e^{\delta A_T}\left|\xi\right|^2+\int_0^T e^{\delta A_s}\frac{\left|\varphi\left(s\right)\right|^2}{a_s^2}ds+\int_0^T e^{\delta A_s}\left|g(s,\omega,0,0)\right|^2 ds\\
&\quad+\frac{1}{\epsilon}\Big(\sup_{0\leq t\leq T}e^{2\delta A_t}\left|L_t^+\right|^2\Big)\Bigg),\\
&\leq C<\infty.
\end{aligned}$$

Hence $||Y||_{\mathcal{M}_\delta^{2,a}}+||Z||_{\mathcal{M}_\delta^2}\leq C<\infty$. Now, we prove that $||Y||_{\mathcal{S}_\delta^2}\leq C<\infty$.

Using Burkholder-Davis-Gundy inequality provides

$$||Y^n||_{\mathcal{S}_\delta^2}+||Y^n||_{\mathcal{M}_\delta^{2,a}}+||Z^n||_{\mathcal{M}_\delta^2}\leq C<\infty. \qquad \square$$

In the following lemma, we prove that the couple (Y^n,Z^n) converge to (Y,Z) in $\mathcal{H}_{\delta,c}^2\left(0,T\right)$.

Lemma 6. *Under* **(A1.6)-(A1.9)** *and* **(H1.2), (H1.3),** *we have*

$$||Y^n-Y||_{\mathcal{S}_\delta^2}+||Y^n-Y||_{\mathcal{M}_\delta^{2,a}}+||Z^n-Z||_{\mathcal{M}_\delta^2}\underset{n\to\infty}{\to}0.$$

Proof. From (11) and (12), there exists a process Y^n such that $Y_t^n\nearrow Y_t$ a.s. for all $t\in[0,T]$. Therefore, it follows from Fatou's lemma together with the Lebesgue's dominated convergence theorem that

$$||Y||_{\mathcal{S}_\delta^2}\leq C,\qquad \textit{and}\qquad \lim_{n\to\infty}||Y^n-Y||_{\mathcal{M}_\delta^{2,a}}=0. \tag{16}$$

Next, for all $n\geq 1$, by Itô's formula applied to discontinuous semimartingale

$e^{\delta A_t}\left|Y_t^{n+1}-Y_t^n\right|^2$, we get

$$\begin{aligned}
&e^{\delta A_t}\left|Y_t^{n+1}-Y_t^n\right|^2+\delta\int_t^T e^{\delta A_s}\left|Y_s^{n+1}-Y_s^n\right|^2 dA_s\\
&\quad+\int_t^T e^{\delta A_s}\left|Z_s^{n+1}-Z_s^n\right|^2 ds+\sum_{t<s\leq T} e^{\delta A_s}\left(\Delta_s Y^{n+1}-\Delta_s Y^n\right)^2\\
&=2\int_t^T e^{\delta A_s}\left(Y_{s-}^{n+1}-Y_{s-}^n\right)\left(f_{n+1}(s,Y_s^{n+1},Z_s^{n+1})-f_n(s,Y_s^n,Z_s^n)\right)ds\\
&\quad+2\int_t^T e^{\delta A_s}\left(Y_{s-}^{n+1}-Y_{s-}^n\right)\left(g(s,Y_s^{n+1},Z_s^{n+1})-g(s,Y_s^n,Z_s^n)\right)d\overleftarrow{B}_s\\
&\quad-2\int_t^T e^{\delta A_s}\left(Y_{s-}^{n+1}-Y_{s-}^n\right)\left(Z_s^{n+1}-Z_s^n\right)dW_s\\
&\quad+\int_t^T e^{\delta A_s}\left|g(s,Y_s^{n+1},Z_s^{n+1})-g(s,Y_s^n,Z_s^n)\right|^2 ds\\
&\quad+2\int_t^T e^{\delta A_s}\left(Y_{s-}^{n+1}-Y_{s-}^n\right)\left(dK_s^{n+1}-dK_s^n\right). \qquad (17)
\end{aligned}$$

Letting $t=0$ and taking the expectation in (17), it follows from property (i) in **Lemma 4**, Cauchy-Schwartz inequality and assumptions **(A1.8), (H1.2)** that

$$\begin{aligned}
&\mathbb{E}\left(\left|Y_0^{n+1}-Y_0^n\right|^2+\delta\int_0^T e^{\delta A_s}a_s^2\left|Y_s^{n+1}-Y_s^n\right|^2 ds+\int_0^T e^{\delta A_s}\left|Z_s^{n+1}-Z_s^n\right|^2 ds\right)\\
&\leq C\left(\mathbb{E}\int_0^T e^{\delta A_s}\left(\frac{|\varphi(s)|^2}{a_s^2}+a^2(s)\left(\left|Y_s^{n+1}\right|^2+|Y_s^n|^2\right)+\left|Z_s^{n+1}\right|^2+|Z_s^n|^2\right)ds\right)^{\frac{1}{2}}\\
&\quad\times\left(\mathbb{E}\int_0^T e^{\delta A_s}\left|Y_s^{n+1}-Y_s^n\right|^2 ds\right)^{\frac{1}{2}}+\mathbb{E}\int_0^T e^{\delta A_s}a^2(s)\left|Y_s^{n+1}-Y_s^n\right|^2 ds\\
&\quad+\alpha\mathbb{E}\int_0^T e^{\delta A_s}\left|Z_s^{n+1}-Z_s^n\right|^2 ds+2\mathbb{E}\int_t^T e^{\delta A_s}\left(Y_{s-}^{n+1}-Y_{s-}^n\right)\left(dK_s^{n+1}-dK_s^n\right).
\end{aligned}$$

Therefore, using the fact that $\int_t^T e^{\delta A_s}\left(Y_{s-}^{n+1}-Y_{s-}^n\right)\left(dK_s^{n+1}-dK_s^n\right)\leq 0$ and from (12) and assumptions **(A1.9)**, we provide the existence of a constant $C>0$ independent of n such that

$$(1-\alpha)\,\mathbb{E}\int_0^T e^{\delta A_s}\left|Z_s^{n+1}-Z_s^n\right|^2 ds\leq C\left(\mathbb{E}\int_0^T e^{\delta A_s}a_s^2\left|Y_s^{n+1}-Y_s^n\right|^2 ds\right)^{\frac{1}{2}}.$$

Consequently, it follows from (16) that $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{M}_\delta^2\left(A,T,\mathbb{R}^d\right)$. Then there exists an process $Z\in\mathcal{M}_\delta^2\left(A,T,\mathbb{R}^d\right)$ such that

$$\lim_{n\to\infty}||Z^n-Z||_{\mathcal{M}_\delta^2}=0.$$

On the other hand, taking supremum and expectation in (17), we deduce from the Burkholder-Davis-Gundy inequality that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left(e^{\delta A_t}\left|Y_t^{n+1}-Y_t^n\right|^2\right)\right)\leq C\left(\mathbb{E}\int_0^T e^{\delta A_s}\left|Y_s^{n+1}-Y_s^n\right|^2ds\right)^{\frac{1}{2}},$$

from (16) the right-hand side converges to 0 as $\to+\infty$, from which we deduce that $\mathbb{P}$-almost surely, Y^n converges uniformly to Y. So Y is rcll and we have $\mathbb{E}\left(\sup_{t\in[0,T]}e^{\delta A_t}\left|Y_t\right|^2\right)\leq C$. □

***Proof of Theorem* 4.** Now, we show that the triplet (Y,Z,K) verifies a reflected BDSDE with one rcll reflecting barrier (1). Since $(Y^n,Z^n)\to(Y,Z)$ in $\mathcal{H}_{\delta,c}^2(a,T)$ (see, **Lemma 6**), along a subsequence which we still denote (Y^n,Z^n), we get

$$(Y^n,Z^n)\to(Y,Z)\ \ dt\otimes d\mathbb{P}\ a.e.,$$

and there exists $\chi\in\mathcal{M}_\delta^2\left(A,T,\mathbb{R}^{k\times d}\right)$ such that for all $n\geq 1$, $|Z^n|<\chi$ $dt\otimes d\mathbb{P}$ *a.e.* Therefore, by property (iii) in **Lemma 4**, we have

$$f_n\left(t,Y_t^n,Z_t^n\right)\to f\left(t,Y_t,Z_t\right)dt\otimes d\mathbb{P}\ a.e.,$$

Moreover, from property (i) in **Lemma 4** and inequality (11), we have

$$\begin{cases}\left|f_n\left(t,Y_t^n,Z_t^n\right)\right|\leq\Sigma\left(t\right)<\infty\ dt\otimes d\mathbb{P}\ a.e.,\\ \text{with}\\ \Sigma\left(t\right)=\varphi\left(t\right)+r\left(t\right)\left(\left|Y_t^1\right|+\left|U_t\right|\right)+\theta\left(t\right)\left|\chi_t\right|.\end{cases}$$

Then it follows from the **(A1.6)** and Lebesgue's dominated convergence theorem that

$$\mathbb{E}\int_0^T\left|f_n\left(s,Y_s^n,Z_s^n\right)-f\left(s,Y_s,Z_s\right)\right|^2ds\to 0,\ \text{as}\ n\to\infty.$$

We have also

$$
\begin{aligned}
&\mathbb{E}\int_t^T \left|g\left(s,Y_s^n,Z_s^n\right)-g\left(s,Y_s,Z_s\right)\right|^2 ds \\
&\leq C\mathbb{E}\left(\int_t^T e^{\delta A_s} a_s^2 \left|Y_s^n-Y_s\right|^2 ds+\alpha\int_t^T e^{\delta A_s}\left|Z_s^n-Z_s\right|^2 ds\right), \\
&\underset{n\to\infty}{\to} 0.
\end{aligned}
$$

Let

$$
\bar{Y}_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T g(s,Y_s,Z_s)d\overleftarrow{B}_s+\int_t^T d\bar{K}_s-\int_t^T \bar{Z}_s dW_s,
$$

where $\bar{K}=\bar{K}^c+\bar{K}^d$ defined as before, $\left(\bar{Y},\bar{Z}\right)\in\mathcal{H}^2_{\delta,c}(a,T)$ such that $\bar{Y}_t\geq L_t$ $\forall t\in[0,T]$ and $\int_0^T\left(\bar{Y}_t-L_t\right)dK_t=0$. By Itô's formula, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(e^{\delta A_t}\left|Y_t^n-\bar{Y}_t\right|^2+\delta\int_t^T e^{\delta A_s}a_s^2\left|Y_s^n-\bar{Y}_s\right|^2 ds+\int_t^T e^{\delta A_s}\left|Z_s^n-\bar{Z}_s\right|^2 ds\right) \\
&\leq 2\mathbb{E}\int_t^T e^{\delta A_s}\left(Y_{s-}^n-\bar{Y}_{s-}\right)\left(f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\right)ds \\
&\quad+2\mathbb{E}\int_t^T e^{\delta A_s}\left(Y_{s-}^n-\bar{Y}_{s-}\right)\left(dK_s^n-d\bar{K}_s\right) \\
&\quad+\mathbb{E}\int_t^T e^{\delta A_s}\left|g\left(s,Y_s^n,Z_s^n\right)-g\left(s,Y_s,Z_s\right)\right|^2 ds.
\end{aligned}
$$

Using the fact that $\int_t^T e^{\delta A_s}\left(Y_{s-}^n-\bar{Y}_{s-}\right)\left(dK_s^n-d\bar{K}_s\right)\leq 0$, we have

$$
\begin{aligned}
&\mathbb{E}\left(e^{\delta A_t}\left|Y_t^n-\bar{Y}_t\right|^2+\int_t^T e^{\delta A_s}\left|Z_s^n-\bar{Z}_s\right|^2 ds\right) \\
&\leq 2\mathbb{E}\int_t^T e^{\delta A_s}\left(Y_{s-}^n-\bar{Y}_{s-}\right)\left(f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\right)ds \\
&\quad+\mathbb{E}\int_t^T e^{\delta A_s}\left|g\left(s,Y_s^n,Z_s^n\right)-g\left(s,Y_s,Z_s\right)\right|^2 ds, \\
&\underset{n\to\infty}{\to} 0.
\end{aligned}
$$

Furthermore,

$$
\left(K_t^n-\bar{K}_t\right)^2=(Y_t^n-\bar{Y}_t-\int_t^T\left(f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\right)ds-K_T^n-\bar{K}_T
$$

$$-\int_t^T (g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s))d\overleftarrow{B}_s+\int_t^T \left(Z_s^n-\bar{Z}_s\right)dW_s)^2.$$

Using **B-D-G** inequality, we get

$$\begin{aligned}\left|\left|K^n-\bar{K}\right|\right|_{\mathcal{S}_0^2} \leq C\Big(&\left|\left|Y^n-\bar{Y}\right|\right|_{\mathcal{S}_\delta^2}+||f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)||_{\mathcal{M}_0^2}\\ &+||(g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s))||_{\mathcal{M}_0^2}+\left|\left|Z^n-\bar{Z}\right|\right|_{\mathcal{M}_\delta^2}\Big).\end{aligned}$$

Finally, passing to the limit we get $K^n \to \bar{K}=\bar{K}^c+\bar{K}^d$. Consequently we obtain $Y_t=\bar{Y}_t$ and $Z_t=\bar{Z}_t \; d\mathbb{P}\otimes dt-a.s.$

Now, we show that the triplet (Y,Z,K) is minimal solution of RBDSDE (1).

Let $\left(\tilde{Y},\tilde{Z},\tilde{K}\right)\in\mathcal{H}_{\delta,c}^2(a,T)\times\mathcal{A}^2$ be any solution of reflected BDSDE (1) and let us consider for any $n\geq 1$ the reflected BDSDE (10) with its solution (Y^n,Z^n,K^n), which converges to (Y,Z,K). Since $f_n\leq f$ for all $n\geq 1$, we get by virtue of the comparison theorem **(Theorem 3)** that $Y^n\leq\tilde{Y}$ for all $n\geq 1$. Therefore, $Y\leq\tilde{Y}$. That proves that (Y,Z,K) is the minimal solution for reflected BDSDE (1). □

6. Minimal Solution of Irregular Barrier Reflected BDSDEs with Left Continuous and Stochastic Linear Growth Generators

In this section, we mainly deal with the result that prove the existence of a minimal solution for reflected BDSDEs (1) under left continuous and stochastic linear growth conditions.

6.1. Assumption and definition

We assume that f and g satisfy the following assumptions:

- **(H1.10)** Stochastic linear growth: There esists a nonnegative process $\varphi_t\in\mathcal{M}_\delta^2\left(A,T,\mathbb{R}^d\right)$ such that f satisfies **(H1.8)**(i).
- **(H1.11)** $f(t,\cdot,z):\mathbb{R}\to\mathbb{R}$ is left continuous and $f(t,y,\cdot)$ is continuous.
- **(H1.12)** There exists a continuous function $\pi:[0,T]\times(\mathbb{R})^2\times\mathbb{R}^d$ satisfying for $y_1\geq y_2$,

$(y_1, y_2) \in (\mathbb{R})^2$, $(z_1, z_2) \in \left(\mathbb{R}^d\right)^2$

$$\begin{cases} |\pi(t, y, z)| \leq r(t)|y| + \theta(t)|z|, \\ f(t, \omega, y_1, z_1) - f(t, \omega, y_2, z_2) \geq \pi(t, y_1 - y_2, z_1 - z_2). \end{cases}$$

- **(H1.13)** g satisfies **(H1.8)**(ii) and $g(t, 0, 0, 0) \equiv 0$.

6.2. Existence result

In this subsection, we show the existence solutions to BDSDEs. Now we prove a technical Lemma before we introduce the main theorem (see [9] and [13]).

Lemma 7. *Let* $\pi(t, y, z)$ *satisfies* **(H1.12)**, g *satisfies* **(H1.13)**, h *belongs in* $\mathcal{M}^2_\delta\left(0, T, \mathbb{R}^d\right)$. *For a discontinuous function of finite variation* S *belong in* $\mathcal{A}^2$, *we consider the processes* $\left(\tilde{Y}, \tilde{Z}\right) \in \mathcal{H}^2_{\delta,c}(a, T)$ *such that:*

$$\begin{cases} \tilde{Y}_t = \xi + \int_t^T \left(\pi\left(s, \omega, \tilde{Y}_s, \tilde{Z}_s\right) + h(s)\right) ds + \int_t^T dS_s \\ + \int_t^T g\left(s, \omega, \tilde{Y}_s, \tilde{Z}_s\right) d\overleftarrow{B}_s - \int_t^T \tilde{Z}_s dW_s, \ 0 \leq t \leq T, \\ \int_t^T e^{\delta A_s} \tilde{Y}_s^- dS_s \geq 0. \end{cases} \tag{18}$$

Then we have

(i) *The BDSDE* (18) *has a least one solution* $\left(\tilde{Y}, \tilde{Z}\right) \in \mathcal{H}^2_{\delta,c}(a, T)$.

(ii) *if* $h(t) \geq 0$ *and* $\xi \geq 0$, *we have* $\tilde{Y}_t \geq 0$, $d\mathbb{P} \times dt - a.s.$

Proof. (i) Obtained from a previous part.
(ii) Applying Itô's- Meyer formula to $e^{\delta A_t}\left|\tilde{Y}_t^-\right|^2$, we have

$$\begin{aligned}
&\mathbb{E}\left(e^{\delta A_t}\left|\tilde{Y}_t^-\right|^2 + \delta\int_t^T e^{\delta A_s}\left|\tilde{Y}_s^-\right|^2 dA_s + \int_t^T 1_{\{\tilde{Y}_s<0\}} e^{\delta A_s}\left|\tilde{Z}_s\right|^2 ds\right) \\
&= \mathbb{E}\left(e^{\delta A_T}\left|\xi^-\right|^2\right) - 2\mathbb{E}\int_t^T e^{\delta A_s}\left(\tilde{Y}_s^- \pi(s, \tilde{Y}_s, \tilde{Z}_s) + h(s)\right) ds - 2\int_t^T e^{\delta A_s}\tilde{Y}_s^- dS_s \\
&\quad + \mathbb{E}\int_t^T 1_{\{\tilde{Y}_s<0\}} e^{\delta A_s}\left|g(s, \tilde{Y}_s, \tilde{Z}_s)\right|^2 ds.
\end{aligned}$$

Since $h(t) \geq 0$, $\xi \geq 0$ and using the fact that $\int_t^T e^{\delta A_s} \tilde{Y}_s^- dS_s \geq 0$, we obtain

$$\mathbb{E}\left(e^{\delta A_t}\left|\tilde{Y}_t^-\right|^2 + \delta\int_t^T e^{\delta A_s}\left|\tilde{Y}_s^-\right|^2 dA_s + \int_t^T 1_{\{\tilde{Y}_s<0\}} e^{\delta A_s}\left|\tilde{Z}_s\right|^2 ds\right)$$
$$\leq \mathbb{E}\left(2\int_t^T e^{\delta A_s}\left|\tilde{Y}_s^-\right|\left|\pi(s,\tilde{Y}_s,\tilde{Z}_s)\right| ds + \int_t^T 1_{\{\tilde{Y}_s<0\}} e^{\delta A_s}\left|g(s,\tilde{Y}_s,\tilde{Z}_s)\right|^2 ds\right)$$

According to assumptions **(H1.12)**, we get $\left|\pi\left(s,\tilde{Y}_s,\tilde{Z}_s\right)\right| \leq r(t)\left|\tilde{Y}_s\right| + \theta(t)\left|\tilde{Z}_s\right|$, by assumption **(H1.13)** for g and Young's inequality, we have

$$\mathbb{E}\left(e^{\delta A_t}\left|\tilde{Y}_t^-\right|^2 + \int_t^T 1_{\{\tilde{Y}_s<0\}} e^{\delta A_s}\left|\tilde{Z}_s\right|^2 ds\right)$$
$$\leq \left(3+\frac{1}{\beta}\right)\mathbb{E}\int_t^T e^{\delta A_s} a_s^2\left|\tilde{Y}_s^-\right|^2 ds + (\alpha+\beta)\,\mathbb{E}\int_t^T 1_{\{\tilde{Y}_s<0\}} e^{\delta A_s}\left|\tilde{Z}_s\right|^2 ds.$$

Therefore, choosing $\beta + \alpha < 1$, we get

$$\mathbb{E}\left(e^{\delta A_t}\left|\tilde{Y}_t^-\right|^2\right) \leq \left(3+\frac{1}{\beta}\right)\mathbb{E}\int_t^T e^{\delta A_s} a_s^2\left|\tilde{Y}_s^-\right|^2 ds.$$

Therefore, choosing β such that $0 < 3+\frac{1}{\beta} < 1$ and using Gronwall's inequality, we have, $\mathbb{E}\left(e^{\delta A_t}\left|\tilde{Y}_t^-\right|^2\right) = 0$, $d\mathbb{P}\times dt - a.s.$ for all $t \in [0,T]$. Finally implies that $\tilde{Y}_t^- \geq 0$, $d\mathbb{P}\times dt - a.s.$ for all $t\in[0,T]$. □

Now by the above theorem, we consider the processes $\left(\tilde{Y}_t^0, \tilde{Z}_t^0, \tilde{K}_t^0\right)$, $\left(Y_t^0, Z_t^0, K_t^0\right)$ and sequence of processes $\left(\tilde{Y}_t^n, \tilde{Z}_t^n, \tilde{K}_t^n\right)_{n\geq 0}$ respectively minimal solution of the following RBDSDEs for all $t\in[0,T]$

$$\begin{cases} Y_t^0 = \xi + \int_t^T \left[r(s)\left|Y_s^0\right| + \theta(s)\left|Z_s^0\right| + \varphi_s\right] ds + \int_t^T g(s,Y_s^0,Z_s^0)d\overleftarrow{B}_s \\ \qquad + \int_t^T dK_s^0 - \int_t^T Z_s^0 dW_s,\ 0\leq t\leq T,\ Y_t^0 \geq L_t,\ \forall t\in[0,T], \\ \text{the Skorokhod condition:} \\ \text{i) } \int_0^T \left(Y_t^0 - L_t\right) dK_t^{0,c} = 0,\ \text{where } K^{0,c} \text{ is the continuous part of } K^0, \\ \text{ii) if } K^{0,d} \text{ is the discontinuous part of } K^0,\ \text{then } K^{0,d} \text{ is predictable} \\ \qquad \text{and } K^{0,d} = \sum_{0<s<t}\left(Y_{s-}^0 - L_{s-}\right)^- \mathbf{1}_{\{\Delta L_s<0\}}. \end{cases} \tag{19}$$

$$\begin{cases} \tilde{Y}_t^0 = \xi + \int_t^T \left[-r(s) \left| \tilde{Y}_s^0 \right| - \theta(s) \left| \tilde{Z}_s^0 \right| - \varphi_s \right] ds + \int_t^T g(s, \tilde{Y}_s^0, \tilde{Z}_s^0) d\overleftarrow{B}_s \\ \qquad + \int_t^T d\tilde{K}_s^0 - \int_t^T \tilde{Z}_s^0 dW_s, \ 0 \le t \le T, \quad \tilde{Y}_t^0 \ge L_t, \ \forall t \in [0,T], \\ \text{the Skorokhod condition:} \\ \text{i) } \int_0^T \left(\tilde{Y}_t^0 - L_t \right) d\tilde{K}_t^{0,c} = 0, \text{ where } \tilde{K}^{0,c} \text{ is the continuous part of } \tilde{K}^0, \\ \text{ii) if } \tilde{K}^{0,d} \text{ is the discontinuous part of } \tilde{K}^0, \text{ then } \tilde{K}^{0,d} \text{ is predictable} \\ \quad \text{and } \ \tilde{K}^{0,d} = \sum_{0<s<t} \left(\tilde{Y}_{s-}^0 - L_{s-} \right)^- \mathbf{1}_{\{\Delta L_s < 0\}}. \end{cases} \tag{20}$$

and

$$\begin{cases} \tilde{Y}_t^n = \xi + \int_t^T \left[f(s, \tilde{Y}_s^{n-1}, \tilde{Z}_s^{n-1}) ds + \pi \left(s, \tilde{Y}_s^n - \tilde{Y}_s^{n-1}, \tilde{Z}_s^n - \tilde{Z}_s^{n-1} \right) \right] ds \\ \qquad + \int_t^T g(s, \tilde{Y}_s^n, \tilde{Z}_s^n) d\overleftarrow{B}_s + \int_t^T d\tilde{K}_s^n - \int_t^T \tilde{Z}_s^n dW_s -, \quad 0 \le t \le T, \\ \qquad \tilde{Y}_t^n \ge L_t, \ \forall t \in [0,T] \\ \text{the Skorokhod condition:} \\ \text{i) } \int_0^T \left(\tilde{Y}_t^n - L_t \right) d\tilde{K}_t^{n,c} = 0, \text{ where } \tilde{K}^{n,c} \text{ is the continuous part of } \tilde{K}^n, \\ \text{ii) if } \tilde{K}^{n,d} \text{ is the discontinuous part of } \tilde{K}^n, \text{ then } \tilde{K}^{n,d} \text{ is predictable} \\ \quad \text{and } \ \tilde{K}^{n,d} = \sum_{0<s<t} \left(\tilde{Y}_{s-}^n - L_{s-} \right)^- \mathbf{1}_{\{\Delta L_s < 0\}}. \end{cases} \tag{21}$$

For these solutions above, we get the following lemma.

Lemma 8. *Under the assumptions* **(H1.1)**–**(H1.3)** *and* **(H1.10)**–**(H1.13)**, *we have for any* $n \ge 1$ *and* $t \in [0,T]$

$$\tilde{Y}_t^0 \le \tilde{Y}_t^n \le \tilde{Y}_t^{n+1} \le Y_t^0.$$

Proof. We will prove $\tilde{Y}_t^n \le \tilde{Y}_t^{n+1}$ at first. For any $n \ge 0$. For any $n \ge 0$, we set

$$\begin{cases} \delta\rho_t^{n+1} = \rho_t^{n+1} - \rho_t^n, \\ \text{and} \\ \Delta\psi^{n+1}(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}) = \psi(s, \delta\tilde{Y}_s^{n+1} + \tilde{Y}_s^n, \delta\tilde{Z}_s^{n+1} + \tilde{Z}_s^n) - \psi(s, \tilde{Y}_s^n, \tilde{Z}_s^n). \end{cases}$$

Using equation (21), we have

$$\delta\bar{Y}_t^{n+1} = \int_t^T \left(\pi \left(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1} \right) + \theta_s^{n+1} \right) ds - \int_t^T \delta\tilde{Z}_s^{n+1} dW_s$$

$$+\int_t^T d\left(\delta\tilde{K}_s^{n+1}\right)+\int_t^T \Delta g^{n+1}(s,\delta\tilde{Y}_s^{n+1},\delta\tilde{Z}_s^{n+1})d\overleftarrow{B}_s,$$

where $\theta_s^{n+1}=\Delta f^n\left(s,\delta\tilde{Y}_s^n,\delta\tilde{Z}_s^n\right)-\pi\left(s,\delta\tilde{Y}_s^n,\delta\tilde{Z}_s^n\right)$ and $\theta_s^0=\Pi_s^1, \forall n\geq 0$. According to the assumptions on f and g, we can show that θ_s^0 and Δg^{n+1}, $\forall n\geq 0$ satisfy all assumptions of above Lemma. Moreover, since $\tilde{K}_\cdot^n = \tilde{K}_\cdot^{n,c}+\tilde{K}_\cdot^{n,d}$ where $\tilde{K}^{n,c}$ (resp. $\tilde{K}^{n,d}$) is the non-discontinuous (resp. purely discontinuous) part of $\tilde{K}^n$. In first part, we can show that

$$\int_0^T\left(\tilde{Y}_{t-}^n-L_{t-}\right)d\tilde{K}_t^n=0.$$

Using the above Lemma, we deduce that $\delta\tilde{Y}_t^{n+1}\geq 0$, i.e. $\tilde{Y}_t^n\leq\tilde{Y}_t^{n+1}$, $\forall t\in[0,T]$. Now we want to prove $\bar{Y}_t^0\leq\bar{Y}_t^n$. By equations (20) and (21), we have

$$\begin{aligned}\tilde{Y}_t^1-\tilde{Y}_t^0 \;=\; &\int_t^T\left(\pi\left(s,\delta\tilde{Y}_s^1,\delta\tilde{Z}_s^1\right)+\Pi_s^1\right)ds+\int_t^T d\left(\tilde{K}_s^1-\tilde{K}_s^0\right)\\ &+\int_t^T\left(g\left(s,\tilde{Y}_s^1,\tilde{Z}_s^1\right)-g\left(s,\tilde{Y}_s^0+\tilde{Z}_s^0\right)\right)d\overleftarrow{B}_s-\int_t^T\delta\tilde{Z}_s^1dW_s,\end{aligned}$$

where $\Pi_s^1=f(s,\tilde{Y}_s^0,\tilde{Z}_s^0)+r\,(s)\left|\tilde{Y}_s^0\right|+\theta\,(s)\left|\tilde{Z}_s^0\right|+\varphi_s$. By hypothesis **(H1.10)** we have $\Pi_s^1\geq 0$, because $\left(\tilde{Y}_t^0,\tilde{Z}_t^0\right)$ is the solution of Eq. (18), we get $\Pi_s^1\in\mathcal{M}_\delta^2\left(0,T,\mathbb{R}^d\right)$. Therefore, from **Lemma 7**, we get $\tilde{Y}_t^1\geq\tilde{Y}_t^0$. So, we have

$$\tilde{Y}_t^0\leq\tilde{Y}_t^n\leq\tilde{Y}_t^{n+1},\quad\forall t\in[0,T].$$

Now we shall prove that $\tilde{Y}_t^{n+1}\leq Y_t^0$, $\forall n\geq 0$, by Eqs. (19) and (21)

$$\begin{aligned}Y_t^0-\tilde{Y}_t^{n+1} \;=\; &\int_t^T\left(-r\,(s)\left|Y_s^0-\tilde{Y}_s^{+1}\right|-\theta\,(s)\left|Z_s^0-\tilde{Z}_s^{n+1}\right|+\Pi_s^{n+1}+\varphi_s\right)ds\\ &+\int_t^T\left(g(s,Y_s^0,Z_s^0)-g(s,\tilde{Y}_s^n,\tilde{Z}_s^n)\right)d\overleftarrow{B}_s+\int_t^T d\left(K_s^0-\tilde{K}_s^{n+1}\right)\\ &+\int_t^T\left(Z_s^0-\tilde{Z}_s^{n+1}\right)dW_s,\end{aligned}$$

where

$$\Pi_s^{n+1}=r\,(s)\left|Y_s^0-\tilde{Y}_s^{+1}\right|+\theta\,(s)\left|Z_s^0-\tilde{Z}_s^{n+1}\right|+r\,(s)\left|Y_s^0\right|+\theta\,(s)\left|Z_s^0\right|$$

$$-f(s,\tilde{Y}_s^n,\tilde{Z}_s^n)-\pi\left(s,\delta\tilde{Y}_s^{n+1},\delta\tilde{Z}_s^{n+1}\right).$$

By **Lemma 7** we deduce that $Y_t^0-\tilde{Y}_t^{n+1}\geq 0$, i.e. $Y_t^0\geq\tilde{Y}_t^{n+1}$, for all $t\in[0,T]$. Thus we have for all $n\geq 0$

$$Y_t^0\geq\tilde{Y}_t^{n+1}\geq\tilde{Y}_t^n\geq\tilde{Y}_t^0,\ d\mathbb{P}\times dt-a.s.\ \forall t\in[0,T].$$

The proof of **Lemma 8** is complete. □

Theorem 5. *Let $\xi\in\mathbb{L}^2(\delta,\mathcal{F}_T,\mathbb{R})$ and $t\in[0,T]$. Under assumption* **(H1.1)–(H1.3)** *and* **(H1.10)–(H1.13)**, *the RBDSDEs* (1) *has a minimal solution $(Y,Z)\in\mathcal{H}^2_{\delta,c}(a,T)$.*

Proof. From **Lemma 8**, we know $\left(\tilde{Y}_t^n\right)_{n\geq 0}$ is increasing and bounded in $\mathcal{M}_\delta^2\left(A,T,\mathbb{R}^d\right)$.

Since $\left|\tilde{Y}_t^n\right|\leq\max\left(\tilde{Y}_t^0,Y_t^0\right)\leq\left|\tilde{Y}_t^0\right|+\left|Y_t^0\right|$ for all $t\in[0,T]$, we have

$$\sup_n\mathbb{E}\left(\sup_{0\leq t\leq T}e^{\delta A_t}\left|\tilde{Y}_t^n\right|^2\right)\leq\mathbb{E}\left(\sup_{0\leq t\leq T}e^{\delta A_t}\left|\tilde{Y}_t^0\right|^2\right)+\mathbb{E}\left(\sup_{0\leq t\leq T}e^{\delta A_t}\left|\tilde{Y}_t^0\right|^2\right)<\infty,$$

then according to the Lebesgue's dominated convergence theorem, we deduce that $\left(\tilde{Y}_t^n\right)_{n\geq 0}$ converges in $\mathcal{S}_\delta^2\left(A,T,\mathbb{R}^d\right)$. We denote by $\tilde{Y}$ the limit of $\left(\tilde{Y}_t^n\right)_{n\geq 0}$. On the other hand from equation (21), we deduce that

$$\begin{aligned}\tilde{Y}_t^{n+1}=\xi+&\int_t^T\left[f(s,\tilde{Y}_s^n,\tilde{Z}_s^n)ds+\pi\left(s,\tilde{Y}_s^{n+1}-\tilde{Y}_s^n,\tilde{Z}_s^{n+1}-\tilde{Z}_s^n\right)\right]ds\\&+\int_t^T g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1})d\overleftarrow{B}_s+\int_t^T d\tilde{K}_s^{n+1}-\int_t^T\tilde{Z}_s^{n+1}dW_s-,\ 0\leq t\leq T,\end{aligned}$$

Applying Itô's formula, we obtain

$$\begin{aligned}&\mathbb{E}\left|\tilde{Y}_0^{n+1}\right|^2+\delta\mathbb{E}\int_0^T e^{\delta A_s}\left|\tilde{Y}_s^{n+1}\right|^2dA_s+\mathbb{E}\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^{n+1}\right|^2ds\\&\leq\ \mathbb{E}\left(e^{\delta A_T}|\xi|^2\right)+2\mathbb{E}\int_0^T e_s^{\delta A_s}\tilde{Y}_s^{n+1}\left(f(s,\tilde{Y}_s^n,\tilde{Z}_s^n)+\pi\left(s,\delta\tilde{Y}_t^{n+1},\delta\tilde{Z}_t^{n+1}\right)\right)ds\\&\quad+\mathbb{E}\int_0^T e^{\delta A_s}\left|g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1})\right|^2ds+2+\int_t^T\tilde{Y}_s^{n+1}d\tilde{K}_s^{n+1}.\end{aligned}$$

From assumptions **(H1.10), (H1.12)** and **(H1.13)** and Young's inequality, we get

$$
\begin{aligned}
&(1-\beta_1-\alpha)\,\mathbb{E}\!\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^{n+1}\right|^2 ds\\
\le&\mathbb{E}\left(e^{\delta A_T}|\xi|^2\right)+\left(6+\frac{1}{\beta_0}+\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)\mathbb{E}\!\int_0^T e^{\delta A_s}a_s^2\left|\tilde{Y}_s^{n+1}\right|^2 ds\\
&+2\mathbb{E}\!\int_0^T e^{\delta A_s}a_s^2\left|\tilde{Y}_s^{n}\right|^2 ds+(\beta_0+\beta_2)\,\mathbb{E}\!\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^{n}\right|^2 ds+\mathbb{E}\!\int_0^T e^{\delta A_s}\frac{\varphi_s^2}{a_s^2}ds,\\
\le& C+(\beta_0+\beta_2)\,\mathbb{E}\!\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^{n}\right|^2 ds.
\end{aligned}
$$

Choosing α such that $1-\beta_1>\alpha$ and divided by it, we obtain

$$
\mathbb{E}\!\left(\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^{n+1}\right|^2 ds\right)\le C\sum_{i=0}^{j=n-1}(\beta_0+\beta_2)^j+(\beta_0+\beta_2)^n\mathbb{E}\!\left(\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^{0}\right|^2 ds\right).
$$

Now choosing β_0 and β_2 such that $\beta_0+\beta_2<1$ and noting $\mathbb{E}\int_0^T\left|\tilde{Z}_s^0\right|^2 ds<\infty$. Obtain

$$
\sup_{n\in\mathbb{N}}\mathbb{E}\left(\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^{n+1}\right|^2 ds\right)<\infty,
$$

Now we shall prove that $\tilde{Z}^n$ is a Cauchy sequence in $\mathcal{M}_\delta^2\left(0,T,\mathbb{R}^d\right)$. Applying Itô's formula to $e^{\delta A_t}\left|\delta\tilde{Y}_t^{n,m}\right|^2=e^{\delta A_t}\left|\tilde{Y}_t^n-\tilde{Y}_t^m\right|^2$, we have

$$
\begin{aligned}
&\mathbb{E}\left(e^{\delta A_t}\left|\tilde{Y}_t^n-\tilde{Y}_t^m\right|^2\right)+\delta\mathbb{E}\!\int_0^T e^{\delta A_s}\left|\tilde{Y}_s^n-\tilde{Y}_s^m\right|^2 dA_s+\mathbb{E}\!\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^n-\tilde{Z}_s^m\right|^2 ds\\
=&2\mathbb{E}\!\int_0^T e^{\delta A_s}\left(\tilde{Y}_s^n-\tilde{Y}_s^m\right)(\Gamma_s^n-\Gamma_s^m)ds+\int_0^T e^{\delta A_s}\left|g\left(s,\tilde{Y}_s^n,\tilde{Z}_s^n\right)-g\left(s,\tilde{Y}_s^m,\tilde{Z}_s^m\right)\right|^2 ds.
\end{aligned}
$$

where $\Gamma_s^n=f(s,\tilde{Y}_s^{n-1},\tilde{Z}_s^{n-1})+\pi\left(s,\delta\tilde{Y}_s^n,\delta\tilde{Z}_s^n\right)$. By the Hölder's inequality and hypothesis **(H1.13)**, we deduce that

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^n-\tilde{Z}_s^m\right|^2 ds\\
&\le 2\mathbb{E}\left(\int_0^T e^{\delta A_s}\left|\tilde{Y}_s^n-\tilde{Y}_s^m\right|^2 ds\right)^{\frac{1}{2}}\mathbb{E}\left(\int_0^T e^{\delta A_s}\left|\Gamma_s^n-\Gamma_s^m\right|^2 ds\right)^{\frac{1}{2}}
\end{aligned}
$$

$$+\mathbb{E}\int_0^T e^{\delta A_s} a_s^2 \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds.$$

The boundedness of the sequence $\left(\tilde{Y}^n, \tilde{Z}^n\right)$, we deduce that the $\Lambda = \sup_{n\in\mathbb{N}} \left[\mathbb{E}\int_0^T |\Gamma_s^n|^2 ds\right] < \infty$, this yields that

$$\begin{aligned}
&(1-\alpha)\,\mathbb{E}\int_0^T e^{\delta A_s} \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds \\
&\leq 4\Lambda\mathbb{E}\left(\int_0^T e^{\delta A_s} \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds\right)^{\frac{1}{2}} + \mathbb{E}\int_0^T e^{\delta A_s} a_s^2 \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds,
\end{aligned}$$

which yields that $\left(\tilde{Z}^n\right)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{M}_\delta^2\left(0,T,\mathbb{R}^d\right)$. Then there exists $\mathcal{M}_\delta^2\left(0,T,\mathbb{R}^d\right)$ such that

$$\mathbb{E}\int_0^T e^{\delta A_s} \left|\tilde{Z}_s^n - Z_s\right|^2 ds \to 0 \ as \ n \to \infty.$$

On the other hand, by Burkholder-Davis-Gundy inequality we get

$$\begin{cases}
\mathbb{E}\sup\limits_{0\leq t\leq T}\left|\int_t^T \tilde{Z}_s^n dW_s - \int_t^T Z_s dW_s\right|^2 \leq \mathbb{E}\int_0^T e^{\delta A_s}\left|\bar{Z}_s^n - Z_s\right|^2 ds \to 0, \text{ as } n\to\infty, \\
\mathbb{E}\sup\limits_{0\leq t\leq T}\left|\int_t^T \left(g(s,\tilde{Y}_s^n,\tilde{Z}_s^n) - g(s,Y_s,Z_s)\right) d\overleftarrow{B}_s\right|^2 \\
\leq \mathbb{E}\int_0^T e^{\delta A_s} a_s^2 \left|\tilde{Y}_s^n - Y_s\right|^2 ds + \alpha\mathbb{E}\int_0^T e^{\delta A_s}\left|\tilde{Z}_s^n - Z_s\right|^2 ds \to 0, \text{ as } n\to\infty.
\end{cases}$$

Therefore, from the properties of f and π

$$\Gamma_s^n = f(s,\tilde{Y}_s^{n-1},\tilde{Z}_s^{n-1}) + \pi\left(s,\delta\tilde{Y}_t^n,\delta\tilde{Z}_t^n\right) \to f(s,Y_s,Z_s),$$

$\bar{\mathbb{P}} - a.s.$, for all $t\in[0,T]$ as $n\to\infty$. Then follows by Lebesgue's dominated convergence theorem that

$$\mathbb{E}\int_0^T e^{\delta A_s}\left|\Gamma_s^n - f(s,Y_s,Z_s)\right|^2 ds \to 0, \ n\to\infty$$

Since $\left(\tilde{Y}_s, \tilde{Z}_s, \Gamma_s^n\right)$ converges in $\mathcal{H}_{\delta,c}^2(a,T)\times\mathcal{M}_\delta^2\left(0,T,\mathbb{R}^d\right)$. Furthermore

$$\left(\tilde{K}_t^n - K_t\right)^2 = (Y_t^n - Y_t + \int_t^T \left(f(s,\tilde{Y}_s^{n-1},\tilde{Z}_s^{n-1}) - f(s,Y_s,Z_s)\right) ds$$

$$+\int_t^T \pi\left(s,\tilde{Y}_s^n-\tilde{Y}_s^{n-1},\tilde{Z}_s^n-\tilde{Z}_s^{n-1}\right)ds-\tilde{K}_T^n-K_T$$
$$+\int_t^T\Big(g(s,\tilde{Y}_s^n,\tilde{Z}_s^n)-g(s,Y_s,Z_s)\Big)\,d\overleftarrow{B}_s-\int_t^T(Z_s^n+Z_s)\,dW_s)^2.$$

Using **B-D-G** inequality, we get

$$\left|\left|\tilde{K}^n-K\right|\right|_{\mathcal{S}_0^2}$$
$$\leq C(||Y^n-Y||_{\mathcal{S}_\delta^2}+\left|\left|f(s,\tilde{Y}_s^{n-1},\tilde{Z}_s^{n-1})-f(s,Y_s,Z_s)\right|\right|_{\mathcal{M}_0^2}$$
$$+\left|\left|\pi\left(s,\tilde{Y}_s^n-\tilde{Y}_s^{n-1},\tilde{Z}_s^n-\tilde{Z}_s^{n-1}\right)\right|\right|_{\mathcal{M}_0^2}+\left|\left|g(s,\tilde{Y}_s^n,\tilde{Z}_s^n)-g(s,Y_s,Z_s)\right|\right|_{\mathcal{M}_0^2}$$
$$+||Z^n-Z||_{\mathcal{M}_\delta^2}).$$

Finally, passing to the limit we get $K^n\to K=K^c+K^d$. Letting $n\to+\infty$ in equation (21), we prove that (Y,Z) is solution to equation (1). Let (Y^*,Z^*) be any solution of the BDSDE (1), we have $Y_\cdot^n\leq Y_\cdot^*$, for all $n\geq 0$ and therefore, $Y_\cdot\leq Y_\cdot^*$ i.e., Y is the minimal solution. □

7. Conclusion

In this work, we have studied, reflected backward doubly stochastic differential equations with one lower rcll reflecting barrier. In the first part, via a penalization method we prove the existence and uniqueness of a solution under stochastic Lipschitz condition, also we proved the comparison theorem. In the second part, by inf-convolution approximation and comparison theorem we proved the existence of a minimal solution when the generator f is assumed to be continuous with stochastic linear growth condition, thus, our method in this case is a similar technique to that done by [7] with some suitable changes. Finally, we studied the existence of the minimal solution for BDSDE with rcll reflecting barrier and left continuous and stochastic linear growth condition.

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