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REMARKS ON STAR PRODUCTS FOR NON-COMPACT CR MANIFOLDS WITH NON-DEGENERATE LEVI FORM

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Abstract

We review results concerning quantization of connected orientable compact CR manifolds with non-degenerate Levi form. We shall also explain how to extend star products for a class of bounded functions on certain non-compact CR manifolds.

1. Introduction

Deformation quantization is a mathematical framework that seeks to associate a non-commutative algebra, known as a star-product algebra, to a given Poisson manifold. The goal is to provide a systematic way of quantizing classical systems, where classical observables are replaced by non-commutative operators on a Hilbert space. More precisely, one wants to define a star product for the algebra of power series $C^\infty(X)[[\nu]]$, where $C^\infty(X)$ is the algebra of smooth functions on a manifold X, and $[[\nu]]$ denotes the power series in a formal parameter ν.

Berezin-Toeplitz operators are a widely studied scheme for quantization on Hodge manifolds. The star products obtained in this way turn out to be of Wick type in the sense that one function is differentiated in holomorphic directions while the other in anti-holomorphic ones only. In this paper we review and extend some results obtained in [8], where we generalize the Berezin-Toeplitz star product to CR manifolds with non-degenerate Levi

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form, with n_- negative and n_+ positive eigenvalues, using the Szegő kernel for $(0, q)$-forms. The main application is a construction of deformation quantization of compact "quantizable" pseudo-Kähler manifolds. In [8] we also investigate the algebra of G-invariant Toeplitz operators on a compact CR manifold X with a Lie group G action and study the Fourier component in the presence of a transversal locally free circle action, which we will not review in this proceeding. We only remark here that the latter results are important for "quantization commutes with reduction" theorems for CR manifolds, see [17], [18] and [9]. An important difference between the CR setting and the symplectic setting is that the quantum spaces in the case of compact symplectic manifolds are finite dimensional, whereas for the compact CR manifolds that we consider, the quantum spaces are infinite dimensional, see also [18]. For a survey of results concerning the proposition that "quantization commutes with reduction" we refer to [20] and [21].

Our results relies on the method of Melin and Sjöstrand [24] and on the microlocal expression of the Szegő kernel [4]. The key result is that the Szegő kernel is a Fourier integral operator of complex type and this allows to apply the calculus of Fourier integral distributions to obtain the properties of algebra of Toeplitz operators. The main technical ingredient used in this paper is the microlocal expression of the Szegő kernel [13], which is in turn based on techniques of microlocal and semiclassical analysis, especially the stationary phase method of Melin and Sjöstrand. This approach is closer in spirit to [16], where Berezin-Toeplitz quantization of open sets of an arbitrary complex manifolds with a Hermitian holomorphic line bundle is considered. The authors study the case where the curvature on the line bundle is non-degenerate. In particular, they quantize any manifold admitting a positive line bundle. The quantum spaces are the spectral spaces corresponding to $[0, k^{-N}]$, where $N > 1$ is fixed, of the Kodaira Laplace operator acting on forms with values in tensor powers of the line bundle. They establish the asymptotic expansion of associated Toeplitz operators and their composition in the semi-classical limit and they define the corresponding star-product.

In [10] Guillemin constructed a star product on compact symplectic manifolds by replacing the CR functions with functions annihilated by a first order pseudodifferential operator introduced in [3], which has the same microlocal structure as the tangential Cauchy-Riemann operator and it is derived actually by first constructing the Szegő kernel. A similar approach was carried over in [1], [26] and [5] for quantization of Kähler manifolds. The quantizing Hilbert spaces of Ma and Marinescu, see [22] and [23], are kernels of the spin^c Dirac operators and their results are based on the asymptotic expansion as of the Bergman kernel of the corresponding Dirac operators from [6].

Karabegov and Schlichenmaier studied star products on compact Kähler manifolds via the asymptotic expansion of the Bergman kernel outside the diagonal in [19]. Engliš, see [7], gave similar results for bounded pseudoconvex domains in $\mathbb{C}^n$. In [25], Paoletti considers a natural variant of Berezin-Toeplitz quantization of compact Kähler manifolds, in the presence of a Hamiltonian circle action lifting to the quantizing line bundle. Paoletti studies the diagonal asymptotics of the associated Toeplitz operators, these expansions can be used to define in a natural manner a $\star$-product on a certain algebra of invariant functions depending on the moment map. There exist other constructions, but the result is only a formal star product without any relation to an operator calculus. We refer to [2], [27], [28] for a review of results.

The paper is split into two parts: we first review Toeplitz operators on compact CR manifolds by [8], in the second part we shall present new results concerning the non-compact case.

1. Berezin-Toeplitz Operators on Compact CR Manifolds, Review of Results

Let $(X, T^{1,0}X)$ be a compact, connected and orientable CR manifold of dimension $2n+1$, $n \geq 1$, of codimension one where $T^{1,0}X$ is a CR structure of X, that is, $T^{1,0}X$ is a integrable subbundle of rank n of the complexified tangent bundle $TX \otimes \mathbb{C}$, satisfying

$$T^{1,0}X \cap T^{0,1}X = \{0\}\,,$$

where $T^{0,1}X$ is the complex conjugate of $T^{1,0}X$. The unique subbundle HX of TX such that $HX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ is called horizontal bundle. Let $J : HX \to HX$ be the complex structure map given by $J(u+\overline{u}) = iu - i\overline{u}$, for every $u \in T^{1,0}X$; by complex linear extension of J to $TX \otimes \mathbb{C}$, the i-eigenspace of J is $T^{1,0}X$. We shall also write (X, HX, J) to denote a CR manifold. Now, fix a real non-vanishing 1 form $\omega_0 \in \mathcal{C}^\infty(X, T^*X)$ so that

$\langle\, \omega_0(x)\,,\, u\,\rangle = 0$, for every $u \in H_xX$ and for every $x \in X$. For each $x \in X$, we define a quadratic form on HX by

$$L_x(U,V) = \frac{1}{2}\mathrm{d}\omega_0(JU,V)\,,$$

for each $U, V \in H_xX$ and we extend L to $HX \otimes \mathbb{C}$ by complex linear extension. The Hermitian quadratic form L_x on $T^{1,0}_xX$ is called Levi form at x. Let $R \in \mathcal{C}^\infty(X,TX)$ be the non-vanishing vector field determined by

$$\omega_0(R) = -1, \quad \mathrm{d}\omega_0(R,\cdot) \equiv 0 \ \text{ on } TX.$$

Note that X is a contact manifold with contact form ω_0, contact distribution HX and R is the Reeb vector field.

Now, fix a smooth Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $TX\otimes\mathbb{C}$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$, $\langle\, u\,|\,v\,\rangle$ is real if u,v are real tangent vectors, $\langle\, R\,|\,R\,\rangle = 1$ and R is orthogonal to $T^{1,0}X \oplus T^{0,1}X$. For $u \in TX \otimes \mathbb{C}$, we write $|u|^2 := \langle\, u\,|\,u\,\rangle$ and denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with subbundles of the complexified cotangent bundle $T^*X \otimes \mathbb{C}$. We are now ready to introduce the vector bundle of $(0,q)$-forms by

$$T^{*0,q}X := \wedge^q T^{*0,1}X$$

whose space of smooth sections is denoted by $\Omega^{0,q}(X)$. The Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $TX \otimes \mathbb{C}$ induces, by duality, a Hermitian metric on $T^*X \otimes \mathbb{C}$ and also on the bundles of $(0,q)$-forms $T^{*0,q}X$ for any $q = 0,1,\cdots,n$; we shall also denote all these induced metrics by $\langle\,\cdot\,|\,\cdot\,\rangle$.

Let $dv(x)$ be the volume form on X induced by the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$. The natural global L^2 inner product $(\,\cdot\,|\,\cdot\,)$ on $\Omega^{0,q}(X)$ induced by $dv(x)$ and $\langle\,\cdot\,|\,\cdot\,\rangle$ is given by

$$(\,u\,|\,v\,) := \int_X \langle\, u(x)\,|\,v(x)\,\rangle\, dv(x)\,, \quad u,v \in \Omega^{0,q}(X)\,.$$

We denote by $L^2_{(0,q)}(X)$ the completion of $\Omega^{0,q}(X)$ with respect to $(\,\cdot\,|\,\cdot\,)$. We extend $(\,\cdot\,|\,\cdot\,)$ and $\overline{\partial}_b$ to $L^2_{(0,q)}(X)$ in the standard way and for $f \in L^2_{(0,q)}(X)$, we denote $\|f\|^2 := (\,f\,|\,f\,)$. Let $\Box^q_b$ denote (the Gaffney extension of) the

Kohn Laplacian. Let

$$S^{(q)} : L^2_{(0,q)}(X) \to \operatorname{Ker}\Box^q_b$$

be the orthogonal projection with respect to the L^2 inner product $(\cdot\,|\,\cdot)$ and let

$$S^{(q)}(x,y) \in \mathcal{D}'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

denote the distribution kernel of $S^{(q)}$. We are now ready to state the main assumptions of this paper.

Assumption 1.1. The Levi form is non-degenerate of constant signature (n_-, n_+), where n_- and n_+ denote the number of negative and respectively positive eigenvalues of the Levi form; and we suppose that $\Box^q_b$ has L^2 closed range.

By [13, Theorem 1.2] and [15, Theorem 4.7] there exist two continuous operators $S_-, S_+ : L^2_{(0,q)}(X) \to \operatorname{Ker}\Box^q_b$ such that

$$S^{(q)} = S_- + S_+, \qquad S_+ \equiv 0 \ \text{ if } q \neq n_+$$

and the wave front sets of S_- and S_+ in the sense of Hörmander are

$$\mathrm{WF}'\,(S_-) = \operatorname{diag}\,(\Sigma^- \times \Sigma^-) \text{ and } \mathrm{WF}'\,(S_+) = \operatorname{diag}\,(\Sigma^+ \times \Sigma^+) \text{ if } q = n_- = n_+,$$

where

$$\Sigma^- = \{(x, \lambda\omega_0(x)) \in T^*X;\, \lambda < 0\} \text{ and } \Sigma^+ = \{(x, \lambda\omega_0(x)) \in T^*X;\, \lambda > 0\}\ .$$

For $m \in \mathbb{R}$, let $L^m_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ denote the space of classical pseudodifferential operators on X of order m from sections of $T^{*0,q}X$ to sections of $T^{*0,q}X$; we write $Op(a)$ to denote the pseudodifferential operator of order m with full symbol a. Let

$$P \in L^{\ell}_{\mathrm{cl}}\,(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

with scalar principal symbol, $\ell \leq 0$, $\ell \in \mathbb{Z}$; we now in a state of readiness to provide a definition for *the Toeplitz operator associated to* P:

$$T^{(q)}_P := S^{(q)} \circ P \circ S^{(q)} : L^2_{(0,q)}(X) \to \operatorname{Ker}\Box^q_b,$$

and we also put

$$T_{P,-} := S_- \circ P \circ S_- : L^2_{(0,q)}(X) \to \operatorname{Ker} \Box^q_b\,,$$

and

$$T_{P,+} := S_+ \circ P \circ S_+ : L^2_{(0,q)}(X) \to \operatorname{Ker} \Box^q_b\,.$$

Now, we notice that if $q \neq n_\pm$, then $T_{P,\pm} \equiv 0$ on X. For ease of exposition we shall consider the case $q = n_-$ and $n_- \neq n_+$ so that $T_P := T_P^{(q)} = T_{P,-}$ and we shall show that the algebra of operators T_P induces a star product for a class of functions in the algebra $\mathcal{C}^\infty(X)$ whose Poisson bracket we shall define in a moment. In fact, for $u \in \mathcal{C}^\infty(X)$, there is a unique vector field X_u such that

$$\iota(X_u)\,\omega_0 = u\,, \text{ and } \iota(X_u)\,\mathrm{d}\omega_0 = (\iota(-R)\,\mathrm{d}u)\,\omega_0 - \mathrm{d}u.$$

Thus, for $u, v \in \mathcal{C}^\infty(X)$, we define the transversal Poisson bracket by

$$\{u, v\} := \mathrm{d}\omega_0(X_u, X_v)\,.$$

Before stating the first main theorem we recall a couple of definitions. Notice that the main consequence of Theorem 1.1 below is a deformation for the algebra defined in Definition 1.2.

Definition 1.1 (Operator of Szegő type)**.** Let $H : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ be a continuous operator with distribution kernel

$$H(x, y) \in \mathcal{D}'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)\,;$$

we say that H is a *complex Fourier integral operator of Szegő type* of order $k \in \mathbb{Z}$ if H is smoothing away the diagonal on X and for every local coordinates patch $D \subset X$ with local coordinates $x = (x_1, \ldots, x_{2n+1})$, we have

$$H(x, y) \equiv H_-(x, y) \equiv \int_0^\infty e^{i\varphi(x,y)t} a(x, y, t) dt$$

where a is classical symbols in the sense of Hörmander of order k, we write

$$a \in S^k_{\mathrm{cl}}\,(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

and φ is the complex phase function of the oscillatory integral defining the Szegő kernel, see [13]. We write $\sigma^0_H(x,y)$ to denote the leading term of the expansion of the symbol $a(x,y,t)$. Note that $\sigma^0_H(x,y)$ depends on the choice of the phase φ but $\sigma^0_H(x,x)$ is independent. Eventually we shall denote by $\Psi_k(X)$ the space of all complex Fourier integral operators of Szegő type of order k.

Definition 1.2 (Poisson algebra $\hat{S}$)**.** Let us define $\triangle_b := \Box^q_b + R^*R : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$, where R^* is the formal adjoint of R, $\sigma^0_{\triangle_b}$ denotes the principal symbol of $\triangle_b$. For every $j \in \mathbb{Z}$, $j \le 0$, put $\hat{S}^j$ to be the smooth functions on T^*X of the form $f \cdot (\sigma^0_{\triangle_b})^{j/2}$ where f is a smooth function on X. Eventually, we are now in position to articulate a definition for the Poisson algebra $\hat{S}$, whose elements are

$$\sum_{l=0}^{N} a_l\, f_l \qquad \text{with } a_l \in \mathbb{C},\ f_l \in \hat{S}^j,\ 0 \le l \le N \text{ and } N \in \mathbb{N}$$

and for $a \in S^k(X)$, $b \in S^\ell(X)$, $k,\ell \in \mathbb{Z}$, $k,\ell \le 0$, the *transversal Poisson bracket* of a and b with respect to Σ^- is defined to be the element $\{a,b\}_- \in \hat{S}^{k+\ell-1}$ such that

$$\{a,b\}_-(x,-\omega_0(x)) = \{\hat{a}_-,\hat{b}_-\}(x), \quad \text{for every } x \in X, \tag{1.1}$$

where $\hat{a}_-(x) := a(x,-\omega_0(x))$ and $\hat{b}_-(x) := b(x,-\omega_0(x))$ are smooth functions on X. Note that $\{a,b\}_-$ is uniquely determined by (1.1). Note that the definitions of $\{\cdot,\cdot\}_-$ can be extended by linearity to all $\hat{S}$. We can identify $\hat{S}^j$ with all homogeneous functions on Σ of degree j. We can define $\hat{+}$ on $\hat{S}$, by summing components of the same degree. Then, $(\hat{S},\hat{+})$ is a vector space and also $\hat{S}$ has natural algebraic structure, that is, if $a,b \in \hat{S}$, then $a \cdot b \in \hat{S}$.

Theorem 1.1. *Let us put $q = n_-$ and $n_+ \ne n_-$, if*

$$P \in L^\ell_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*) \text{ and } Q \in L^k_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

have scalar principal symbols, $\ell,k \le 0$, $\ell,k \in \mathbb{Z}$; then we get

$$[T_P,T_Q] \in \Psi_{n-1+\ell+k}(X) \text{ and } [T_P,T_Q] - T_{\mathrm{Op}\,(i\{\sigma^0_P,\sigma^0_Q\})} \in \Psi_{n-2+\ell+k}(X)$$

where $\{\sigma^0_P,\sigma^0_Q\}$ denotes the transversal Poisson bracket of σ^0_P and σ^0_Q with respect to Σ^-.

When X is strictly pseudoconvex, Theorem 1.1 was obtained by Boutet de Monvel and Guillemin, [3, Chapter 2], for $q = 0$. The proof of the previous theorem is based on the calculus of complex Fourier integral operators as developed in [24]. The big difference between $q = 0$ and $q > 0$ case is that the proof is closely related to the calculation of the second coefficient of the Szegő kernel asymptotic expansion which is more complicated for $q > 0$.

To establish star product for a certain class of functions on X, we first need to define a deformation for the algebra of symbols $\hat{S}$; we first recall the definition of star product.

Definition 1.3. A star product for the graded algebra $\hat{S}$ with respect to Σ^- is given by the power series

$$a \star b = \sum_{j=0}^{+\infty} C_j(a,b)\, \nu^{-j}$$

such that $\star$ is an associative $\mathbb{C}[[\nu]]$-linear product, that is, $(a\star b)\star c = a\star(b\star c)$, for all $a, b, c \in \hat{S}$ and $C_0(a,b) = a \cdot b$, $C_1(a,b) - C_1(b,a) = i\{a,b\}$, for all $a, b \in \hat{S}$, where $\{a,b\}$ denotes the transversal Poisson bracket of a and b with respect to Σ^-.

To simplify the notations, for $a \in \hat{S}$, we denote $T_a := T_{\mathrm{Op}\,(a)}$. Our second main result is about the existence of star product for $\hat{S}$. One can also state Theorem 1.1 for $n_- = n_+$ and as a consequence we can proof the following theorem.

Theorem 1.2. *Let $q = n_-$. Let $a \in \hat{S}^{\ell}$, $b \in \hat{S}^k$, $\ell, k \in \mathbb{Z}$, $\ell, k \leq 0$, we have*

$$T_a \circ T_b - \sum_{j=0}^{N} T_{C_j(a,b)} \in \Psi_{n-N-1+\ell+k}(X),$$

for every $N \in \mathbb{N}_0$, where $C_j(a,b) \in \hat{S}^{\ell+k-j}$, C_j is a universal bi-differential operator of order $\leq 2j$, $j = 0, 1, \ldots$. Moreover,

$$a \star b := \sum_{j=0}^{+\infty} C_j(a,b)\, \nu^{-j},$$

$a, b \in \hat{S}$, is an associative star product for the graded algebra $\hat{S}$ with respect to Σ^-.

We shall now briefly explain the idea of the proof. The key remark is that Toeplitz operators looks microlocally like the Szegő kernel, hence we define a class $\Psi_k(X)$ of a complex Fourier integral operator as in Definition 1.1. In [8, Lemma 4.1] we establish a criterion for an operator $B \in \Psi_k(X)$ to be in $B \in \Psi_{k-1}(X)$. The main tools in the proof of [8, Lemma 4.1] are the calculus defined in [24], the microlocal behavior of the Szegő kernel in [13] and [15, Theorem 5.4]. Now, [8, Lemma 4.1] is the main ingredient in the proof of Theorem 1.1. We observe that, when studying the commutator of Toeplitz operators for the Szegő kernel for $(0,q)$-forms, a cancellation property occurs, making it unnecessary to study the sub-leading term in the asymptotic expansions of Toeplitz operators. This observation is important because it is closely related to the calculation of the second coefficient of the asymptotic expansion of the Szegő kernel. In cases where the Levi form has negative eigenvalues, this calculation is highly non-trivial, see [14].

1.1. Application I: Star product for Reeb invariant functions

The main application of Theorem 1.2 is Theorem 1.4: when X admits a transversal and CR $\mathbb{R}$-action, we establish star product for $\mathbb{R}$-invariant smooth functions. First, let

$$\hat{R} := \frac{1}{2} S^{(q)}(-iR + (-iR)^*) S^{(q)} : \Omega^{0,q}(X) \to \Omega^{0,q}(X),$$

where $(-iR)^*$ is the adjoint of $-iR$ with respect to $(\,\cdot\,|\,\cdot\,)$. Then, note that $\hat{R} \in \Psi_{n+1}(X)$ with $\sigma^0_{\hat{R},-}(x,x) \neq 0$. Let $\hat{H} \in \Psi_{n-1}(X)$ with

$$S^{(q)}\hat{H} = \hat{H} = \hat{H}S^{(q)} \quad \text{and} \quad \hat{H}\hat{R} \equiv \hat{R}\hat{H} \equiv S^{(q)}. \tag{1.2}$$

Note that $\hat{H}$ is uniquely determined by (1.2), up to some smoothing operators. We can adapt the proof of Theorem 1.1 and deduce the following result.

Theorem 1.3. *Recall that we work with the assumption that $q = n_-$. Let $f, g \in \mathcal{C}^\infty(X)$. We have*

$$T_f \circ T_g - \sum_{j=0}^{N} \hat{H}^j T_{\hat{C}_j(f,g)} \in \Psi_{n-N-1}(X),$$

for every $N \in \mathbb{N}_0$*, where* $\hat{C}_j(f,g) \in \mathcal{C}^\infty(X)$*,* $\hat{C}_j$ *is a universal bidifferential operator of order* $\leq 2j$*,* $j = 0, 1, \ldots,$ *and*

$$\hat{C}_0(f,g) = f \cdot g\,, \quad \hat{C}_1(f,g) - \hat{C}_1(g,f) = i\,\{f,g\}.$$

As in the Theorem 1.2, for each $f, g \in \mathcal{C}^\infty(X)$ we can define

$$f \,\hat{\star}\, g = \sum_{j=0}^{+\infty} \hat{C}_j(f,g)\,\nu^{-j}.$$

but now, in general, $\hat{\star}$ is not associative. When $\hat{R}$ commutes with all T_f, we can show that $\star$ is associative. As a concrete case, assume that X admits a transversal CR $\mathbb{R}$-action η and take R so that R is induced by the $\mathbb{R}$-action. Suppose that X admits a $\mathbb{R}$-invariant Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ and let $(\,\cdot\,|\,\cdot\,)$ be the L^2 inner product for $\Omega^{0,q}(X)$ induced by $\langle \cdot \,|\, \cdot \rangle$. If we put

$$\mathcal{C}^\infty(X)^{\mathbb{R}} := \{f \in \mathcal{C}^\infty(X) \,:\, \eta^* f = f\}$$

we can check that

$$\hat{H} T_f \equiv T_f \hat{H} \;\text{ for all } f \in \mathcal{C}^\infty(X)^{\mathbb{R}}.$$

Moreover, it is straightforward to see that for every $j = 0, 1, \ldots$ and for all $f, g \in \mathcal{C}^\infty(X)^{\mathbb{R}}$

$$\hat{C}_j(f,g) \in \mathcal{C}^\infty(X)^{\mathbb{R}}\,.$$

Theorem 1.4. *The star product*

$$f \,\hat{\star}\, g = \sum_{j=0}^{+\infty} \hat{C}_j(f,g)\,\nu^{-j}\,,$$

$f,\, g \in \mathcal{C}^\infty(X)^{\mathbb{R}}$*, is associative.*

1.2. Application II: Star product for compact quantizable pseudo-Kähler manifolds

In this section, we explain how to apply the results of Section 1 in the situation of deformation quantization of compact "quantizable" pseudo-Kähler manifolds, this result was proved in [22, Theorem 8.2.5] using a different approach. Let (L, h^L) be a holomorphic line bundle over a compact complex manifold M and let (L^k, h^{L^k}) be the k-th power of (L, h^L), where h^L denotes the Hermitian metric of L. Let R^L be the curvature of L induced by h^L. Fix a Hermitian metric $\langle \cdot | \cdot \rangle$ on the holomorphic tangent bundle $T^{1,0}M$ of M and let $(\cdot | \cdot)_k$ be the L^2 inner product of $\Omega^{0,q}(M, L^k)$ induced by $\langle \cdot | \cdot \rangle$ and h^{L^k}, where $\Omega^{0,q}(M, L^k)$ denotes the space of smooth $(0, q)$-forms of M with values in L^k. Let

$$\Box_k^q := \overline{\partial}^* \, \overline{\partial} + \overline{\partial} \, \overline{\partial}^* : \Omega^{0,q}(M, L^k) \to \Omega^{0,q}(M, L^k)$$

be the Kodaira Laplacian, where $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$ with respect to $(\cdot | \cdot)_k$. Let

$$B_k^{(q)} : L^2_{(0,q)}(M, L^k) \to \operatorname{Ker} \Box_k^q$$

be the orthogonal projection with respect to $(\cdot | \cdot)_k$ (Bergman projection). For $f \in \mathcal{C}^\infty(M)$, let M_f denote the operator given by the multiplication f. The Toeplitz operator is given by

$$T_{f,k} := B_k^{(q)} \circ M_f \circ B_k^{(q)} : L^2_{(0,q)}(M, L^k) \to \operatorname{Ker} \Box_k^q.$$

Applying Theorem 1.3 to the circle bundle of (L^*, h^{L^*}), we get

Theorem 1.5. *With the same assumptions as above, suppose that the curvature $R^L = -2i\omega$ is non-degenerate of constant signature (n_-, n_+) and let $q = n_-$. For each $f, g \in \mathcal{C}^\infty(M)$, as $k \gg 1$,*

$$\left\| T_{f,k} \circ T_{g,k} - T_{g,k} \circ T_{f,k} - \frac{1}{k} T_{i\{f,g\},k} \right\| = O(k^{-2}),$$

and

$$\left\| T_{f,k} \circ T_{g,k} - \sum_{j=0}^{N} k^{-j} T_{C_j(f,g),k} \right\| = O(k^{-N-1})$$

in L^2 operator norm, for every $N \in \mathbb{N}$, where $C_j(f, g) \in \mathcal{C}^\infty(M)$, C_j is a universal bidifferential operator of order $\leq 2j$, $j = 0, 1, \ldots,$ and

$$C_0(f, g) = f \cdot g,$$
$$C_1(f, g) - C_1(g, f) = i\{f, g\},$$

where $\{\cdot,\cdot\}$ *denotes the Poisson bracket:* $\{f,g\} := \omega(X_f, X_g)$, $f,g \in \mathcal{C}^\infty(M)$; X_f *(respectively* X_g*) denotes the Hamiltonian vector field of* f *(respectively* g*) on* (M,ω).

Moreover, the star product

$$f \star g = \sum_{j=0}^{+\infty} C_j(f,g)\,\nu^{-j},$$

$f,g \in \mathcal{C}^\infty(M)$, *is associative.*

This theorem can be thought of as an analogue of Theorem 2.2 in [26] in the more general setting of complex manifolds. We remark that the quantization Hilbert spaces of $(0,q)$-forms we consider here are kernels of Dolbeault-Dirac operators.

2. Toeplitz Operators on Non-Compact CR Manifolds

In this section, we shall explain how to adapt the proof of Theorem 1.2 to non-compact manifolds: we shall adopt the same notation and assumptions as in Section 1 except for the fact that here (X, HX, J) is non-compact; we shall also impose a natural local analytic condition (weaker than L^2-closed range hypothesis for the Kohn Laplacian) which implies that the Szegő kernel admits a local asymptotic expansion. First, we need to recall some definitions and theorems from [15]. Without any regularity assumption, $\ker \Box_b^q$ could be trivial and, therefore, we consider the spectral projections $S^{(q)}_{\leq\lambda} := E([0,\,\lambda])$ for $\lambda > 0$, where E denotes the spectral measure of $\Box_b^q$.

Theorem 2.1 ([15, Theorem 1.5]). *For every* $\lambda > 0$ *the restriction of the spectral projector* $S^{(q)}_{\leq\lambda}$ *is a smoothing operator for* $q \neq n_-,\, n_+$ *and is a Fourier integral operator with complex phase for* $q = n_-,\, n_+$*; in the latter case, the singularity of* $S^{(q)}_{\leq\lambda}$ *does not depend on* λ*. Moreover, on any local neighborhood* $D \Subset X$ *we have*

$$S_{\leq\lambda}(x,\,y) = \int_0^{+\infty} e^{i\,t\varphi_-(x,y)} s_-(x,y,t)\,\mathrm{d}t + \int_0^{+\infty} e^{i\,t\varphi_+(x,y)} s_+(x,y,t)\,\mathrm{d}t + R(x,\,y) \tag{2.1}$$

where the integrals are oscillatory integrals, φ_- and φ_+ are complex phase functions, s_- and s_+ classical symbol of type $(1,0)$ and order $n-1$, $s_- = 0$ if $q \neq n_-$, $s_+ = 0$ if $q \neq n_+$ and R is a smooth function.

A detailed version of Theorem 2.1 is given in [15, Theorems 4.1, 4.7 and 4.8]. Notice that Theorem 1.5 is deduced by chapter 6, 7 and 8 of part I in [13]. In fact, the existence of the microlocal Hodge decomposition in [13] is stated for compact CR manifolds, but the construction and arguments used are essentially local. The key feature is that the complex phase functions appearing in (2.1) own the same properties as the Szegő kernel defined in Section 1.

Now, in view of Theorem 2.1 and the remark of the previous paragraph, we can carry over to non-compact case the local computations in [8, Lemma 4.1] and the ones following [8, Theorem 4.4]. In order to give a global meaning to the operators C_j's of the $\star$ product we need to introduce a class of bounded functions on X.

Definition 2.1. Let us denote by $\mathcal{B}(X)$ the space of smooth functions on X such that their derivatives are all bounded. Assume that X admits a transversal CR $\mathbb{R}$-action η and take R so that R is induced by the $\mathbb{R}$-action; then we put $\mathcal{B}(X)^{\mathbb{R}}$ to be the space of all $f \in \mathcal{B}(X)$ such that $\eta^* f = f$.

In view of the Definition 2.1, for each $f \in \mathcal{B}(X)^{\mathbb{R}}$, we define the corresponding Toeplitz operator

$$T_f := S^{(q)}_{\leq\lambda} \circ f \circ S^{(q)}_{\leq\lambda}\,.$$

Assume that X admits a transversal CR $\mathbb{R}$-action η as in Section 1.1 and let $\hat{H}$ be uniquely determined by (1.2), up to some smoothing operators. We can adapt the proof of Theorem 1.1 and deduce an analogue of Theorem 1.3. In fact, put $q = n_-$. Let $f, g \in \mathcal{B}(X)^{\mathbb{R}}$. We have

$$T_f \circ T_g - \sum_{j=0}^{N} \hat{H}^j T_{\hat{C}_j(f,g)} \in \Psi_{n-N-1}(X),$$

for every $N \in \mathbb{N}_0$, where $\hat{C}_j(f,g)$ lies in $\mathcal{B}(X)^{\mathbb{R}}$, $\hat{C}_j$ is a universal bidifferential operator of order $\leq 2j$, $j = 0, 1, \ldots$. Here is where we are using the fact that derivatives of f and g are all bounded, in fact by the definition of $\hat{C}_j$ we have that $\hat{C}_j(f,g)$ lies in $\mathcal{B}(X)^{\mathbb{R}}$. Furthermore, we can check that

$$\hat{C}_0(f,g) = f \cdot g\,, \quad \hat{C}_1(f,g) - \hat{C}_1(g,f) = i\,\{f,g\}\,.$$

Thus, we get an associative star product for the algebra $\mathcal{B}(X)^{\mathbb{R}}[[\nu]]$.

Example 2.1. Let $\mathbb{H}_{n+1} = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group. Let $x = (z,\, x_{2n+1})$ denote the coordinates on $\mathbb{H}_{n+1}$ where $z = (z_1\,,\,\dots,\,z_n)$ and $z_j = x_{2j-1} + i\,x_{2j}$ for each $0 \leq j \leq n$. The standard CR structure $T^{1,0}\mathbb{H}_{n+1}$ is given by

$$T^{1,0}\mathbb{H}_{n+1} := \operatorname{span}_{\mathbb{C}}\left\{Z_j \,:\, Z_j = \frac{\partial}{\partial z_j} - i\,\lambda_j\,\overline{z}_j\frac{\partial}{\partial x_{2n+1}},\, j = 1,\dots,n\right\}$$

where $\lambda_j \in \mathbb{R}$ are given real numbers, see [11]. Here the space $\mathcal{B}(X)^{\mathbb{R}}$ is given by all smooth functions on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ whose derivatives are bounded.

As a corollary of the main result of this section we can construct star product for non-compact “quantizable” pseudo-Kähler manifolds, see also [23].

Eventually we mentioned that in [14] Szegő kernel asymptotics for high power of CR line bundles are studied. In an ongoing project with Hsiao and Herrmann, we are investigating the existence of star product for CR line bundles.

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