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SEMI-CLASSICAL ASYMPTOTICS OF BERGMAN AND SPECTRAL KERNELS FOR $(0,q)$-FORMS

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Abstract

In this paper, we develop a new scaling method to study spectral and Bergman kernels for the k-th tensor power of a line bundle over a complex manifold under local spectral gap condition. In particular, we establish a simple proof of the pointwise asymptotics of spectral and Bergman kernels. As a new result, in the function case, we obtain the leading term of Bergman kernel under spectral gap with exponential decay. Moreover, in the general cases of $(0,q)$-forms, the asymptotics remain valid while the curvature of the line bundle is degenerate.

1. Introduction

Let M be a Hermitian complex manifold with $\dim_{\mathbb{C}} M = n$ and equip M with a positive Hermitian $(1,1)$-form ω. Consider a holomorphic line bundle L over M with a locally defined weight function ϕ that gives L a Hermitian metric h. The Hermitian form ω and the metric h endow the space of L-valued $(0,q)$-forms with a L^2-inner product. By taking the completion of this space with respect to the inner product, we obtain the Hilbert space $L^2_{\omega,\phi}(M, T^{*,(0,q)}M \otimes L)$. Consider $\Box^{(q)}_{\omega,\phi}$ to be the Kodaira Laplacian induced by the Hermitian structures ω and h. The *Bergman projection*

$$\mathcal{B}^{(q)}_{\omega,\phi} : L^2_{\omega,\phi}(M, T^{*,(0,q)}M \otimes L) \to \operatorname{Ker} \Box^{(q)}_{\omega,\phi}$$

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is the orthogonal projection from the space of L^2-integrable sections of $T^{*,(0,q)}M \otimes L$ onto the space of harmonic sections with respect to Kodaira Laplacian $\Box^{(q)}_{\omega,\phi}$. For a Borel set $B \subset \mathbb{R}$, we denote by $\mathbb{1}_B(\Box^{(q)}_{\omega,\phi})$ the functional calculus of $\Box^{(q)}_{\omega,\phi}$ with respect to the indicator function $\mathbb{1}_B$ (cf. [11, section 2]). Given a non-negative constant c, the *spectral projection*

$$\mathcal{P}^{(q)}_{\omega,\phi,c} := \mathbb{1}_{[0,c]}(\Box^{(q)}_{\omega,k\phi}) : L^2_{\omega,\phi}(M, T^{*,(0,q)}M \otimes L) \to E^{(q)}_{\leq c}$$

is the orthogonal projection onto the space Rang $\left(\mathbb{1}_{[0,c]}(\Box^{(q)}_{\omega,k\phi})\right)$ denoted by $E^{(q)}_{\leq c}$. The *Bergman kernel* $B^{(q)}_{\omega,\phi}(z,w)$ is the Schwartz kernel of $\mathcal{B}^{(q)}_{\omega,\phi}$ and the *spectral kernel* $P^{(q)}_{\omega,\phi,c}(z,w)$ is the Schwartz kernel of $\mathcal{P}^{(q)}_{\omega,\phi,c}$.

The Bergman kernel is a fundamental object in complex analysis and geometry, which plays a central role in some important problems in complex geometry, geometric quantization, and mathematical physics. However, it is challenging to study the Bergman kernel directly. Inspired by quantum mechanics and semi-classical analysis, if we consider the k-th tensor power L^k of L and replace the Hermitian metric ϕ by $k\phi$, it is possible to handle the asymptotic behavior of the Bergman kernel as k goes to infinity. Therefore, the study of the large k behavior of the Bergman kernel $B^{(q)}_{\omega,k\phi}(z,w)$ has become prominent in modern research. The asymptotic behavior of the Bergman kernel $B^{(q)}_{\omega,k\phi}(z,w)$ is rich in geometrical meaning and closely related to index theory and algebraic geometry. In [1], R. Berman obtained the local holomorphic Morse inequalities by analyzing the Bergman kernel on the diagonal part. In [20], C.-Y. Hsiao illustrated a proof of the Kodaira embedding theorem by the full expansion. Furthermore, the approximation of Kähler metrics(e.g., [5], [28]), existence of canonical Kähler metrics (e.g., [7], [8], [13], [14]) and the Berezin-Toeplitz quantization (e.g., [4], [22], [23], [27]) are impressive applications. We refer readers to the book [25] of X. Ma and G. Marinescu for a comprehensive study of Bergman kernel and relative subjects.

For a compact manifold M with a positive line bundle L, T. Bouche (1990, [5]) and G. Tian (1990, [28]) obtained the leading term of the Bergman kernel, and D. Catlin proved the full expansion (1997, [6]) later. More precisely, D.Catlin claimed that

$$B^{(q)}_{\omega,k\phi}(z,z) \sim k^n b^{(q)}_n + k^{n-1} b^{(q)}_{n-1} + \cdots + b^{(q)}_0 \quad \text{as } k \to \infty \tag{1.1}$$

for the case $q = 0$. Furthermore, X. Dai, K. Liu and X. Ma gave another proof of the full expansion based on localized techniques and heat kernel methods (2004, [9]),(2008, [10]) and B. Berndtsson, R. Berman and J. Sjöstrand also offered a different proof (2008, [2]).

In the case of non-degenerate line bundle L which may not be positive, if M is compact and $M = M(q)$ (cf. Def. 1.1), there is a full asymptotic expansion of $B^{(q)}_{\omega,k\phi}(z,w)$ proven by R. Berman and J. Sjöstrand (2007, [3]). Moreover, in (2006, [24]), X. Ma and G. Marinescu established similar results in the context of spin^c-Dirac operators in compact symplectic manifolds. In a later work, C.-Y. Hsiao and G. Marinescu (2014, [21]) demonstrated that the Bergman kernel has a local asymptotic expansion at all non-degenerate points under the local spectral gap condition (cf. Def. 1.2). Also, they showed that the spectral kernel $P^{(q)}_{\omega,k\phi,k^{-N}}$ has an analogous result.

In this paper, we derive the leading term $b^{(q)}_n$ of the asymptotic expansion (cf. (1.1)) by scaling method under the local spectral gap condition (cf. Def.1.2). For the function case, we can loosen the spectral gap condition to an exponential decay rate (cf. Def.1.3). It is noteworthy that we do not require the curvature to be non-degenerate.

As for the spectral kernel, we fix a sequence c_k satisfying $\limsup_{k\to\infty} k^{-1}c_k = 0$ and consider the asymptotic behavior of $P^{(q)}_{\omega,k\phi,c_k}(z,z)$. If there exists an integer d such that $\liminf_{k\to\infty} k^d c_k > 0$, then we can also obtain the leading term of the expansion of $P^{(q)}_{\omega,k\phi,c_k}(z,z)$. Furthermore, in the case $q = 0$, we only need a weaker condition of c_k that

$$\exists c < 1 \,\text{such that}\;\; \liminf e^{2c\min\lambda_i\cdot k^{1/2}} c_k > 0 \quad \text{where}\;\; \lambda_i \,\text{are defined in (1.2).}$$

1.1. Set-up and the main results

Let (M,ω) be a Hermitian manifold with complex dimension n where ω is a positive Hermitian $(1,1)$-form. Denote by $\langle\cdot|\cdot\rangle_\omega$ the pointwise Hermitain inner product induced by ω on $T_{\mathbb{C}}M$ and dV_ω the induced Riemannian volume form given by $\frac{\omega^n}{n!}$.

We consider a holomorphic Hermitian line bundle (L, h^L) over the manifold M, and denote its k-th tensor power $L^{\otimes k}$ by L^k. Let s be a local holomorphic trivializing section of L over an open subset U of M. The Hermitian metric h^L corresponds locally to a *weight function* $\phi : U \to \mathbb{R}$ such that $|s|^2_{h^L} = e^{-2\phi}$. Denote by s^k the k-th tensor power $s^{\otimes k}$ of s. Then the metric of L^k in U can be described as $|s^k|^2_{k\phi} := |s^k|^2_{h^{L^k}} = e^{-2k\phi}$ where s^k trivializes L^k in U with its weight function $k\phi$. Denote by $\langle\cdot|\cdot\rangle_{k\phi} := \langle\cdot|\cdot\rangle_{h^{L^k}}$ the pointwise Hermitian inner product h^{L^k} on L^k for convenience.

We also introduce the holomorphic Hermitian connection ∇^L on (L, h^L) that has a curvature form denoted by Θ^L. We identify Θ^L with a Hermitian matrix $\dot{\Theta}^L \in \mathcal{C}^\infty(M, \mathrm{End}\,(T^{(1,0)}M))$ that satisfies the following equation:

$$\langle \dot{\Theta}^L(z)v_1 \mid v_2\rangle_\omega := \Theta^L(z)(v_1 \wedge \overline{v_2}) \quad \text{for all } \ v_1, v_2 \in T_z^{(1,0)}M, \quad z \in M.$$

Next, we set the notation describing the signature of the curvature.

Definition 1.1. For any $q \in \{0, 1, \ldots, n\}$, we denote

$$M(q) := \{z \in M \mid \dot{\Theta}^L(z) \in \mathrm{End}\,(T_z^{(1,0)}M) \text{ is non-degenerate and has exactly } q \text{ negative eigenvalues}\}.$$

There is a natural Hermitian structure denoted by $\langle\cdot|\cdot\rangle_{\omega,k\phi}$ on the vector bundle $T^{*,(0,q)}M \otimes L^k$ over M obtained by the Hermitian pointwise inner product on $T^{*,(0,q)}M$ induced by ω (cf. (2.2)) and the local weight functions $k\phi$ of the Hermitian metric h^{L^k} of L^k, where $T^{*,(0,q)}M$ denotes the bundle of $(0,q)$-forms on M (cf. (2.1)). Let $\Omega^{(0,q)}(M, L^k)$ be the space of smooth $(0,q)$-forms on M with values in L^k, and let $\Omega_c^{(0,q)}(M, L^k)$ be the subspace of $\Omega^{(0,q)}(M, L^k)$ consisting of elements with compact support in M. The pointwise inner product $\langle\cdot|\cdot\rangle_{\omega,k\phi}$ on $T^{*,(0,q)}M \otimes L^k$ induces a $L^2_{\omega,k\phi}$-inner product $(\cdot|\cdot)_{\omega,k\phi}$ on the space $\Omega_c^{(0,q)}(M, L^k)$ (cf. (2.5)). Denote $L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k)$ as the completion of $\Omega_c^{(0,q)}(M, L^k)$ with respect to $(\cdot|\cdot)_{\omega,k\phi}$ and denote $\|\cdot\|_{\omega,k\phi}$ as its norm.

Let $\bar{\partial}_k^{(q)} : \Omega^{(0,q)}(M, L^k) \to \Omega^{(0,q+1)}(M, L^k)$ be the Cauchy-Riemann operator with values in L^k and $\bar{\partial}_k^{*,(q+1)} : \Omega^{(0,q+1)}(M, L^k) \to \Omega^{(0,q)}(M, L^k)$ be the formal adjoint of $\bar{\partial}_k^{(q)}$ with respect to $(\cdot|\cdot)_{\omega,k\phi}$. Recall that the *Kodaira Laplacian* is given by

$$\Box^{(q)}_{\omega,k\phi} := \bar{\partial}_k^{*,(q+1)}\bar{\partial}_k^{(q)} + \bar{\partial}_k^{(q-1)}\bar{\partial}_k^{*,(q)} : \Omega^{(0,q)}(M, L^k) \to \Omega^{(0,q)}(M, L^k)$$

and it has the Gaffney extension(cf. (2.6)):

$$\square^{(q)}_{\omega,k\phi} : \operatorname{Dom}\square^{(q)}_{\omega,k\phi} \subset L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k) \to L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k).$$

Denote by $E^{(q)}_{k,\leq c}$ the image of $\mathbb{1}_{[0,c]}\left(\square^{(q)}_{\omega,k\phi}\right)$ which is the functional calculus of $\square^{(q)}_{\omega,k\phi}$ with respect to the indicator function $\mathbb{1}_{[0,c]}$. We specify a non-negative sequence c_k and denote by (cf. (2.7))

$$\mathcal{P}^{(q)}_{k,c_k} := \mathbb{1}_{[0,c_k]}\left(\square^{(q)}_{\omega,k\phi}\right) : L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k) \to E^{(q)}_{k,\leq c_k}$$

the *spectral projection* which is the orthogonal projection. Specifically, in the case $c_k = 0$, denote

$$\mathcal{B}^{(q)}_k : L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k) \to \operatorname{Ker}\square^{(q)}_k$$

to be the *Bergman projection.* Define $P^{(q)}_{k,c_k}(z,w)$ to be the *spectral kernel* and $B^{(q)}_k(z,w)$ to be the *Bergman kernel* which are the Schwartz kernels of $\mathcal{P}^{(q)}_{k,c_k}$ and $\mathcal{B}^{(q)}_k$, respectively. Now, we choose a suitable holomorphic coordinate chart U centered at $p \in M$ and a holomorphic trivialization s on U such that (cf. Lemma 2.1)

$$\phi = \sum_{i=1}^n \lambda_i |z^i|^2 + O(|z|^3); \quad \omega = \sqrt{-1}\sum_{i=1}^n dz^i \wedge d\bar{z}^i + O(|z|). \tag{1.2}$$

Moreover, if $\lambda_i > 0$ for all $i = 1, \ldots, n$, we take the trivialization such that

$$\phi = \sum_{i=1}^n \lambda_i |z^i|^2 + O(|z|^4).$$

Note that if $p \in M(q')$ for some $q' \in \{0, \ldots, n\}$, then

$$q' = \#\{i\,;\lambda_i < 0\} \quad \text{and} \quad n - q' = \#\{i\,;\lambda_i > 0\}.$$

In this paper, we always assume that $\lambda_i < 0$ for all $i = 1, \ldots, q'$ by rearrangement. Next, we introduce the spectral gap conditions.

Definition 1.2 (spectral gap condition 1)**.** For any $q \in \{0, \ldots, n\}$ and an open set $U \subset M$, we say $\square^{(q)}_{\omega,k\phi}$ has a small local spectral gap condition of polynomial rate on U if there exist $d \in \mathbb{N}$ and $C > 0$ such that for all large enough k,

$$\left\| \left(I - \mathcal{B}_k^{(q)}\right) u\right\|^2_{\omega,k\phi} \leq Ck^d \left(\Box^{(q)}_{\omega,k\phi} u \mid u\right)_{\omega,k\phi} \quad \text{for all } u \in \Omega_c^{(0,q)}(U, L^k).$$

For the function case $q = 0$, we introduce a relaxed condition that allows for a *narrower* spectral gap.

Definition 1.3 (spectral gap condition 2)**.** For an open set $U \subset M$, we say $\Box^{(0)}_{\omega,k\phi}$ has a small local spectral gap condition of suitable exponential rate on U if there are constants $0 < c < 1$ and $C > 0$ such that for large enough k,

$$\left\| \left(I - \mathcal{B}_k^{(0)}\right) u\right\|^2_{\omega,k\phi} \leq Ce^{2c\min\lambda_i \cdot k^{1/2}} \left(\Box^{(0)}_{\omega,k\phi} u \mid u\right)_{\omega,k\phi} \text{ for all } u \in \mathcal{C}_c^\infty(U, L^k).$$

Let $s : U \to L$ be a local non-vanishing holomorphic section defined on an open set $U \subset M$. We can locally express the spectral and Bergman kernels on $U \times U$ as

$$\begin{aligned} P^{(q)}_{k,c_k}(z,w) &= P^{(q),s}_{k,c_k}(z,w)\, s^k(z) \otimes (s^k(w))^*; \\ B^{(q)}_k(z,w) &= B^{(q),s}_k(z,w)\, s^k(z) \otimes (s^k(w))^*. \end{aligned} \tag{1.3}$$

Here, $P^{(q),s}_{k,c_k}(z,w)$ and $B^{(q),s}_k(z,w)$ are elements in $\mathcal{C}^\infty(U \times U, T^{*,(0,q)}M \boxtimes T^{*,(0,q)}M)$ where $T^{*,(0,q)}M \boxtimes T^{*,(0,q)}M$ is the vector bundle over $U \times U$ whose fiber at $(z,w) \in U \times U$ is the space of linear transformations from $T^{*,(0,q)}_w M$ to $T^{*,(0,q)}_z M$. We now introduce the primary object in our approach.

Definition 1.4. We treat U as a subset in $\mathbb{C}^n$ and assume that U is convex. The scaled spectral kernel $P^{(q),s}_{(k),c_k} \in \mathcal{C}^\infty\left(\sqrt{k}U \times \sqrt{k}U, T^{*,(0,q)}\mathbb{C}^n \boxtimes T^{*,(0,q)}\mathbb{C}^n\right)$ is defined by

$$P^{(q),s}_{(k),c_k}(z,w) := k^{-n} P^{(q),s}_{k,c_k}\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right).$$

Similarly, the scaled Bergman kernel is defined by

$$B^{(q),s}_{(k)}(z,w) := k^{-n} B^{(q),s}_{k}\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right).$$

We are ready to illustrate the main results of this paper.

Theorem 1.1 (main theorem for Bergman kernel)**.** *If $p \notin M(q)$, the scaled Bergman kernel $B^{(q),s}_{(k)}(z,w) \to 0$ locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$. If $p \in M(q)$ and $\Box^{(q)}_{\omega,k\phi}$ has local small spectral gap condition of polynomial rate in U (cf. Def.* 1.2*), then $B^{(q),s}_{(k)}(z,w)$ converges to*

$$\frac{|\lambda_1 \cdots \lambda_n|}{\pi^n}\, e^{2(\sum_{i=1}^q |\lambda_i|\bar{z}^i w^i + \sum_{i=q+1}^n |\lambda_i| z^i \bar{w}^i - \sum_{i=1}^n |\lambda_i||w^i|^2)}(d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q) \otimes (\frac{\partial}{\partial \bar{w}^1} \wedge \cdots \wedge \frac{\partial}{\partial \bar{w}^q})$$

locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$. Here, we identify $(d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q) \otimes (\frac{\partial}{\partial \bar{w}^1} \wedge \cdots \wedge \frac{\partial}{\partial \bar{w}^q})$ as a section of $T^{*,(0,q)}\mathbb{C}^n \boxtimes T^{*,(0,q)}\mathbb{C}^n$ over $\mathbb{C}^n$ defined by

$$\eta \mapsto (d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q) \otimes \eta(\frac{\partial}{\partial \bar{w}^1} \wedge \cdots \wedge \frac{\partial}{\partial \bar{w}^q}) \quad \text{for all } \eta \in T^{*,(0,q)}\mathbb{C}^n.$$

In particular, in the case $p \in M(0)$, the convergence above for the function case $q = 0$ remains valid if $\Box^{(0)}_{\omega,k\phi}$ has only local small spectral gap condition of suitable exponential rate in U (cf. Def. 1.3*).*

Next, the second main theorem is the spectral kernel version. The spectral gap conditions can be dropped and conditions can be imposed on the sequence c_k since

$$\| \left(I - \mathcal{P}^{(q)}_{k,c_k}\right) u\|^2_{\omega,k\phi} \le c_k \left(\Box^{(q)}_{\omega,k\phi} u \mid u\right) \text{ for all } u \in L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k).$$

This estimate plays the role of a spectral gap condition.

Theorem 1.2 (main theorem for spectral kernel)**.** *Assume that the non-negative sequence c_k satisfies*

$$\limsup_{k\to\infty} \frac{c_k}{k} = 0.$$

If $p \notin M(q)$, the scaled spectral kernel $P^{(q),s}_{(k),c_k}(z,w) \to 0$ locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$. If $p \in M(q)$ and there exists $d \in \mathbb{N}$ such that $\liminf_{k\to\infty} k^d c_k > 0$, then $P^{(q),s}_{(k),c_k}(z,w)$ converges to

$$\frac{|\lambda_1 \cdots \lambda_n|}{\pi^n}\, e^{2(\sum_{i=1}^q |\lambda_i|\bar{z}^i w^i + \sum_{i=q+1}^n |\lambda_i| z^i \bar{w}^i - \sum_{i=1}^n |\lambda_i||w^i|^2)}(d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q) \otimes (\frac{\partial}{\partial \bar{w}^1} \wedge \cdots \wedge \frac{\partial}{\partial \bar{w}^q}).$$

locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$.

Particularly, in the case $p \in M(0)$*, the convergence above of the function case* $q=0$ *still holds under a weaker condition of* c_k *that* $\liminf e^{2c\min\{\lambda_i\}k^{1/2}} c_k > 0$ *for some* $c<1$.

Remark 1.1. For a fixed point $p \in M$, observe that $B_k^{(q),s}(p,p) = k^n B_{(k)}^{(q),s}(0,0)$. By Theorem 1.1, under spectral gap conditions, we deduce

$$B_k^{(q),s}(p,p) = k^n \frac{|\lambda_1 \cdots \lambda_n|}{\pi^n}(d\bar z^1 \wedge \cdots \wedge d\bar z^q) \otimes (\frac{\partial}{\partial \bar w^1} \wedge \cdots \wedge \frac{\partial}{\partial \bar w^q}) + o(k^n) \quad \text{if} \quad p \in M(q);$$

$$B_k^{(q),s}(p,p) = o(k^n) \quad \text{if } p \notin M(q).$$

In a similar way, by Theorem 1.2, we are able to conclude the same asymptotic behavior for the diagonal part of the spectral kernels $P_{k,c_k}^{(q),s}(p,p)$ under the suitable conditions on c_k. From our results, if the expansion (1.1) exists, we can conclude that

$$b_n^{(q)}(p,p) = \frac{|\lambda_1 \cdots \lambda_n|}{\pi^n}(d\bar z^1 \wedge \cdots \wedge d\bar z^q) \otimes (\frac{\partial}{\partial \bar w^1} \wedge \cdots \wedge \frac{\partial}{\partial \bar w^q}) \otimes s^k \otimes \left(s^k\right)^* \quad \text{if} \quad p \in M(q);$$

$$b_n^{(q)}(p,p) = 0 \quad \text{if } p \notin M(q).$$

Remark 1.2. Theorem 1.1 provides a purely analytic proof of the Kodaira embedding theorem (cf. [20]), while Theorem 1.2 can be used to establish the Demaillys Morse inequality (cf. [21, section 10.5]).

We divide the proof of the main theorems into two steps. First, in Chapter 3, we try to establish local uniform bounds of $B_{(k)}^{(q),s}(z,w)$ and $P_{(k),c_k}^{(q),s}(z,w)$ on $\mathbb{C}^n$ (cf. Theorem 3.5). In this way, we can infer that any subsequence of $B_{(k)}^{(q),s}$ (or $P_{(k),c_k}^{(q),s}$) has a $\mathcal{C}^\infty$ uniformly convergent subsequence by the Arzelà-Ascoli theorem.

Next, in Chapter 4, we prove that every convergent subsequence of $B_{(k)}^{(q),s}$ (or $P_{(k),c_k}^{(q),s}$) must converge to the Bergman kernel of the model case on $\mathbb{C}^n$ (cf. Theorem 4.5, Theorem 4.10, Theorem 4.21), which is exactly

$$\frac{|\lambda_1 \cdots \lambda_n|}{\pi^n} e^{2(\sum_{i=1}^q |\lambda_i| \bar z^i w^i + \sum_{i=q+1}^n |\lambda_i| z^i \bar w^i - \sum_{i=1}^n |\lambda_i||w^i|^2)}(d\bar z^1 \wedge \cdots \wedge d\bar z^q) \otimes (\frac{\partial}{\partial \bar w^1} \wedge \cdots \wedge \frac{\partial}{\partial \bar w^q}).$$

2. Preliminaries and Terminology

2.1. Standard notations

Let $\mathbb{N}_0$ be the set $\mathbb{N} \cup \{0\}$, and a multi-index α is of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N}_0)^n$. Denote $|\alpha| := \sum_i \alpha_i$ and $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$. For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Let M be a n-dimensional complex manifold and TM be the real tangent bundle of the underlying smooth manifold. Denote by $T_{\mathbb{C}}M$ the complexified tangent bundle $TM \otimes \mathbb{C}$ and $\bigwedge^l T^*_{\mathbb{C}}M$ the l-th exterior algebra of the cotangent bundle $T^*_{\mathbb{C}}M$. For a local holomorphic coordinate $(z^1, \ldots, z^n)$ that has an underlying real coordinate $(x^1, \ldots, x^{2n})$ with $z^j = x^{2j-1} + \sqrt{-1}x^{2j}$, let

$$\frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} - \sqrt{-1}\frac{\partial}{\partial x^{2j}}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} + \sqrt{-1}\frac{\partial}{\partial x^{2j}}\right)$$

as sections of $T_{\mathbb{C}}M$. Therefore, $dz^j := dx^{2j-1} + \sqrt{-1}dx^{2j}$ and $d\bar{z}^j := dx^{j-1} - \sqrt{-1}dx^{2j}$ are sections of $T^*_{\mathbb{C}}M$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0)^n$, we denote $\frac{\partial}{\partial z^\alpha} := (\frac{\partial}{\partial z^1})^{\alpha_1} \cdots (\frac{\partial}{\partial z^n})^{\alpha_n}$ and $\frac{\partial}{\partial \bar{z}^\alpha} := (\frac{\partial}{\partial \bar{z}^1})^{\alpha_1} \cdots (\frac{\partial}{\partial \bar{z}^n})^{\alpha_n}$. Sometimes, we simply write them as ∂_z^α and $\partial_{\bar{z}}^\alpha$, respectively. Also, for $\alpha \in \mathbb{N}_0^{2n}$, we denote $\frac{\partial}{\partial x^\alpha} := (\frac{\partial}{\partial x^1})^{\alpha_1} \cdots (\frac{\partial}{\partial x^{2n}})^{\alpha_{2n}}$ and sometimes write it as ∂_x^α.

Define

$$\mathcal{J}_{q,n} := \{I = (i_1, \ldots, i_q) : 1 \le i_1 < i_2 < \cdots < i_q \le n\} \subset (\mathbb{N}_0)^q.$$

For any element $I = (i_1, \ldots, i_q) \in \mathcal{J}_{q,n}$, we denote the q-forms $dz^I := dz^{i_1} \wedge \cdots \wedge dz^{i_q}$ and $d\bar{z}^I := d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_q}$.

Consider an open subset U of M. Denote by $\mathcal{C}^\infty(U)$ the space of smooth functions on U and by $\mathcal{C}_c^\infty(U)$ the subspace of $\mathcal{C}^\infty(U)$ whose elements have compact support in U. For a vector bundle E over M, we denote $\mathcal{C}^\infty(U, E)$ as the space of smooth sections of E over U and $\mathcal{C}_c^\infty(U, E)$ as the subspace of $\mathcal{C}^\infty(U, E)$ whose every element has compact support in U. Let dm be the standard Lebesgue measure on $\mathbb{C}^n$, and let $B(r)$ be the set $\{z \in \mathbb{C}^n; |z| < r\}$.

2.2. Complex geometry and Hermitian holomorphic line bundle

Let M be a complex manifold of dimension n. There is a natural complex structure $J : TM \to TM$ such that $J^2 = -\mathrm{Id}$. Then $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ where $T^{(1,0)}M$ and $T^{(0,1)}M$ are the i-eigenbundle and $-i$-eigenbundle of J, respectively. Similarly, $T^*_{\mathbb{C}}M = T^{*,(1.0)}M \oplus T^{*,(0,1)}M$ where $T^{*,(1.0)}M$ and $T^{*,(0,1)}M$ are dual bundles of $T^{(1,0)}M$ and $T^{(0,1)}M$, respectively. The splitting of the complexified tangent bundle can be extended to the exterior algebra of the complexified cotangent bundle. Namely,

$$\bigwedge^k T^*_{\mathbb{C}}M = \bigoplus_{p+q=k} \left(\bigwedge^p T^{*,(1,0)}M\right) \bigwedge \left(\bigwedge^q T^{*,(0,1)}M\right). \tag{2.1}$$

Define $T^{*,(p,q)}M := \left(\bigwedge^p T^{*,(1,0)}M\right) \bigwedge \left(\bigwedge^q T^{*,(0,1)}M\right)$ and hence $\bigwedge^k T^*_{\mathbb{C}}M = \bigoplus_{p+q=k} T^{*,(p,q)}M$. Let $\Omega^{(p,q)}(M)$ be the space of smooth (p,q)-forms which are smooth sections of $T^{*,(p,q)}M$ and $\Omega_c^{(p,q)}(M)$ be the subspace of $\Omega^{(p,q)}(M)$ consisting of elements with compact support in M. For a local holomorphic coordinate $(z^1, \dots, z^n)$ in $U \subset M$, we have a local frame for $T^{*,(p,q)}M$ given by

$$T^{*,(p,q)}M\mid_U = \mathrm{span}\{dz^I \wedge d\bar{z}^J\}_{I\in\mathcal{J}_{p,n}, J\in\mathcal{J}_{q,n}}.$$

Next, we call ω a positive Hermitian $(1,1)$-form if:

(i) $\omega \in \Omega^{(1,1)}(M)$;

(ii) For any local holomorphic coordinate $(z^1, \dots, z^n)$, ω can be written as

$$\omega = \sqrt{-1}\sum_{i,j=1}^{n} h_{i,j}dz_i \wedge d\bar{z}_j$$

where $[h_{i,j}]$ is a positive Hermitian matrix.

A positive Hermitian $(1,1)$-form ω induces pointwise Hermitian inner products $\langle\cdot|\cdot\rangle_\omega$ on $T^{(1,0)}M$ and $T^{(0,1)}M$ that are locally given by $\langle\frac{\partial}{\partial z^i}|\frac{\partial}{\partial z^j}\rangle_\omega := h_{i,j}$ and $\langle\frac{\partial}{\partial \bar{z}^i}|\frac{\partial}{\partial \bar{z}^j}\rangle_\omega := \bar{h}_{i,j}$, respectively. Thus, we have a Hermitian inner product $\langle\cdot|\cdot\rangle_\omega$ on the complexified cotangent bundle $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$. When we restrict the domain of $\langle\cdot|\cdot\rangle_\omega$ to the subbundle $TM \subset T_{\mathbb{C}}M$, we obtain a Riemannian metric called g_ω on the underlying real manifold. The Riemannian volume form dV_ω associated with g_ω is given by $dV_\omega = \frac{\omega^n}{n!}$.

Moreover, The Hermitian inner product $\langle\cdot|\cdot\rangle_\omega$ can be naturally extended to $T^{*,(0,q)}M$ by

$$\langle d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q} \mid d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q}\rangle_\omega = \frac{1}{q!}\overline{\det[h^{i_l,j_k}]_{l,k=1\dots q}} \tag{2.2}$$

where $[h^{i,j}]$ is the inverse matrix of $[h_{i,j}]$. We can now define the L^2-inner product on the space $\Omega_c^{(0,q)}(M)$ by

$$(\eta_1 \mid \eta_2)_\omega = \int_M \langle\eta_1 \mid \eta_2\rangle_\omega dV_\omega \quad \text{for all } \eta_1,\eta_2\in\Omega_c^{(0,q)}(M). \tag{2.3}$$

Let $L^2_\omega(M,T^{*,(0,q)}M)$ be the completion of $\Omega_c^{(0,q)}(M)$ with respect to the inner product $(\cdot|\cdot)_\omega$ and denote by $\|\cdot\|_\omega$ the corresponding norm. For an open set $U\subset M$, we define the restriction of the L^2-inner product by

$$(\eta_1 \mid \eta_2)_{\omega,U} := \int_U \langle\eta_1 \mid \eta_2\rangle_\omega dV_\omega \quad \text{for all } \eta_1,\eta_2\in\Omega_c^{(0,q)}(U). \tag{2.4}$$

In the same manner, we can define $L^2_\omega(U,T^{*,(0,q)}M)$ to be the completion of $\Omega_c^{(0,q)}(U)$ with respect to $(\cdot|\cdot)_{\omega,U}$ and denote $\|\cdot\|_{\omega,U}$ to be the corresponding norm.

Recall that a holomorphic Hermitian line bundle (L,h^E) is a 1-dimensional holomorphic Hermitian vector bundle. Let (U,s) be a local trivialization where U is a holomorphic chart and $s: U\subset M\to L$ is a holomorphic local non-vanishing section. Then there exists a local weight $\phi: U\to\mathbb{R}$ such that $\langle s|s\rangle_{h^L} = e^{-2\phi(z)}$. The Chern connection is locally given by the connection 1-form $\theta = -2\partial\phi$ and the curvature Θ^L is locally given by the $(1,1)$-form

$$\Theta^L = -2\bar{\partial}\partial\phi = 2\sum_{i,j=1}^n \frac{\partial^2\phi}{\partial z^i\partial\bar{z}^j}dz^i\wedge d\bar{z}^j.$$

Define $\dot{\Theta}^L\in\mathcal{C}^\infty(M,\operatorname{End}(T^{(1,0)}M))$ to be the curvature operator such that

$$\langle\dot{\Theta}^L(p)v_1 \mid v_2\rangle_\omega = \Theta^L(p)(v_1\wedge\overline{v_2}) \quad \text{for all } v_1,v_2\in T_p^{(1,0)}M, \quad p\in M.$$

Now, we introduce a lemma that allows us to simplify the information on curvature.

Lemma 2.1 (cf. [26], Lemma III,2.3)**.** *Let $L \to M$ be a holomorphic line bundle over a complex manifold M. For any fixed $p \in M$, there exists a trivialization (U, s) where $U \subset \mathbb{C}^n$ is a holomorphic chart centered at p and $s : U \to L$ is a non-vanishing holomorphic section such that the Hermitian form ω and the local weight ϕ with respect to s can be written as*

$$\omega(z) = \sqrt{-1}\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i + O(|z|); \quad \phi(z) = \sum_{i=1}^{n} \lambda_i |z^i|^2 + O(|z|^3).$$

Remark 2.1. If $\lambda_i \neq 0$ for all $i = 1, \ldots, n$, then the trivialization in Lemma 2.1 can be chosen such that

$$\phi(z) = \sum_{i=1}^{n} \lambda_i |z^i|^2 + O(|z|^4).$$

Observe that the Hermitian metric h^L on L can be identified by a family of local wights $\{\phi_i\}$ with respect of a family of trivializing sections $\{s_i\}$. We will alternatively denote $\langle\cdot|\cdot\rangle_\phi := \langle\cdot|\cdot\rangle_{h^L}$ if there is no risk of ambiguity. We can define the k-th tensor power of L as $L^k := L^{\otimes k}$, and denote the corresponding trivializing section as $s^k := s^{\otimes k}$. It follows that the local weight of s^k with respect to the induced metric h^{L^k} is given by $k\phi$. The norm of s^k is $|s^k|_{k\phi} := |s^k|_{h^{L^k}} = e^{-k\phi}$.

Fix a positive Hermitian $(1, 1)$-form ω on M and a Hermitian metric h^L with local weights ϕ on L. They induce a pointwise Hermitian inner product $\langle\cdot|\cdot\rangle_{\omega,k\phi}$ on the bundle $T^{*,(0,q)}M \otimes L^k$. For a fixed trivialization $s : U \to L$ with local weight ϕ, if $u_1 = \eta_1 \otimes s^k$ and $u_2 = \eta_2 \otimes s^k$ where $\eta_i \in \Omega^{(0,q)}(U)$, then

$$\langle u_1 \mid u_2\rangle_{\omega,k\phi} = \langle \eta_1 \otimes s^k \mid \eta_2 \otimes s^k\rangle_{\omega,k\phi} = \langle \eta_1 \mid \eta_2\rangle_\omega e^{-2k\phi}.$$

Then we can define the L^2-inner product on the space $\Omega_c^{(0,q)}(M, L^k)$ by

$$(u_1 \mid u_2)_{\omega,k\phi} := \int_M \langle u_1 \mid u_2\rangle_{\omega,k\phi} dV_\omega \quad \text{for } u_1, u_2 \in \Omega_c^{(0,q)}(M, L^k). \tag{2.5}$$

Denote $\|u\|^2_{\omega,k\phi} := (u \mid u)_{\omega,k\phi}$ and $L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k)$ as the Hilbert space which is the completion of $\Omega_c^{(0,q)}(M, L^k)$ with respect to the inner product $(\cdot|\cdot)_{\omega,k\phi}$.

2.3. The spectral and Bergman kernels

Let $\bar{\partial}_k^{(q)} : \Omega^{(0,q)}(M, L^k) \to \Omega^{(0,q+1)}(M, L^k)$ be the Cauchy-Riemann operator and $\bar{\partial}_k^{*,(q+1)} : \Omega^{(0,q+1)}(M, L^k) \to \Omega^{(0,q)}(M, L^k)$ be its formal adjoint with respect to $(\cdot|\cdot)_{\omega,k\phi}$. The *Kodaira Laplacian* is given by

$$\Box_k^{(q)} = \Box_{\omega,k\phi}^{(q)} := \bar{\partial}_k^{*,(q+1)}\bar{\partial}_k^{(q)} + \bar{\partial}_k^{(q-1)}\bar{\partial}_k^{*,(q)} : \Omega^{(0,q)}(M, L^k) \to \Omega^{(0,q)}(M, L^k).$$

Next, we define

$$\mathrm{Dom}\ \bar{\partial}_k^{(q)} := \{u \in L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k); \bar{\partial}_k^{(q)} u \in L^2_{\omega,k\phi}(M, T^{*,(0,q+1)}M \otimes L^k)\}$$

where $\bar{\partial}_k^{(q)} u$ is defined in distribution sense. Then we can extend $\bar{\partial}_k^{(q)}$ as

$$\bar{\partial}_k^{(q)} : \mathrm{Dom}\ \bar{\partial}_k^{(q)} \subset L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k) \to L^2_{\omega,k\phi}(M, T^{*,(0,q+1)}M \otimes L^k).$$

Let $\bar{\partial}_k^{*,(q+1)} : \mathrm{Dom}\ \bar{\partial}_k^{*,(q+1)} \subset L^2_{\omega,k\phi}(M, T^{*,(0,q+1)}M \otimes L^k)$
$\to L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k)$ be the $L^2_{\omega,k\phi}$-adjoint of $\bar{\partial}_k^{(q)}$ and denote

$$\mathrm{Dom}\,\Box_k^{(q)} := \{u \in \mathrm{Dom}\,\bar{\partial}_k^{(q)} \cap \mathrm{Dom}\,\bar{\partial}_k^{*,(q)} \mid \bar{\partial}_k^{(q)} u \in \mathrm{Dom}\,\bar{\partial}_k^{*,(q+1)} \text{ and } \bar{\partial}_k^{*,(q)} u \in \mathrm{Dom}\,\bar{\partial}_k^{(q-1)}\}.$$

We have the *Gaffney extension*(cf. [15])

$$\Box_k^{(q)} : \mathrm{Dom}\ \Box_k^{(q)} \subset L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k) \to L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k). \tag{2.6}$$

It is well-known that the extension is semi-positive and self-adjoint(cf. [25, proposition 3.1.2]). Next, we introduce the spectral theorem.

Theorem 2.2 ([11], Theorem 2.5.1)**.** *Let* $A : \mathrm{Dom}\,A \subset \mathcal{H} \to \mathcal{H}$ *be a self-adjoint operator on a Hilbert space* $\mathcal{H}$*. Then there exists a spectrum set* $\mathrm{Spec}A \subset \mathbb{R}$ *, a finite measure* μ *on* $\mathrm{Spec}A \times \mathbb{N}$ *and a unitary operator*

$$H : \mathcal{H} \to L^2(\mathrm{Spec}A \times \mathbb{N}, d\mu)$$

with the following properties: Set $h : \mathrm{Spec}\,A \times \mathbb{N} \to \mathbb{R}$ *by* $h(s, n) := s$*. Then an element* $f \in \mathcal{H}$ *is in* $\mathrm{Dom}\,A$ *if and only if* $h \cdot H(f) \in L^2(\mathrm{Spec}A \times \mathbb{N}, d\mu)$*.*

In addition, we have

$$Af = H^{-1} \circ (h \cdot Hf) \quad \text{for all } f \in \operatorname{Dom} A.$$

By Theorem 2.2, we know that $\Box_k^{(q)}$ has the spectrum set $\operatorname{Spec} \Box_k^{(q)}$ that lies in $[0, \infty)$ since $\Box_k^{(q)}$ is semi-positive. Moreover, there is a unitary map

$$H_k : L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k) \to L^2(\operatorname{Spec} \Box_k^{(q)} \times \mathbb{N}, d\mu_k)$$

such that $\Box_k^{(q)} u = H_k^{-1} \circ (h \cdot H_k u)$ for all $u \in \operatorname{Dom} \Box_k^{(q)}$.

Given non-negative constants c_k, we define the *spectral projections* by

$$\mathcal{P}^{(q)}_{k,c_k} u := H_k^{-1} \circ (\mathbb{1}_{[0,c_k]\times\mathbb{N}} \cdot H_k u) \tag{2.7}$$

where $\mathbb{1}_{[0,c_k]\times\mathbb{N}}$ is the indicator function defined on $\operatorname{Spec} \Box_k^{(q)} \times \mathbb{N}$ by

$$\begin{cases} \mathbb{1}_{[0,c_k]\times\mathbb{N}}(s, l) = 1 & \text{if } s \in [0, c_k]; \\ \mathbb{1}_{[0,c_k]\times\mathbb{N}}(s, l) = 0 & \text{if } s \notin [0, c_k]. \end{cases}$$

Clearly, $\mathcal{P}^{(q)}_{k,c_k}$ is an orthogonal projection since H_k is a unitary map. In fact, the construction of $\mathcal{P}^{(q)}_{k,c_k}$ above coincides with the functional calculus $\mathbb{1}_{[0,c_k]}(\Box_k^{(q)})$ with respect to the indicator function $\mathbb{1}_{[0,c_k]}$ (cf. [11, section 2]). We may denote by $E^{(q)}_{k,\le c_k}$ the image of $\mathbb{1}_{[0,c_k]}(\Box_k^{(q)})$ and then

$$\mathcal{P}^{(q)}_{k,c_k} = \mathbb{1}_{[0,c_k]}(\Box_k^{(q)}) : L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k) \to E^{(q)}_{k,\le c_k}.$$

For the case $c_k = 0$, we denote

$$\mathcal{B}_k^{(q)} := \mathcal{P}^{(q)}_{k,0} : L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k) \to \operatorname{Ker} \Box_k^{(q)}$$

to be the *Bergman projection*. To introduce the spectral and Bergman kernels, we need the following theorem (cf. [18, section 5.2]). Denote by $\mathcal{D}'_c(M, E)$ the space of distribution sections of a vector bundle E over M whose elements have compact support in M.

Theorem 2.3 (Schwartz Kernel Theorem for smoothing operators)**.** *Let E and F be two vector bundles on a manifold M with a volume form dV. Then* *for any continuous linear operator* $\mathcal{P} : \mathcal{D}'_c(M, E) \to \mathcal{C}^\infty(M, F)$*, there exists a unique smooth kernel* $K_{\mathcal{P}} \in \mathcal{C}^\infty(M \times M, F \boxtimes E)$ *such that*

$$\mathcal{P}u(x_0) = \int_M K_{\mathcal{P}}(x_0, y)(u(y))dV(y)$$

for all $u \in \mathcal{D}'_c(M; E)$*. Here, we denote* $F \boxtimes E$ *as a vector bundle on* $M \times M$ *whose fiber at* $(x, y) \in M \times M$ *is the space of linear transformations from* E_x *to* F_y*.*

Since Kodaira Laplacian is elliptic, the spectral projection $\mathcal{P}^{(q)}_{k,c_k}$ and the Bergman projection $\mathcal{B}^{(q)}_k$ are smoothing operators in the sense that

$$\begin{aligned}
\mathcal{P}^{(q)}_{k,c_k} &: \mathcal{D}'_c(M, T^{*,(0,q)}M \otimes L^k) \to \mathcal{C}^\infty(M, T^{*,(0,q)}M \otimes L^k),\\
\mathcal{B}^{(q)}_k &: \mathcal{D}'_c(M, T^{*,(0,q)}M \otimes L^k) \to \mathcal{C}^\infty(M, T^{*,(0,q)}M \otimes L^k)
\end{aligned}$$

are continuous maps. In conclusion, the conditions of Theorem 2.3 hold for $\mathcal{P}^{(q)}_{k,c_k}$ and $\mathcal{B}^{(q)}_k$ and hence their distribution kernels are smooth.

Definition 2.1. Define the *spectral kernel* $P^{(q)}_{k,c_k}(z, w)$ and *Bergman kernel* $B^{(q)}_k(z, w)$ which are in $\mathcal{C}^\infty\left(M \times M, (T^{*,(0,q)}M \otimes L^k) \boxtimes (T^{*,(0,q)}M \otimes L^k)\right)$ to be the Schwartz kernels of the spectral projection $\mathcal{P}^{(q)}_{k,c_k}$ and Bergman projection $\mathcal{B}^{(q)}_k$, respectively. In this way, for all $u \in L^2_{\omega,k\phi}(M, T^{*,(0,q)}M \otimes L^k)$, we have

$$\begin{aligned}
\mathcal{P}^{(q)}_{k,c_k}u(z) &= \int_M P^{(q)}_{k,c_k}(z, w)u(w)dV_\omega(w);\\
\mathcal{B}^{(q)}_k u(z) &= \int_M B^{(q)}_k(z, w)u(w)dV_\omega(w).
\end{aligned}$$

2.4. The Sobolev and Gårding inequalities

In this section, we consider the $\binom{n}{q}$-dimensional trivial complex vector bundle $T^{*,(0,q)}\mathbb{C}^n$ over an open subset U of $\mathbb{C}^n$ with a global trivializing frame $\{d\bar{z}^I\}_{I\in\mathcal{J}_{q,n}}$. Let

$$u := \sum_{I\in\mathcal{J}_{q,n}} u_I d\bar{z}^I \in \Omega^{(0,q)}_c(U)$$

be a smooth section of $T^{*,(0,q)}\mathbb{C}^n$. We consider u as a smooth vector-valued function

$$(u_I)_{I\in\mathcal{J}_{q,n}} : U \subset \mathbb{C}^n \simeq \mathbb{R}^{2n} \to \mathbb{C}^{\binom{n}{q}}$$

by fixing an order of $\mathcal{J}_{q,n}$. Recall that the Fourier transform of $u = (u_I)_{I\in\mathcal{J}_{q,n}}$ is

$$\widehat{u}(\xi) := (\widehat{u}_I(\xi))_{I\in\mathcal{J}_{q,n}}$$

where $\widehat{u}_I(\xi) := (2\pi)^{-n/2}\int_{\mathbb{R}^{2n}} u_I(x)e^{-i\xi\cdot x}dm$. For any $s \in \mathbb{R}$, the Sobolev s-norm $\|\cdot\|_{s,U}$ is

$$\|u\|_{s,U}^2 = \|u\|_s^2 := \int_{\mathbb{R}^{2n}} (1+|\xi|^2)^s|\widehat{u}(\xi)|^2dm(\xi). \tag{2.8}$$

The *Sobolev space* $H_s(U, T^{*,(0,q)}\mathbb{C}^n)$ is the completion of $\Omega_c^{(0,q)}(U)$ with respect to the norm $\|\cdot\|_s$. Since $\int_{\mathbb{R}^{2n}} |\widehat{u_I}|^2dm = \int_{\mathbb{R}^{2n}} |u_I|^2dm$ for all $I \in \mathcal{J}_{q,n}$, we have $\|\cdot\|_{0,U} = \|\cdot\|_0 = \|\cdot\|_{dm} = \|\cdot\|_{dm,U}$ where $\|\cdot\|_{dm}$ is the L^2-norm with respect to standard Lebesgue measure dm in Euclidean space. The following proposition induces a variant of the Sobolev norm.

Proposition 2.4 (compatibility)**.** *Given* $u = \sum_{I\in\mathcal{J}_{q,n}} u_I d\bar{z}^I \in \Omega_c^{(0,q)}(U)$ *and* $s \in \mathbb{N}$, *there exist positive constants* C_1 *and* C_2 *independent of* u *such that*

$$C_1 \sum_{I\in\mathcal{J}_{q,n}} \sum_{|\alpha|\le s} \|\partial_x^\alpha u_I\|_0^2 \le \|u\|_s^2 \le C_2 \sum_{I\in\mathcal{J}_{q,n}} \sum_{|\alpha|\le s} \|\partial_x^\alpha u_I\|_0^2.$$

The proof of Proposition 2.4 is simply by the fact that $|\widehat{\partial_x^\alpha u_I}(\xi)| = |\xi^\alpha \widehat{u_I}(\xi)|$ where $\xi^\alpha := \xi_1^{\alpha_1}\cdots\xi_{2n}^{\alpha_{2n}}$. Next, we introduce a basic proposition in the Sobolev theory.

Proposition 2.5. *For any* $s \in \mathbb{R}$, $H_{-s}(U, T^{*,(0,q)}\mathbb{C}^n)$ *is the dual space of* $H_s(U, T^{*,(0,q)}\mathbb{C}^n)$ *and*

$$|\,(u\mid v)_0\,| \le \|u\|_{-s}\|v\|_s$$

for all $u \in H_{-s}(U, T^{*,(0,q)}\mathbb{C}^n)$ *and* $v \in H_s(U, T^{*,(0,q)}\mathbb{C}^n)$.

Let $\mathcal{C}^d(U, T^{*,(0,q)}\mathbb{C}^n)$ be the space of d-th differentiable sections. For any point $x \in U$, define

$$|u|^2_{\mathcal{C}^d}(x) := \sum_{I\in\mathcal{J}_{q,n}} \sum_{|\alpha|\le d} |\partial^\alpha u_I(x)|^2 \quad \text{for all } u \in \mathcal{C}^d(U, T^{*,(0,q)}\mathbb{C}^n). \tag{2.9}$$

Next, define a norm $\|\cdot\|_{\mathcal{C}^d(U)}$ on the space $\mathcal{C}^d(U, T^{*,(0,q)}\mathbb{C}^n)$ by

$$\|u\|^2_{\mathcal{C}^d(U)} := \sup_{x\in U} |u|^2_{\mathcal{C}^d}(x). \tag{2.10}$$

The following theorem is well-known and will be applied in Section 3.3.

Theorem 2.6 (Sobolev inequality)**.** *Let $d \in \mathbb{N}_0$ and $s \in \mathbb{R}$ such that $s > d+n$. If $u \in H_s(U, T^{*,(0,q)}\mathbb{C}^n)$, then $u \in \mathcal{C}^d(U, T^{*,(0,q)}\mathbb{C}^n)$ and there exists a constant $C_{s,d}$ independent of u such that*

$$\|u\|_{\mathcal{C}^d(U)} \le C_{s,d}\|u\|_s.$$

We now consider a second-order differential operator $P : \Omega^{(0,q)}(U) \to \Omega^{(0,q)}(U)$. By ordering the basis $\{d\bar{z}^I\}_{I\in\mathcal{J}_{q,n}}$, we can treat P as a $\binom{n}{q} \times \binom{n}{q}$ matrix $[P_{i,j}]$ of second-order differential operators $P_{i,j} : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$. Let $(x^1, \ldots, x^{2n})$ be the standard coordinate on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. We can represent $P_{i,j}$ as

$$P_{i,j} = \sum_{|\alpha|\le 2} a_{i,j,\alpha}(x)\partial_x^\alpha \quad \text{where } a_{i,j,\alpha} \in \mathcal{C}^\infty(U).$$

Define the symbol $\sigma(P_{i,j})$ of $P_{i,j}$ by

$$\sigma(P_{i,j})(x,\xi) := \sum_{|\alpha|\le 2} \sqrt{-1}^{|\alpha|} a_{i,j,\alpha}(x)\xi^\alpha \quad \text{where } x \in U\,,\ \xi \in \mathbb{R}^{2n}.$$

$\sigma(P_{i,j})$ is a polynomial of ξ of degree 2 for any fixed $x \in U$. Furthermore, we define the symbol $\sigma(P)(x,\xi)$ of P as the $\binom{n}{q} \times \binom{n}{q}$ matrix $[\sigma(P_{i,j})(x,\xi)]$ of polynomials of ξ for any $x \in U$.

Definition 2.2 (elliptic operator)**.** We call a second-order differential operator $P : \Omega^{(0,q)}(U) \to \Omega^{(0,q)}(U)$ is *uniform elliptic* on U if there exists $C > 0$

A basic reference for this section is [16], sections 1.1-1.3

such that

$$|\sigma(P)(z,\xi)v| \geq C|\xi|^2|v| \quad \forall x \in U, \xi \neq 0 \text{ and } v \in \mathbb{R}^{2n}. \tag{2.11}$$

Theorem 2.7 (Gårding inequality)**.** *Let P be a second-order differential operator which is uniform elliptic on an open set $U \subset\subset \mathbb{C}^n$. Then for any $m \in \mathbb{N}$, there exists a positive constant $\tilde{C}$ such that*

$$\|u\|_{2m,U} \leq \tilde{C}\left(\|u\|_{0,U} + \|P^m u\|_{0,U}\right) \quad \text{for all } u \in H_{2m}(U, T^{*,(0,q)}\mathbb{C}^n).$$

Remark 2.2. The settings of this section can be modified to any trivial vector bundles. In particular, the cotangent bundle $T^{*,(p,q)}\mathbb{C}^n \to \tilde{U} \subset \mathbb{C}^n$ with the trivializing frame $\{dz^I \wedge d\bar{z}^J\}_{I\in\mathcal{J}_{p,n}, J\in\mathcal{J}_{q,n}}$. For example, if $u = \sum_{i,j=1}^n u_{i,j} dz^i \wedge d\bar{z}^j$ is a smooth $(1,1)$-form with compact support in $\mathbb{C}^n$, we can define

$$|u|^2_{\mathcal{C}^d}(x) := \sum_{i,j=1}^{n} \sum_{|\alpha|\leq d} |\partial^\alpha u_{i,j}(x)|^2 \tag{2.12}$$

and the norm

$$\|u\|^2_{\mathcal{C}^d(U)} := \sup_{x\in U} |u|^2_{\mathcal{C}^d}(x).$$

3. The Local Uniform Bounds for Scaled Spectral and Bergman Kernels

In this chapter, our aim is to analyze the behavior of the scaled spectral and Bergman kernels. Our objective is to establish local uniform bounds on the scaled kernels, which will allow us to investigate their local convergence properties. To this end, we will apply the Arzelà-Ascoli theorem.

In Section 3.1, we recall the set-up which has been mentioned in Section 1.1 and construct the scaled bundles. In Section 3.2, we compute the Kodaira Laplacian on the trivial line bundle and apply the results on the cases of scaled bundles. In Section 3.3, under the framework set in Sections 3.1 and 3.2, we can eventually control the local behavior of scaled spectral and Bergman kernels by the analytic tools of Sobolev theory.

3.1. The scaled bundles

Let (M,ω) be a Hermitian manifold of dimension n and $(L,h^L)\to M$ be a holomorphic Hermitian line bundle. Given a non-vanishing holomorphic section s of L on a holomorphic chart U that trivializes L, there exists a local weight $\phi: U\to\mathbb{R}$ such that $|s|^2_{h^L}=e^{-2\phi}$. Denote $|s|_\phi := |s|_{h^L}$ for convenience.

Recall that $(\cdot|\cdot)_\omega$ is the L^2-inner product of the Hilbert space $L^2_\omega(M,T^{*,(0,q)}M)$ (cf. (2.3)) and $(\cdot|\cdot)_{\omega,U}$ is the restriction of the inner product (cf. (2.4)). Also, we can define another Hilbert space $L^2_{\omega,k\phi}(M,T^{*,(0,q)}M \otimes L^k)$ which has the inner product $(\cdot|\cdot)_{\omega,k\phi}$(cf. (2.5)).

Denote $\bar\partial_k^{(q)}:\Omega^{(0,q)}(M,L^k)\to\Omega^{(0,q+1)}(M,L^k)$ to be the *Cauchy-Riemann operator* and $\bar\partial_k^{*,(q+1)}:\Omega^{(0,q+1)}(M,L^k)\to\Omega^{(0,q)}(M,L^k)$ to be the formal adjoint of $\bar\partial_k$ with respect to $(\cdot|\cdot)_{\omega,k\phi}$. Recall that we have the *Kodaira Laplacian* $\Box_k^{(q)}$ (or $\Box^{(q)}_{\omega,k\phi}$) given by

$$\Box_k^{(q)} := \bar\partial_k^{*,(q+1)}\bar\partial_k^{(q)}+\bar\partial_k^{(q-1)}\bar\partial_k^{*,(q)} : \mathrm{Dom}\,\Box_k^{(q)}\to L^2_{\omega,k\phi}(M,T^{*,(0,q)}M\otimes L^k)$$

which is the *Gaffney extension*(cf. (2.6)).

Fix a non-negative sequence c_k. Denote by $E^{(q)}_{k,\le c_k}$ the image of the functional calculus $\mathbb{1}_{[0,c_k]}(\Box_k^{(q)})$ with respect to the indicator function $\mathbb{1}_{[0,c_k]}$. The *spectral projection* is the orthogonal projection

$$\mathcal{P}^{(q)}_{k,c_k} := \mathbb{1}_{[0,c_k]}(\Box_k^{(q)}) : L^2_{\omega,k\phi}(M,T^{*,(0,q)}M\otimes L^k)\to E^{(q)}_{k,\le c_k}$$

and the *Bergman projection* is the orthogonal projection

$$\mathcal{B}_k^{(q)} := \mathcal{P}^{(q)}_{k,0} : L^2_{\omega,k\phi}(M,T^{*,(0,q)}M\otimes L^k)\to \mathrm{Ker}\,\Box_k^{(q)}$$

which is a special case of spectral projection by taking $c_k=0$. Readers may consult (2.7) for an explicit definition.

For $\eta_1\otimes s^k$ and $\eta_2\otimes s^k$ in $L^2_{\omega,k\phi}(U,T^{*,(0,q)}M\otimes L^k)$ where $\eta_i\in\Omega_c^{(0,q)}(U)$, observe that

$$(\eta_1\otimes s^k|\eta_2\otimes s^k)_{\omega,k\phi} = \int_U \langle\eta_1|\eta_2\rangle_\omega e^{-2k\phi}dV_\omega = (\eta_1 e^{-k\phi}|\eta_2 e^{-k\phi})_{\omega,U}.$$

This induces a unitary identification

$$L^2_{\omega,k\phi}(U,T^{*,(0,q)}M\otimes L^k)\cong L^2_\omega(U,T^{*,(0,q)}M) \quad \text{by } \eta\otimes s^k\leftrightarrow \eta e^{-k\phi}. \tag{3.1}$$

Define the *localized spectral projection*

$$\mathcal{P}^{(q)}_{k,c_k,s}: L^2_\omega(U,T^{*,(0,q)}M)\to L^2_\omega(U,T^{*,(0,q)}M)$$

satisfying $\mathcal{P}^{(q)}_{k,c_k}(\eta\otimes s^k)=e^{k\phi}\mathcal{P}^{(q)}_{k,c_k,s}(\eta e^{-k\phi})\otimes s^k$ for all $\eta\in L^2_\omega(U,T^{*,(0,q)}M)$. In the case $c_k=0$, we denote $\mathcal{B}^{(q)}_{k,s}:=\mathcal{P}^{(q)}_{k,0,s}$ as the *localized Bergman projection.*

Next, let $P^{(q)}_{k,c_k}(z,w)$ and $B^{(q)}_k(z,w)$ be the *spectral* and *Bergman kernels* which are the Schwartz kernels of $\mathcal{P}^{(q)}_{k,c_k}$ and $\mathcal{B}^{(q)}_k$, respectively. We may also define the *localized spectral kernel* $P^{(q)}_{k,c_k,s}(z,w)$ and *localized Bergman kernel* $B^{(q)}_{k,s}(z,w)$ to be the Schwartz kernels of $\mathcal{P}^{(q)}_{k,c_k,s}$ and $\mathcal{B}^{(q)}_{k,s}$, respectively. The relation between $P^{(q),s}_{k,c_k}(z,w)$ and $P^{(q)}_{k,c_k,s}(z,w)$ is given by

$$P^{(q)}_{k,c_k,s}(z,w)=P^{(q),s}_{k,c_k}(z,w)\cdot|s^k(z)|_{h^L}\cdot|(s^k)^*(w)|_{h^{L^*}}=e^{-k\phi(z)}P^{(q),s}_{k,c_k}(z,w)e^{k\phi(w)}, \tag{3.2}$$

where $P^{(q),s}_{k,c_k}(z,w)$ is defined in (1.3).

From now on, we fix a point $p\in M$ throughout this paper and apply Lemma 2.1 to obtain a trivialization (U,s) centered at p such that

$$\phi(z)=\sum_{i=1}^n\lambda_i|z|^2+O(|z|^3) \quad\text{and}\quad \omega(z)=\sqrt{-1}\sum_{i=1}^n dz^i\wedge d\bar z^i+O(|z|).$$

Recall that the set of points with signature q is defined by

$$M(q):=\{p'\in M\mid \dot\Theta^L(p')\text{ is non-degenerate and has exactly } q \text{ negative eigenvalues}\}$$

and observe that $p\in M(q)$ means $q=\#\{i\mid\lambda_i<0\}$ and $n-q=\#\{i\mid \lambda_i>0\}$. In the case of $p\in M(0)$, we choose the trivialization such that $\phi=\sum_{i=1}^n\lambda_i|z|^2+O(|z|^4)$ throughout this paper. Without loss of generality, we assume $B(1) \subset U \subset \mathbb{C}^n$ and make the following observations. Denote

$$\phi_0(z) := \sum_{i=1}^{n} \lambda_i |z^i|^2 \quad \text{and} \quad \phi_{(k)}(z) := k\phi(z/\sqrt{k}).$$

Then for any $\epsilon < 1/2$, there exists a constant C independent of k such that

$$|\phi_{(k)} - \phi_0|_{\mathcal{C}^2}(z) \leq C\frac{|z|^3 + 1}{\sqrt{k}} \quad \forall\ |z| \leq k^{\epsilon}. \tag{3.3}$$

where $|\cdot|_{\mathcal{C}^2}$ is defined in (2.9). Also, set

$$\omega_0 := \sqrt{-1}\sum_i dz_i \wedge d\bar{z}_i \quad \text{and} \quad \omega_{(k)} := \omega(\frac{z}{\sqrt{k}}).$$

Then there also exists a constant C' such that

$$|\omega_{(k)} - \omega_0|_{\mathcal{C}^2}(z) \leq C'\frac{|z| + 1}{\sqrt{k}} \quad \forall\ |z| \leq k^{\epsilon}. \tag{3.4}$$

where $|\cdot|_{\mathcal{C}^2}$ for $(1,1)$-forms is defined in (2.12). Furthermore, $\phi_{(k)} \to \phi_0$ and $\omega_{(k)} \to \omega_0$ locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$. $\phi_{(k)}$ and $\omega_{(k)}$ are defined on $B(\sqrt{k})$ and are called the *scaled metric* and *scaled Hermitian form*, respectively. Inspired by the observations above, we construct the *scaled line bundles*. Define

$$s^{(k)}(z) := s^k(\frac{z}{\sqrt{k}}) : B(\sqrt{k}) \to L^k.$$

This makes L^k a trivial line bundle over $B(\sqrt{k})$ with a trivializing section $s^{(k)}$ for any $k \in \mathbb{N}$. We denote the scaled line bundle as

$$L^{(k)} := L^k \to B(\sqrt{k}) \subset \mathbb{C}^n$$

which equipped with the scaled metric $\phi_{(k)}$ by

$$\langle s^{(k)} \mid s^{(k)} \rangle_{\phi_{(k)}} := e^{-2\phi_{(k)}} = e^{-2k\phi(z/\sqrt{k})}.$$

More generally, we consider the trivial vector bundle

$$T^{*,(0,q)}\mathbb{C}^n \otimes L^{(k)} \to B(\sqrt{k}) \subset \mathbb{C}^n$$

which is a $\binom{n}{q}$-dimensional complex vector bundle with the space of smooth sections $\Omega^{(0,q)}(B(\sqrt{k}), L^{(k)})$ and trivializing frames $\{d\bar{z}^I \otimes s^{(k)}\}_{I\in\mathcal{J}_{q,n}}$.

We endow the vector bundle $T^{*,(0,q)}\mathbb{C}^n \otimes L^{(k)} \to B(\sqrt{k})$ with a pointwise Hermitian structure by the scaled metric $\phi_{(k)} = k\phi(z/\sqrt{k})$ on $L^{(k)}$ and the scaled Hermitian form $\omega_{(k)} = \omega(z/\sqrt{k})$ on $T^{*,(0,q)}B(\sqrt{k})$. That is, for all $\eta_1, \eta_2 \in \Omega^{(0,q)}(B(\sqrt{k}))$,

$$\langle \eta_1 \otimes s^{(k)} | \eta_2 \otimes s^{(k)} \rangle_{\omega_{(k)},\phi_{(k)}}(z) := \langle \eta_1(z) | \eta_2(z) \rangle_{\omega_{(k)}} \, e^{-2\phi_{(k)}(z)}.$$

Similar to the identification (3.1), there is a unitary correspondence

$$L^2_{\omega_{(k)},\phi_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n \otimes L^{(k)}) \cong L^2_{\omega_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n) \quad \text{by} \quad \eta \otimes s^{(k)} \leftrightarrow \eta e^{-k\phi(z/\sqrt{k})}. \tag{3.5}$$

In the meantime, by changing variable, there are unitary identifications

$$L^2_{\omega,k\phi}(B(1), T^{*,(0,q)}\mathbb{C}^n \otimes L^k) \cong L^2_{\omega_{(k)},\phi_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n \otimes L^{(k)}) \quad \text{by} \quad \eta \otimes s^k \leftrightarrow k^{-n/2}\eta(z/\sqrt{k}) \otimes s^{(k)} \tag{3.6}$$

and

$$L^2_{\omega}(B(1), T^{*,(0,q)}\mathbb{C}^n) \cong L^2_{\omega_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n) \quad \text{by} \quad \eta \leftrightarrow k^{-n/2}\eta(z/\sqrt{k}). \tag{3.7}$$

So far, we have four unitary identifications (3.1),(3.5),(3.6),(3.7) between the spaces of sections. In fact, the identifications form a commutative diagram. We can transform the localized spectral (or Bergman) kernels defined on $B(1)$ to kernels on the scaled bundles over $B(\sqrt{k})$ by (3.7).

Define the *scaled localized spectral projection* $\mathcal{P}^{(q)}_{(k),c_k,s} : L^2_{\omega_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n) \to L^2_{\omega_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n)$ such that

$$\left(\mathcal{P}^{(q)}_{(k),c_k,s} u\right)(\sqrt{k}z) = \mathcal{P}^{(q)}_{k,c_k,s}\left(u(\sqrt{k}w)\right) \tag{3.8}$$

and the *scaled localized Bergman projection* $\mathcal{B}^{(q)}_{(k),s} := \mathcal{P}^{(q)}_{(k),0,s}$. Define the *scaled localized spectral kernel* $P^{(q)}_{(k),c_k,s}(z,w)$ to be the Schwartz kernel of

$\mathcal{P}^{(q)}_{(k),c_k,s}$ which is given by

$$P^{(q)}_{(k),c_k,s}(z,w) = k^{-n} P^{(q)}_{k,c_k,s}(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}})$$

and the *scaled localized Bergman kernel* $B^{(q)}_{(k),s}(z,w) := P^{(q)}_{(k),0,s}$. In this way, we have

$$\mathcal{P}^{(q)}_{(k),c_k,s} u(z) = \int_{B(\sqrt{k})} P^{(q)}_{(k),c_k,s}(z,w) u(w) dV_{\omega_{(k)}} \quad \text{where } dV_{\omega_{(k)}} := \frac{\omega^n_{(k)}}{n!}.$$

The relation between $P^{(q),s}_{(k),c_k}(z,w)$ and $P^{(q)}_{(k),c_k,s}(z,w)$ is given by

$$P^{(q)}_{(k),c_k,s}(z,w) = e^{-\phi_{(k)}(z)} P^{(q),s}_{(k),c_k}(z,w) e^{\phi_{(k)}(w)} \tag{3.9}$$

where $P^{(q),s}_{(k),c_k}$ is defined in (1.4).

3.2. The Laplacians

The goal of this section is to compute Kodaira Laplacian on a trivial line bundle. We now temporarily forget the set-up in Section 3.1. Let $\tilde{U}$ be an open set in $\mathbb{C}^n$ and $\tilde{L} \to \tilde{U}$ be a trivial line bundle over $\tilde{U}$ with a trivializing section $\tilde{s}$. Fix a positive Hermitian $(1,1)$-form $\tilde{\omega}$ on $\tilde{U}$ and a weight function $\tilde{\phi}$ such that $\langle \tilde{s} | \tilde{s} \rangle_{\tilde{\phi}} = e^{-2\tilde{\phi}}$. Consider

$$T^{*,(0,q)}\mathbb{C}^n \otimes \tilde{L} \to \tilde{U} \subset \mathbb{C}^n$$

to be the Hermitian vector bundle with the pointwise Hermitian structure $\langle \cdot | \cdot \rangle_{\tilde{\omega},\tilde{\phi}}$ induced by $\tilde{\omega}$ and $\tilde{\phi}$. That is, for $\eta_1, \eta_2 \in \Omega^{(0,q)}_c(\tilde{U})$,

$$\langle \eta_1 \otimes \tilde{s} \mid \eta_2 \otimes \tilde{s} \rangle_{\tilde{\omega},\tilde{\phi}}(z) = \langle \eta_1(z) \mid \eta_2(z) \rangle_{\tilde{\omega}} e^{-2\tilde{\phi}(z)} \quad \text{for all } z \in \tilde{U}. \tag{3.10}$$

This defines an inner product on the space on $\Omega^{(0,q)}_c(\tilde{U}, \tilde{L})$. Namely,

$$(\eta_1 \otimes \tilde{s} \mid \eta_2 \otimes \tilde{s})_{\tilde{\omega},\tilde{\phi}} := \int_{\tilde{U}} \langle \eta_1 \otimes \tilde{s} \mid \eta_2 \otimes \tilde{s} \rangle_{\tilde{\omega},\tilde{\phi}} dV_{\tilde{\omega}} \quad \text{where } dV_{\tilde{\omega}} := \frac{\tilde{\omega}^n}{n!}. \tag{3.11}$$

Let $L^2_{\tilde{\omega},\tilde{\phi}}(\tilde{U}, T^{*,(0,q)}\mathbb{C}^n \otimes \tilde{L})$ be the completion of $\Omega^{(0,q)}_c(\tilde{U})$ with respect to $(\cdot|\cdot)_{\tilde{\omega},\tilde{\phi}}$. Similarly, we have another Hilbert space $L^2_{\omega}(\tilde{U}, T^{*,(0,q)}\mathbb{C}^n)$ with its inner product $(\cdot|\cdot)_{\tilde{\omega}}$ (cf. (2.3)). There is a unitary identification

$$L^2_{\tilde{\omega},\tilde{\phi}}(\tilde{U}, T^{*,(0,q)}\mathbb{C}^n \otimes \tilde{L}) \cong L^2_{\tilde{\omega}}(\tilde{U}, T^{*,(0,q)}\mathbb{C}^n), \quad \eta \otimes \tilde{s} \leftrightarrow \eta e^{-\tilde{\phi}}. \tag{3.12}$$

For any smooth $(0,1)$-form η on $\tilde{U}$, we can consider the wedge operator $\eta \wedge \cdot : T_p^{*,(0,q)}\tilde{U} \to T_p^{*,(0,q+1)}\tilde{U}$. Moreover, for the positive Hermitian $(1,1)$-form $\tilde{\omega}$, we let $\eta \wedge^*_{\tilde{\omega}} \cdot : T_p^{*,(0,q+1)}\tilde{U} \to T_p^{*,(0,q)}\tilde{U}$ to be the adjoint of $\eta\wedge\cdot$ via the pointwise inner product $\langle\cdot|\cdot\rangle_{\tilde{\omega}}$. For $\eta_1, \eta_2 \in T_p^{*,(0,1)}\tilde{U}$, we have the identity $(\eta_1 \wedge^*_{\tilde{\omega}})\eta_2 \wedge \cdot + \eta_2 \wedge (\eta_1 \wedge^*_{\tilde{\omega}})\cdot = \langle \eta_1|\eta_2\rangle_{\tilde{\omega}} \cdot$.

Let $\bar{\partial}^{(q)}_{\tilde{L}} : \Omega^{(0,q)}(\tilde{U}, \tilde{L}) \to \Omega^{(0,q+1)}(\tilde{U}, \tilde{L})$ be the Cauchy-Riemann operator with values in $\tilde{L}$ and $\bar{\partial}^{*,(q+1)}_{\tilde{L}} : \Omega^{(0,q+1)}(\tilde{U}, \tilde{L}) \to \Omega^{(0,q)}(\tilde{U}, \tilde{L})$ be the formal adjoint of $\bar{\partial}^{(q)}_{\tilde{L}}$ with respect to $(\cdot|\cdot)_{\tilde{\omega},\tilde{\phi}}$. Under identification (3.12), it is natural to define the *localized Cauchy-Riemann operator* $\bar{\partial}_{\tilde{s}} : \Omega^{(0,q)}(\tilde{U}) \to \Omega^{(0,q+1)}(\tilde{U})$ such that $\bar{\partial}_{\tilde{L}}(\eta \otimes \tilde{s}) = e^{\tilde{\phi}}\bar{\partial}_{\tilde{s}}(\eta e^{-\tilde{\phi}}) \otimes \tilde{s}$. Denote by $\bar{\partial}^{(q)}$ the standard Cauchy-Riemann operator on $\Omega^{(0,q)}(\tilde{U})$. Note that

$$\bar{\partial}^{(q)}_{\tilde{s}} = e^{-\tilde{\phi}}\bar{\partial}^{(q)}e^{\tilde{\phi}}. \tag{3.13}$$

By direct computation, we have

$$\bar{\partial}^{(q)}_{\tilde{s}} = \bar{\partial}^{(q)} + (\bar{\partial}\tilde{\phi}) \wedge \cdot. \tag{3.14}$$

Of course, we can also define $\bar{\partial}^{*,(q)}_{\tilde{s}} : \Omega^{(0,q-1)}(\tilde{U}) \to \Omega^{(0,q)}(\tilde{U})$ satisfying $\bar{\partial}^*_{\tilde{L}}(\eta \otimes \tilde{s}) = e^{\tilde{\phi}}\ \bar{\partial}^*_{\tilde{s}}(\eta e^{-\tilde{\phi}}) \otimes \tilde{s}$. Then $\bar{\partial}^{*,(q)}_{\tilde{s}}$ is the formal adjoint of $\bar{\partial}^{(q-1)}_{\tilde{s}}$ with respect to $(\cdot|\cdot)_{\tilde{\omega}}$. Next, define $\bar{\partial}^{*,(q)}_{\tilde{\omega}}$ to be the formal adjoint of $\bar{\partial}^{(q-1)}$ with respect to $(\cdot|\cdot)_{\tilde{\omega}}$. Note that

$$\bar{\partial}^{*,(q)}_{\tilde{s}} = \bar{\partial}^{*,(q)}_{\tilde{\omega}} + (\bar{\partial}\tilde{\phi}) \wedge^*_{\tilde{\omega}} \cdot. \tag{3.15}$$

Recall that $\Box^{(q)}_{\tilde{L}} := \bar{\partial}^{*,(q+1)}_{\tilde{L}}\bar{\partial}^{(q)}_{\tilde{L}} + \bar{\partial}^{(q-1)}_{\tilde{L}}\bar{\partial}^{*,(q)}_{\tilde{L}}$ is the Kodaira Laplacian. We can define the *localized Kodaira Laplacian* $\Box^{(q)}_{\tilde{s}} := \bar{\partial}^{*,(q+1)}_{\tilde{s}}\bar{\partial}^{(q)}_{\tilde{s}} + \bar{\partial}^{(q-1)}_{\tilde{s}}\bar{\partial}^{*,(q)}_{\tilde{s}}$ that acts on $\Omega^{(0,q)}(\tilde{U})$. $\Box^{(q)}_{\tilde{L}}$ and $\Box^{(q)}_{\tilde{s}}$ are compatible under the identification (3.12) in the sense that

$$\Box^{(q)}_{\tilde{L}}(\eta \otimes \tilde{s}) = e^{\tilde{\phi}}(\Box^{(q)}_{\tilde{s}}\eta e^{-\tilde{\phi}}) \otimes \tilde{s}. \tag{3.16}$$

We can consider the Gaffney extansions of $\Box^{(q)}_{\tilde{L}}$ and $\Box^{(q)}_{\tilde{s}}$ which preserve the relation (3.16) and $\eta \otimes \tilde{s} \in \operatorname{Dom} \square_{\tilde{L}}^{(q)} \iff \eta e^{-\tilde{\phi}} \in \operatorname{Dom} \square_{\tilde{s}}^{(q)}$. Next, we compute the localized Kodaira Laplacian using the settings above.

Lemma 3.1. *The localized Kodaira Laplacian can be expressed as*

$$\square_{\tilde{s}}^{(q)} = \Delta_{\tilde{\omega}} + \bar{\partial}\left((\bar{\partial}\tilde{\phi}) \wedge_{\tilde{\omega}}^{*} \cdot\right) + (\bar{\partial}\tilde{\phi}) \wedge_{\tilde{\omega}}^{*} \bar{\partial} + \bar{\partial}_{\tilde{\omega}}^{*}\left((\bar{\partial}\tilde{\phi}) \wedge \cdot\right) + (\bar{\partial}\tilde{\phi}) \wedge \bar{\partial}_{\tilde{\omega}}^{*} + \langle \bar{\partial}\tilde{\phi} | \bar{\partial}\tilde{\phi} \rangle \cdot, \tag{3.17}$$

where $\Delta_{\tilde{\omega}}^{(q)} := \bar{\partial}_{\tilde{\omega}}^{*,(q+1)} \bar{\partial}^{(q)} + \bar{\partial}^{(q-1)} \bar{\partial}^{*,(q)} : \Omega^{(0,q)}(\tilde{U}) \to \Omega^{(0,q)}(\tilde{U})$ *is the Hodge Laplacian with respect to* $\tilde{\omega}$*. Furthermore, assume that* $\tilde{\omega}^n/n! = e^{\tilde{\varphi}} dm$ *for some function* $\tilde{\varphi}$ *and let* θ *denote the matrix of connection forms of* ∇ *on* $T^{(q,0)}\tilde{U}$ *with respect to the frame* $\{dz^I\}_{I \in \mathcal{J}_{q,n}}$*. Then for* $f \in \mathcal{C}^\infty(\tilde{U})$ *and* $I \in \mathcal{J}_{q,n}$*,*

$$\bar{\partial}_{\tilde{\omega}}^{*} f d\bar{z}^I = \left(-\frac{\partial f}{\partial z^i} - f \frac{\partial \tilde{\varphi}}{\partial z^i}\right)(d\bar{z}^i) \wedge_{\tilde{\omega}}^{*} d\bar{z}^I - f\,(d\bar{z}^i \wedge_{\tilde{\omega}}^{*}) \overline{\theta}_{\partial/\partial \bar{z}^i}^{*} d\bar{z}^I. \tag{3.18}$$

Proof. First, by (3.14) and 3.15),

$$\begin{aligned}
\square_{\tilde{s}} &= \bar{\partial}_{\tilde{s}}^{*} \bar{\partial}_{\tilde{s}} + \bar{\partial}_{\tilde{s}} \bar{\partial}_{\tilde{s}}^{*} = \left(\bar{\partial}_{\tilde{\omega}}^{*} + (\bar{\partial}\tilde{\phi}) \wedge_{\tilde{\omega}}^{*} \cdot\right)\left(\bar{\partial} + (\bar{\partial}\tilde{\phi}) \wedge \cdot\right) \\
&\qquad + \left(\bar{\partial} + (\bar{\partial}\tilde{\phi}) \wedge \cdot\right)\left(\bar{\partial}_{\tilde{\omega}}^{*} + (\bar{\partial}\tilde{\phi}) \wedge_{\tilde{\omega}}^{*} \cdot\right) \\
&= \Delta_{\tilde{\omega}} + \bar{\partial}\left((\bar{\partial}\tilde{\phi}) \wedge_{\tilde{\omega}}^{*} \cdot\right) + \left((\bar{\partial}\tilde{\phi}) \wedge_{\tilde{\omega}}^{*}\right) \bar{\partial} + \bar{\partial}_{\tilde{\omega}}^{*}\left((\bar{\partial}\tilde{\phi}) \wedge \cdot\right) + (\bar{\partial}\tilde{\phi}) \wedge \bar{\partial}_{\tilde{\omega}}^{*} + \langle \bar{\partial}\tilde{\phi} | \bar{\partial}\tilde{\phi} \rangle \cdot .
\end{aligned}$$

Now, we compute $\bar{\partial}_{\tilde{\omega}}^{*} f d\bar{z}^I$. By the locality of differential operator, we may assume $f \in \mathcal{C}_c^\infty(\tilde{U})$. Let $g \in \mathcal{C}^\infty(\tilde{U})$ and $J \in \mathcal{J}_{q-1,n}$ then

$$\begin{aligned}
&\left(\bar{\partial}_{\tilde{\omega}}^{*}(f(z) d\bar{z}^I) \big| g(z) d\bar{z}^J\right)_{\tilde{\omega}} = \left(f(z) d\bar{z}^I \big| \bar{\partial}(g(z) d\bar{z}^J)\right)_{\tilde{\omega}} \\
&= \int_{\tilde{U}} f(z) \overline{\left(\frac{\partial g(z)}{\partial \bar{z}^i}\right)} \langle d\bar{z}^I | d\bar{z}^i \wedge d\bar{z}^J \rangle_{\tilde{\omega}} e^{\tilde{\varphi}} dm \\
&= \int_{\tilde{U}} \left(\frac{\partial \bar{g}}{\partial z_i}(z)\right) f(z)\, e^{\tilde{\varphi}(z)} \langle d\bar{z}^I | d\bar{z}^i \wedge d\bar{z}^J \rangle_{\tilde{\omega}} dm \\
&= \int_{\tilde{U}} -\bar{g}(z) \frac{\partial}{\partial z^i}\left(f(z)\, e^{\tilde{\varphi}(z)} \langle d\bar{z}^I | d\bar{z}^i \wedge d\bar{z}^J \rangle_{\tilde{\omega}}\right) dm \\
&= \int_{\tilde{U}} -\bar{g}(z)\left(\frac{\partial f}{\partial z^i}(z) + f(z) \frac{\partial \tilde{\varphi}}{\partial z^i}(z)\right) \langle (d\bar{z}^i) \wedge_{\tilde{\omega}}^{*} d\bar{z}^I | d\bar{z}^J \rangle_{\tilde{\omega}}\, e^{\tilde{\varphi}(z)} dm \\
&\qquad - \int_{\tilde{U}} \bar{g}(z) f(z)\left(\frac{\partial}{\partial z^i} \langle d\bar{z}^I \mid d\bar{z}^i \wedge d\bar{z}^J \rangle_{\tilde{\omega}}\right) e^{\tilde{\varphi}(z)} dm.
\end{aligned}$$

By direct computation,

$$\frac{\partial}{\partial z^i}\langle d\bar{z}^I|(d\bar{z}^i)\wedge d\bar{z}^J\rangle_{\tilde{\omega}} = \overline{\frac{\partial}{\partial \bar{z}^i}\langle dz^I|(dz^i)\wedge dz^J\rangle_{\tilde{\omega}}} = \overline{\langle dz^I|\nabla_{\partial/\partial z^i}(dz^i)\wedge dz^J\rangle_{\tilde{\omega}}}$$
$$= \overline{\langle (dz^i\wedge^*_{\tilde{\omega}})\theta^*_{\partial/\partial z^i}dz^I|dz^J\rangle_{\tilde{\omega}}} = \langle (d\bar{z}^i\wedge^*_{\tilde{\omega}})\overline{\theta}^*_{\partial/\partial \bar{z}^i}d\bar{z}^I|d\bar{z}^J\rangle_{\tilde{\omega}}.$$

So, we can conclude that

$$\bar{\partial}^*_{\tilde{\omega}} f d\bar{z}^I = \left(-\frac{\partial f}{\partial z^i} - f\frac{\partial\tilde{\varphi}}{\partial z^i}\right)(d\bar{z}^i)\wedge^*_{\tilde{\omega}} d\bar{z}^I - f\,(d\bar{z}^i\wedge^*_{\tilde{\omega}})\overline{\theta}^*_{\partial/\partial\bar{z}^i}d\bar{z}^I. \qquad \square$$

So far, we establish a framework for the localized Cauchy-Riemann operator $\bar{\partial}_{\tilde{s}}$ and the localized Kodaira Laplacian $\square_{\tilde{s}}$. They depend only on the local data $\tilde{\omega}$ and $\tilde{\phi}$ on $\mathbb{C}^n$. Next, we are going to apply this framework to the configuration in Section 3.1.

Recall the trivializing neighborhood $B(1)\subset U\subset M$ taken in the previous section. We insert $\tilde{U} = B(1)$ and $\tilde{s} = s^k$ into the above framework. Denote $\bar{\partial}_{k,s} := \bar{\partial}_{s^k}$ as the localized Cauchy-Riemann operator. By (3.13), we have $\bar{\partial}_{k,s} = e^{-k\phi}\bar{\partial}e^{k\phi}$, which is an analogy of the Witten deformation of exterior derivative on real manifolds (cf. [29]). Moreover, by (3.14) and (3.15),

$$\bar{\partial}_{k,s} = \bar{\partial} + (\bar{\partial}k\phi)\wedge\cdot; \quad \bar{\partial}^*_{k,s} = \bar{\partial}^*_{\omega} + (\bar{\partial}k\phi)\wedge^*_{\omega}\cdot.$$

Denote $\square^{(q)}_{k,s} := \square^{(q)}_{s^k}$ to be the *localized Kodaira Laplacian*. The expression of $\square^{(q)}_{k,s}$ is given in Lemma 3.1. On the other hand, we can consider the scaled vector bundle $T^{*,(0,q)}\mathbb{C}^n\otimes L^{(k)}\to B(\sqrt{k})$ and insert $\tilde{U} = B(\sqrt{k})$ and $\tilde{s} = s^{(k)}$. Thus, $\tilde{\omega} = \omega_{(k)}$ and $\tilde{\phi} = \phi_{(k)}$. Denote $\bar{\partial}_{(k),s} := \bar{\partial}_{s^{(k)}}$ and compute that

$$\bar{\partial}_{(k),s} = \bar{\partial} + (\bar{\partial}\phi_{(k)})\wedge\cdot; \quad \bar{\partial}^*_{(k),s} = \bar{\partial}^*_{\omega_{(k)}} + (\bar{\partial}\phi_{(k)})\wedge^*_{\omega_{(k)}}\cdot.$$

Denote $\square^{(q)}_{(k),s} := \square^{(q)}_{s^{(k)}}$ to be the *scaled localized Kodaira Laplacian*. The relations between the two sets of operators established above are given by

$$(\bar{\partial}^{(q)}_{(k),s}u)(\sqrt{k}z) = \frac{1}{\sqrt{k}}\bar{\partial}^{(q)}_{k,s}(u(\sqrt{k}z)); \quad (\bar{\partial}^{*,(q)}_{(k),s}u)(\sqrt{k}z) = \frac{1}{\sqrt{k}}\bar{\partial}^{*,(q)}_{k,s}(u(\sqrt{k}z)) \tag{3.19}$$

and hence

$$\left(\Box^{(q)}_{(k),s}u\right)(\sqrt{k}z) = \frac{1}{k}\Box^{(q)}_{k,s}(u(\sqrt{k}z)). \tag{3.20}$$

Now, recall the Sobolev space $H_s(\tilde{U}, T^{*,(0,q)}\mathbb{C}^n)$ with its norm $\|\cdot\|_s$(cf. (2.8)). By the fact that $\omega_{(k)} \to \omega_0$ and $\phi_{(k)} \to \phi_0$ locally uniformly in $\mathcal{C}^\infty$, we know that the coefficients of $\Box^{(q)}_{(k)}$ converge locally uniformly. Therefore, we can apply Theorem 2.7 to obtain the following proposition.

Proposition 3.2 (k-uniform Gårding inequalities)**.** *For any fixed radius $r \geq 0$ and integers $m \in \mathbb{N}$, there is a constant C independent of k such that*

$$\|u\|_{2m} \leq C\left(\|u\|_0 + \|(\Box^{(q)}_{(k)})^m u\|_0\right)$$

for all $u \in H_{2m}(B(r), T^{*,(0,q)}\mathbb{C}^n)$ and $k \geq r^2$.

Remark 3.1. In fact, for any cut-off functions $\rho \in \mathcal{C}^\infty_c(B(r),[0,1])$, $\tilde{\rho} \in \mathcal{C}^\infty_c(B(r),[0,\infty))$ with $\operatorname{supp}\rho \subset\subset \operatorname{supp}\tilde{\rho}$, there is a $C > 0$ such that

$$\|\rho u\|_{2m} \leq C\left(\|\tilde{\rho}u\|_0 + \|\tilde{\rho}(\Box^{q}_{(k)})^m u\|_0\right) \quad \text{for all } u \in \Omega^{(0,q)}(B(r)).$$

This property makes the Gårding inequality applicable to sections without compact support.

3.3. The uniform bounds

To begin with, we make some observations about compatibility of norms. Note that there exist positive constants C_1 and C_2 such that

$$C_1\|u\|_{\omega_0,B(1)} \leq \|u\|_{\omega,B(1)} \leq C_2\|u\|_{\omega_0,B(1)}$$

for all $u \in \Omega^{(0,q)}(B(1))$ since ω and ω_0 are both positive bounded Hermitian $(1,1)$-forms on the precompact domain $B(1)$. By scaling the metric, it follows that

$$C_1\|u\|_{\omega_0,B(\sqrt{k})} \leq \|u\|_{\omega_{(k)},B(\sqrt{k})} \leq C_2\|u\|_{\omega_0,B(\sqrt{k})} \tag{3.21}$$

for all $u \in \Omega^{(0,q)}(B(k))$. Moreover, by the fact that $\omega_{(k)}$ converges to ω_0 locally uniformly, for any fixed radius $r > 0$ and positive number $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$(1-\varepsilon)|\,(u \mid v)_{\omega_0,B(r)}\,| \le |\,(u \mid v)_{\omega_{(k)},B(r)}\,| \le (1+\varepsilon)|\,(u \mid v)_{\omega_0,B(r)}\,| \tag{3.22}$$

for all $u, v \in \Omega_c^{(0,q)}(B(r))$ and $k \ge k_0$. Now, we enter the core of this section.

Lemma 3.3. *Given $u \in \Omega_c^{(0,q)}(B(\sqrt{k}))$ for some $k \in \mathbb{N}$, we have*

$$\|\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_{(k)},B(\sqrt{k})} \le \|u\|_{\omega_{(k)},B(\sqrt{k})}.$$

Consequently, there exists a constant C independent of k such that the four inequalities hold:

$$\|\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_0 \text{ or } \omega_{(k)},B(\sqrt{k})} \le C\|u\|_{\omega_0 \text{ or } \omega_{(k)},B(\sqrt{k})},$$

for all $u \in \Omega_c^{(0,q)}(B(\sqrt{k}))$ and $k \in \mathbb{N}$. Moreover, for any radius $r > 0$ and a number $\varepsilon > 0$, there exists k_0 such that

$$\|\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_{(k)},B(\sqrt{k})} \le (1+\varepsilon)\|u\|_{\omega_0,B(r)},$$

for all $u \in \Omega_c^{(0,q)}(B(r))$ and $k > k_0$.

Proof. Let $u \in \Omega_c^{(0,q)}(B(\sqrt{k}))$. Inspired by identification (3.7), we define

$$u_k(z) := k^{n/2}u(\sqrt{k}z) \in \Omega_c^{(0,q)}(B(1))$$

which satisfies $\|u_k\|_\omega = \|u\|_{\omega_{(k)}}$. Since $B(1) \subset M$, we can treat u_k as an element of $\Omega_c^{(0,q)}(M)$. By (3.8),

$$\mathcal{P}^{(q)}_{(k),c_k,s}u(z) = k^{-n/2}\mathcal{P}^{(q)}_{k,c_k,s}u_k(\sqrt{k}z).$$

This means $\mathcal{P}^{(q)}_{(k),c_k,s}u$ corresponds to $\mathcal{P}^{(q)}_{k,c_k,s}u_k$ under the identification (3.7) and hence

$$\|\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_{(k)},B(\sqrt{k})} = \|\mathcal{P}^{(q)}_{k,c_k,s}u_k\|_{\omega,B(1)}.$$

By the identification (3.1) and the definition of $\mathcal{P}^{(q)}_{k,c_k}$,

$$\|\mathcal{P}^{(q)}_{k,c_k,s}u_k\|_{\omega,B(1)} = \|\mathcal{P}^{(q)}_{k,c_k}(u_k e^{k\phi} \otimes s^k)\|_{\omega,k\phi,B(1)}$$

$$
\begin{aligned}
&\leq \|\mathcal{P}^{(q)}_{k,c_k}(u_k e^{k\phi}\otimes s^k)\|_{\omega,k\phi}\\
&\leq \|u_k e^{k\phi} s^k\|_{\omega,k\phi} = \|u_k\|_{\omega}.
\end{aligned}
$$

Combine the above arguments and get

$$\|\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_{(k)},B(\sqrt{k})} \leq \|u\|_{\omega_{(k)},B(\sqrt{k})}.$$

Finally, we apply (3.21) and (3.22) to complete the proof. □

Observe that

$$dV_{\omega_0} = \frac{\omega_0^n}{n!} = 2^n dx^1 \wedge \cdots \wedge dx^{2n} = 2^n dm.$$

The volume form induced by ω_0 coincides with the standard Lebesgue measure dm up to a constant 2^n. Consequently, we know that the induced L^2-norms $\|\cdot\|_{\omega_0}$ and $\|\cdot\|_{dm}$ (or $\|\cdot\|_0$) are equivalent. Here, $\|\cdot\|_0$ means the Sobelov 0-norm which is exactly $\|\cdot\|_{dm}$. We now verify an essential result of this paper.

Theorem 3.4. *(k-uniform smoothing property) Fix functions χ and ρ in $\mathcal{C}^\infty_c(\mathbb{C}^n)$ and real numbers $s,t\in\mathbb{R}$. If $\limsup_{k\to\infty}\frac{c_k}{k}<\infty$, there exists a constant C independent of k such that*

$$\|\chi\mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_s \leq C\|u\|_t$$

for all $u\in H_t(B(r),T^{*,(0,q)}\mathbb{C}^n)$ and $k\in\mathbb{N}$ with $\operatorname{supp}\chi\cup\operatorname{supp}\rho\subset B(\sqrt{k})$.

Proof. It is sufficient to show that for each $m\in\mathbb{N}$,

$$\chi\mathcal{P}^{(q)}_{(k),c_k,s}\rho : H_{-2m}(B(r),T^{*,(0,q)}\mathbb{C}^n)\to H_{2m}(B(2r),T^{*,(0,q)}\mathbb{C}^n) \tag{3.23}$$

is a k-uniformly bounded linear map for all k with $\operatorname{supp}\chi\cup\operatorname{supp}\rho\subset B(\sqrt{k})$.

We may assume $u\in\Omega^{(0,q)}_c(\mathbb{C}^n)$ by density argument. By Proposition 3.2 and Remark 3.1,

$$
\begin{aligned}
\|\chi\mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_{2m} &\lesssim \|\tilde{\chi}\mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_0 + \|\tilde{\chi}(\Box^{(q)}_{(k),s})^m\mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_0\\
&\lesssim \|\tilde{\chi}\mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_{\omega_{(k)},B(\sqrt{k})} + \|\tilde{\chi}\left(\Box^{(q)}_{(k),s}\right)^m\mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_{\omega_{(k)},B(\sqrt{k})}.
\end{aligned} \tag{3.24}
$$

The second inequality is from (3.21). Define $u_k(z) := k^{n/2}u(\sqrt{k}z)$, $\rho_k(z) := \rho(\sqrt{k}z)$ and $\tilde{\chi}_k(z) := \tilde{\chi}(\sqrt{k}z)$. For large enough k with $B(\sqrt{k}) \supset \operatorname{supp}\rho$, we observe that $\rho_k u_k \in \Omega_c^{(0,q)}(B(1)) \subset \Omega_c^{(0,q)}(M)$ and $\|\rho_k u_k\|_{\omega,B(1)}$ $= \|\rho u\|_{\omega_{(k)},B(\sqrt{k})}$. By Lemma 3.3,

$$\|\tilde{\chi}\mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_{\omega_{(k)},B(\sqrt{k})} \lesssim \|u\|_0. \tag{3.25}$$

Next, from the relations (3.8) and (3.20), we can see

$$k^{n/2}\left(\Box^{(q)}_{(k),s}\right)^m \mathcal{P}^{(q)}_{(k),c_k,s}\rho u(\sqrt{k}z) = k^{-m}\left(\Box^{(q)}_{k,s}\right)^m \mathcal{P}^{(q)}_{k,c_k,s}\rho_k u_k(z).$$

By changing variable again, we compute that

$$\|\left(\Box^{(q)}_{(k),s}\right)^m \mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_{\omega_{(k)},B(\sqrt{k})} = \|k^{-m}\left(\Box^{(q)}_{k,s}\right)^m P^{(q)}_{k,c_k,s}\rho_k u_k\|_{\omega,B(1)}. \tag{3.26}$$

Moreover, by the property of spectral projection, we estimate that

$$\begin{aligned}\|k^{-m}\left(\Box^{(q)}_{k,s}\right)^m \mathcal{P}^{(q)}_{k,c_k,s}\rho_k u_k\|_{\omega,B(1)} &\le \|k^{-m}\left(\Box^{(q)}_{k}\right)^m \mathcal{P}^{(q)}_{k,c_k}\rho_k e^{k\phi}u_k\otimes s^k\|_{\omega,k\phi,M}\\ &\le (\frac{c_k}{k})^m\|\rho_k e^{k\phi}u_k\otimes s^k\|_{\omega,k\phi,B(1)}\\ &= (\frac{c_k}{k})^m\|\rho_k u_k\|_{\omega,B(1)} \lesssim \|u\|_0. \end{aligned}\tag{3.27}$$

The last inequality is from the fact that $\limsup_{k\to\infty} c_k/k < \infty$. Combining estimates (3.24)-(3.27), we have

$$\|\chi\mathcal{P}^{(q)}_{(k),c_k,s}\rho u\|_{2m} \lesssim \|u\|_0. \tag{3.28}$$

Next, by the self-adjointness of spectral projection, for all $v \in \Omega_c^{(0,q)}(\mathbb{C}^n)$,

$$\begin{aligned}\left(\chi\mathcal{P}^{(q)}_{(k),c_k,s}\rho u \mid v\right)_{\omega_{(k)}} &= \left(\chi_k\mathcal{P}^{(q)}_{k,c_k}\rho_k e^{k\phi}u_k\otimes s^k \mid e^{k\phi}v_k\otimes s^k\right)_{\omega,k\phi}\\ &= \left(e^{k\phi}u_k\otimes s^k \mid \rho_k\mathcal{P}^{(q)}_{k,c_k}\chi_k e^{k\phi}v_k\otimes s^k\right)_{\omega,k\phi}\\ &= \left(u \mid \rho\mathcal{P}^{(q)}_{(k),c_k,s}\chi v\right)_{\omega_{(k)}}\end{aligned}$$

where $v_k(z) := k^{n/2}v(\sqrt{k}z)$ and $\chi_k(z) := \chi(\sqrt{k}z)$. By Proposition 2.5 and

(3.28),

$$|\left(u \mid \rho\mathcal{P}^{(q)}_{(k),c_k,s}\chi v\right)_0| \leq \|u\|_{-2m}\|\rho\mathcal{P}^{(q)}_{(k),c_k,s}\chi v\|_{2m} \lesssim \|u\|_{-2m}\|v\|_0.$$

By the arguments above and (3.22), we have

$$\|\chi P^{(q)}_{(k),c_k,s}\rho u\|_0 \lesssim \|u\|_{-2m} \tag{3.29}$$

since v is arbitrary. By (3.24), it remains to show the following fact:

Claim. $\tilde{\chi}\left(\Box^{(q)}_{(k),s}\right)^m\mathcal{P}^{(q)}_{(k),c_k,s}\rho : H_{-2m}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \to L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$ *is a k-uniformly bounded map.*

To prove the claim, we observe that $\tilde{\chi}_k u_k \in \Omega^{(0,q)}_c(B(1)) \subset \Omega^{(0,q)}_c(M)$ for large enough k, and $\|\tilde{\chi}_k u_k\|_{\omega,B(1)} = \|\tilde{\chi}u\|_{\omega_{(k)}}$. By Proposition 3.2 and (3.21),

$$\begin{aligned}\|\rho\mathcal{P}^{(q)}_{(k),c_k,s}(\Box^{(q)}_{(k),s})^m\tilde{\chi}u\|_{2m} \lesssim &\|\tilde{\rho}\mathcal{P}^{(q)}_{(k),c_k,s}(\Box^{(q)}_{(k),s})^m\tilde{\chi}u\|_{\omega_{(k)}}\\ &+ \|\tilde{\rho}(\Box^{(q)}_{(k),s})^m\mathcal{P}^{(q)}_{(k),c_k,s}(\Box^{(q)}_{(k),s})^m\tilde{\chi}u\|_{\omega_{(k)}}.\end{aligned}$$

By rescaling, the first term on the right-hand side above is $k^{-m}\|\tilde{\rho}_k\mathcal{P}^{(q)}_{k,c_k,s}(\Box^{(q)}_{k,s})^m\tilde{\chi}_k u_k\|_{\omega,B(1)}$ where $\tilde{\rho}_k(z) := \tilde{\rho}(\sqrt{k}z)$. This can be dominated by

$$k^{-m}\|\tilde{\rho}_k\mathcal{P}^{(q)}_{k,c_k}(\Box^{(q)}_k)^m\tilde{\chi}_k e^{k\phi}u_k \otimes s^k\|_{\omega,k\phi,M} \leq (\frac{c_k}{k})^m\|\tilde{\chi}_k e^{k\phi}u_k \otimes s^k\|_{\omega,k\phi} \lesssim \|u\|_0$$

since $\limsup_{k\to\infty} c_k/k < \infty$. For the second term, by rescaling, we write

$$\begin{aligned}&\|\tilde{\rho}(\Box^{(q)}_{(k),s})^m\mathcal{P}^{(q)}_{(k),c_k,s}(\Box^{(q)}_{(k),s})^m\tilde{\chi}u\|_{\omega_{(k)}}\\ &\quad = k^{-2m}\|\tilde{\rho}_k(\Box^{(q)}_{k,s})^m\mathcal{P}^{(q)}_{k,c_k,s}(\Box^{(q)}_{k,s})^m\tilde{\chi}_k u_k\|_{\omega,B(1)}\end{aligned}$$

which is smaller than

$$\begin{aligned}&k^{-2m}\|\tilde{\rho}_k(\Box^{(q)}_k)^m\mathcal{P}^{(q)}_{k,c_k}(\Box^{(q)}_k)^m\tilde{\chi}_k e^{k\phi}u_k \otimes s^k\|_{\omega,k\phi,M}\\ &\quad \leq (\frac{c_k}{k})^{2m}\|\tilde{\chi}_k e^{k\phi}u_k \otimes s^k\|_{\omega,k\phi} \lesssim \|u\|_0.\end{aligned}$$

Combining arguments above, we get

$$\|\rho\mathcal{P}^{(q)}_{(k),c_k,s}(\Box^{(q)}_{(k),s})^m\tilde{\chi}u\|_{2m} \lesssim \|u\|_0. \tag{3.30}$$

By the self-adjointness of $\Box_k^{(q)}$ and $\mathcal{P}_{k,c_k}^{(q)}$, for any test function $v \in \Omega_c^{(0,q)}(\mathbb{C}^n)$,

$$\begin{aligned}\Big(\tilde{\chi}(\Box_{(k),s}^{(q)})^m \mathcal{P}_{(k),c_k,s}^{(q)} \rho u \mid v\Big)_{\omega_{(k)}} &= \Big(k^{-m}\tilde{\chi}_k(\Box_{k,s}^{(q)})^m \mathcal{P}_{k,c_k,s}^{(q)} \rho_k u_k \mid v_k\Big)_{\omega} \\ = \Big(u_k \mid k^{-m}\rho_k \mathcal{P}_{k,c_k,s}^{(q)}(\Box_{k,s}^{(q)})^m \tilde{\chi}_k v_k\Big)_{\omega} &= \Big(u \mid \rho \mathcal{P}_{(k),c_k,s}^{(q)}(\Box_{(k),s}^{(q)})^m \tilde{\chi} v\Big)_{\omega_{(k)}},\end{aligned}$$

where $v_k(z) := k^{n/2} v(\sqrt{k}z)$. Again, by Proposition 2.5 and (3.30), we have

$$\begin{aligned}\left|\Big(u \mid \rho \mathcal{P}_{(k),c_k,s}^{(q)}(\Box_{(k),s}^{(q)})^m \tilde{\chi} v\Big)_0\right| &\leq \|u\|_{-2m} \|\rho \mathcal{P}_{(k),c_k,s}^{(q)}(\Box_{(k),s}^{(q)})^m \tilde{\chi} v\|_{2m} \\ &\lesssim \|u\|_{-2m}\|v\|_0.\end{aligned}$$

Hence, we obtain $\Big(\tilde{\chi}(\Box_{(k),s}^{(q)})^m \mathcal{P}_{(k),c_k,s}^{(q)} \rho u \mid v\Big)_{\omega_{(k)}} \lesssim \|u\|_{-2m}\|v\|_0$ by combining above arguments. This completes the proof of the claim since v is arbitrary. Finally, by estimates (3.24), (3.29) and the claim,

$$\begin{aligned}\|\chi \mathcal{P}_{(k),c_k,s}^{(q)} \rho u\|_{2m} \lesssim &\|\tilde{\chi} \mathcal{P}_{(k),c_k,s}^{(q)} \rho v\|_{\omega_{(k)}, B(\sqrt{k})} \\ &+ \|\tilde{\chi}\left(\Box_{(k),s}^{(q)}\right)^m \mathcal{P}_{(k),c_k,s}^{(q)} \rho v\|_{\omega_{(k)}, B(\sqrt{k})} \lesssim \|u\|_{-2m}\end{aligned}$$

for all $u \in \Omega_c^{(0,q)}(\mathbb{C}^n)$. The theorem follows. $\square$

With the preliminary work out of the way, we can now address the local uniform bound of $P_{(k),c_k,s}^{(q)}(z,w)$. To do so, we represent $P_{(k),c_k,s}^{(q)}(z,w)$ as the form

$$P_{(k),c_k,s}^{(q)}(z,w) = \sum_{I,J \in \mathcal{J}_{q,n}} P_{(k),c_k,s}^{(q),I,J}(z,w) d\bar{z}^I \otimes (\frac{\partial}{\partial \bar{w}})^J,$$

where $P_{(k),c_k,s}^{(q),I,J}(z,w) \in \mathcal{C}^\infty(B(\sqrt{k}) \times B(\sqrt{k}))$. Also, we define the $\mathcal{C}^d$-norm of $P_{(k),c_k,s}^{(q)}(z,w)$ on the bounded domain $B(r) \times B(r)$ as

$$\|P_{(k),c_k,s}^{(q)}(z,w)\|_{\mathcal{C}^d(B(r)\times B(r))}^2 := \sup_{x,y \in B(r)} \sum_{I,J \in \mathcal{J}_{q,n}} \sum_{|\alpha|+|\beta| \leq d} \Big(|\partial_x^\alpha \partial_y^\beta P_{(k),c_k,s}^{(q),I,J}(x,y)|^2\Big)$$

where the variables x, y represent the real coordinates of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. For a $K(z,w) \in \mathcal{C}^\infty(\mathbb{C}^n \times \mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n \boxtimes T^{*,(0,q)}\mathbb{C}^n)$, we say $P_{(k),c_k,s}^{(q)}(z,w) \to K(z,w)$ as $k \to \infty$ locally uniformly in $\mathcal{C}^\infty$ if

$$\|P_{(k),c_k,s}^{(q)}(z,w) - K(z,w)\|_{\mathcal{C}^d(B(r)\times B(r))} \to 0$$

as $k \to \infty$ for all $d \in \mathbb{N}$ and $r > 0$.

Theorem 3.5 (The local uniform bounds for scaled spectral and Bergman kernels)**.** *Suppose c_k is a non-negative sequence such that*

$$\limsup_{k\to\infty} \frac{c_k}{k} < \infty.$$

Fix a radius $r > 0$. For any multi-indices $\alpha, \beta \in \mathbb{N}_0^{2n}$ and $I, J \in \mathcal{J}_{q,n}$, there exists a constant C independent of k such that

$$\sup_{B(r)\times B(r)} |\partial_x^\alpha \partial_y^\beta P^{(q),I,J}_{(k),c_k,s}(x,y)| < C.$$

Proof. We start from the approximation of identity. For any fixed point $y_0 \in B(r)$, we set f_l as an approximation of identity with its mass concentrated at y_0 as $l \to \infty$. For example, let $f_l = l^n f(\sqrt{l}(y - y_0))$ where $f \in \mathcal{C}_c^\infty(B(r); [0,\infty))$ and $\int_{B(r)} f dm = 1$. By the property of approximation of identity, it is sufficient to establish the following estimate:

$$\sup_{\substack{x\in B(r), k>r^2,\\ l\in\mathbb{N}}} |\int_{B(r)} \partial_x^\alpha \partial_y^\beta P^{(q),I,J}_{(k),c_k,s}(x,y) f_l(y) dm(y)| < C.$$

We hope the C is independent of u and the point $y_0 \in B(r)$ chosen above. By integration by part, we just need to find an upper bound of

$$\sup_{\substack{x\in B(r), k>r^2,\\ l\in\mathbb{N}}} |\partial_x^\alpha \int_{B(r)} P^{(q),I,J}_{(k),c_k,s}(x,y) \partial_y^\beta f_l(y) dm(y)|.$$

Choose $\chi \in \mathcal{C}_c^\infty(B(2r))$ so that $\chi \equiv 1$ on $B(r)$. Observe that

$$\begin{aligned}
&\sup_{\substack{x\in B(r), k>r^2,\\ l\in\mathbb{N}}} |\partial_x^\alpha \int_{B(r)} P^{(q),I,J}_{(k),c_k,s}(x,y)(\partial_y^\beta f_l(y)) dm(y)| \\
&\le \sup_{k>r^2, l\in\mathbb{N}} \|\mathcal{P}^{(q)}_{(k),c_k,s}\left((\partial^\beta f_l) d\bar{z}^J\right)\|_{\mathcal{C}^{|\alpha|}(B(r))} \\
&\le \sup_{k>r^2, l\in\mathbb{N}} \|\chi \mathcal{P}^{(q)}_{(k),c_k,s}\left((\partial^\beta f_l) d\bar{z}^J\right)\|_{\mathcal{C}^{|\alpha|}(B(2r))},
\end{aligned}$$

where the norm $\|\cdot\|_{\mathcal{C}^{|\alpha|}(B(2r))}$ we adopted is defined in (2.10). By Sobolev inequality, for any integer m with $2m \geq |\alpha| + n$, we have

$$\|\chi \mathcal{P}^{(q)}_{(k),c_k,s}\left((\partial^\beta f_l) d\bar{z}^J\right)\|_{\mathcal{C}^{|\alpha|}(B(2r))} \lesssim \|\chi \mathcal{P}^{(q)}_{(k),c_k,s}\left((\partial^\beta f_l) d\bar{z}^J\right)\|_{2m}.$$

Note that $|\hat{f}_l(\xi)| \lesssim |\int_{\mathbb{R}^{2n}} e^{-\sqrt{-1}x\cdot\xi} f_l(x) dx| \lesssim O(1)$ and hence $|\widehat{(\partial^\beta f_l)}| \lesssim |\xi|^{|\beta|} |\hat{f}_l| \lesssim |\xi|^{|\beta|}$. This implies that for large enough $m \in \mathbb{N}$,

$$\|(\partial^\beta f_l) d\bar{z}^J\|_{-2m} \lesssim O(1).$$

After combining this fact with Theorem 3.4, we know that for large enough $m \in \mathbb{N}$,

$$\begin{aligned}\|\chi \mathcal{P}^{(q)}_{(k),c_k,s}(\partial^\beta f_l) d\bar{z}^J\|_{2m} &\lesssim \|\chi \mathcal{P}^{(q)}_{(k),c_k,s} \rho (\partial^\beta f_l) d\bar{z}^J\|_{2m} \\ &\lesssim \|(\partial^\beta f_l) d\bar{z}^J\|_{-2m} \lesssim O(1),\end{aligned}$$

where ρ is a bump function which has value 1 around the point y_0. We have completed the proof. □

We end this section with the following extremely important corollary, which is an immediate consequence of the Arzelà-Ascoli theorem and Theorem 3.5.

Corollary 3.6. *If c_k is a non-negative sequence such that*

$$\limsup_{k\to\infty} \frac{c_k}{k} < \infty,$$

then any subsequence of the scaled localized spectral kernel $P^{(q)}_{(k),c_k,s}(z,w)$ (or Bergman kernel $B^{(q)}_{(k),s}(z,w)$ in the case $c_k = 0$) has a $\mathcal{C}^\infty$ locally uniformly convergent subsequence in $\mathbb{C}^n$.

By the identity (3.9)*, we have the same results for $P^{(q),s}_{(k),c_k}(z,w)$ and $B^{(q),s}_{(k)}(z,w)$.*

4. Asymptotics of Spectral and Bergman Kernels

Recall the Corollary 3.6. We have established that $P^{(q)}_{(k),c_k,s}(z,w)$ (or $P^{(q),s}_{(k),c_k}(z,w)$) is a sequence such that every subsequence has a $\mathcal{C}^\infty$ uniformly convergent subsequence. To show that $P^{(q)}_{(k),c_k,s}(z,w)$ (or $P^{(q),s}_{(k),c_k}(z,w)$) is itself a uniformly convergent sequence in $\mathcal{C}^\infty$, it suffices to demonstrate that every convergent subsequence of $P^{(q)}_{(k),c_k,s}(z,w)$ converges to the same limit. Therefore, without loss of generality, we may assume that $P^{(q)}_{(k),c_k,s}(z,w)$ is a $\mathcal{C}^\infty$ uniformly convergent sequence, and our goal is to prove that the limit must be the Bergman kernel in a model case which will be thoroughly investigated in Section 4.1. From now on, we make the following assumption throughout this chapter.

Assumption 4.1. The scaled localized spectral kernel $P^{(q)}_{(k),c_k,s}(z,w)$ converges to $B^{(q)}_s(z,w)$ locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$, where $B^{(q)}_s(z,w) \in \mathcal{C}^\infty(\mathbb{C}^n \times \mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n \boxtimes T^{*,(0,q)}\mathbb{C}^n)$. Equivalently, the scaled spectral kernel $P^{(q),s}_{(k),c_k}(z,w)$ converges to $B^{(q),s}(z,w) := e^{\phi_0(z)}B^{(q)}_s(z,w)e^{-\phi_0(w)}$ locally uniformly in $\mathcal{C}^\infty$.

To maintain the validity of Corollary 3.6, we must specify a non-negative sequence c_k such that $\limsup_{k\to\infty}\frac{c_k}{k} < \infty$. However, we will require a stronger condition that $\limsup_{k\to\infty}\frac{c_k}{k} = 0$. We will see the reason in Lemma 4.4. Clearly, the scaled localized Bergman kernels $B^{(q)}_{(k),s}(z,w)$ can be treated as a spacial case of localized spectral kernels when $c_k = 0$ and they satisfy the above condition. Before investigating the properties of $B^{(q)}_s(z,w)$, we study the space of sections in the *model case* on $\mathbb{C}^n$.

4.1. The model case

We now consider the trivial vector bundle $T^{*,(0,q)}\mathbb{C}^n \otimes \mathbb{C} \to \mathbb{C}^n$ which is equipped with a pointwise Hermitian structure induced by the weight function $\phi_0 = \sum_{i=1}^n \lambda_i |z^i|^2$ on the trivial line bundle $\mathbb{C} \to \mathbb{C}^n$ and the standard Hermitian form $\omega_0 = \sqrt{-1}\sum_{i=1}^n dz^i \wedge d\bar{z}^i$ on $T^{*,(0,q)}\mathbb{C}^n \to \mathbb{C}^n$(cf. (3.10)).

We can define the Hilbert space $L^2_{\omega_0,\phi_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$ with an inner product $(\cdot|\cdot)_{\omega_0,\phi_0}$ (cf. (2.5)) and another space $L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$ with $(\cdot|\cdot)_{\omega_0}$ as its inner product (cf. (2.3)). There is an unitary identification

$$L^2_{\omega_0,\phi_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n \otimes \mathbb{C}) \cong L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \quad \text{by } \eta \leftrightarrow \eta e^{-\phi_0}.$$

Let $\bar{\partial}^{(q)}_0 : \Omega^{(0,q)}(\mathbb{C}^n,\mathbb{C}) \to \Omega^{(0,q+1)}(\mathbb{C}^n,\mathbb{C})$ and $\bar{\partial}^{*,(q+1)}_0 : \Omega^{(0,q+1)}(\mathbb{C}^n,\mathbb{C}) \to \Omega^{(0,q)}(\mathbb{C}^n,\mathbb{C})$ be the Cauchy-Riemann operator and its formal adjoint with respect to $(\cdot|\cdot)_{\omega_0,\phi_0}$, respectively. Then

$$\Box_0^{(q)} := \bar{\partial}_0^{*,(q+1)}\bar{\partial}_0^{(q)} + \bar{\partial}_0^{(q-1)}\bar{\partial}_0^{*,(q)} : \mathrm{Dom}\,\Box_0^{(q)} \to L^2_{\omega_0,\phi_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n \otimes \mathbb{C})$$

is the Gaffney extension of the Kodaira Laplacian with respect to the Hermitian structure. As in (3.14) and (3.15), we can define the localized Cauchy-Riemann operator $\bar{\partial}_{0,s}$ and its formal adjoint $\bar{\partial}^*_{0,s}$ with respect to $(\cdot|\cdot)_{\omega_0}$ which are given by

$$\bar{\partial}_{0,s}^{(q)} = \bar{\partial}^{(q)} + (\bar{\partial}\phi_0) \wedge \cdot; \quad \bar{\partial}_{0,s}^{*,(q)} = \bar{\partial}_{\omega_0}^{*,(q)} + (\bar{\partial}\phi_0)\wedge^*_{\omega_0},$$

respectively. The localized Kodaira Laplacian is

$$\Box_{0,s}^{(q)} := \bar{\partial}_{0,s}^{*,(q+1)}\bar{\partial}_{0,s}^{(q)} + \bar{\partial}_{0,s}^{(q-1)}\bar{\partial}_{0,s}^{*,(q)} : \mathrm{Dom}\,\Box_{0,s}^{(q)} \to L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$$

which satisfies $\Box_0^{(q)}(\eta \otimes 1) = e^{\phi_0}\Box_{0,s}^{(q)}(\eta e^{-\phi_0}) \otimes 1$. Next, we denote

$$\mathcal{B}_0^{(q)} : L^2_{\omega_0,\phi_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n \otimes \mathbb{C}) \to \mathrm{Ker}\,\Box_0^{(q)} \subset L^2_{\omega_0,\phi_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n \otimes \mathbb{C})$$

to be the Bergman projection and

$$\mathcal{B}_{0,s}^{(q)} : L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \to \mathrm{Ker}\,\Box_{0,s}^{(q)} \subset L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$$

to be the localized Bergman projection satisfying $\mathcal{B}_0^{(q)}(\eta\otimes 1) = e^{\phi_0}\mathcal{B}_{0,s}^{(q)}(\eta e^{-\phi_0}) \otimes 1$. Furthermore, denote by $B_0^{(q)}(z,w)$ the *Bergman kernel* and $B_{0,s}^{(q)}(z,w)$ the *localized Bergman kernel* which are Schwartz kernels of $\mathcal{B}_0^{(q)}$ and $\mathcal{B}_{0,s}^{(q)}$, respectively. Our main goal of this section is to compute the localized Bergman kernel $B_{0,s}^{(q)}(z,w)$. Proposition 4.1 below tells us that $\Box_{0,s}^{(q)}$ is *diagonal* with respect to the basis $\{d\bar{z}^I\}_{I\in\mathcal{J}_{q,n}}$.

Proposition 4.1. *For $f \in \mathcal{C}^\infty(\mathbb{C}^n, \mathbb{C})$ and $I \in \mathcal{J}_{q,n}$,*

$$\begin{aligned}\Box_{0,s}^{(q)} f d\bar{z}^I = &\sum_{i=1}^n \left(-\frac{\partial^2 f}{\partial \bar{z}^i \partial z^i} + \frac{\partial \phi_0}{\partial z^i}\frac{\partial f}{\partial \bar{z}^i} - \frac{\partial \phi_0}{\partial \bar{z}^i}\frac{\partial f}{\partial z^i}\right) d\bar{z}^I \\ &+ \left(\sum_{i\in I}\lambda_i - \sum_{i\notin I}\lambda_i + |\bar{\partial}\phi_0|^2_{\omega_0}\right) f d\bar{z}^I.\end{aligned}$$

Proof. Since the metric ω_0 is flat, the Hodge Laplacian $\Delta_{\omega_0} u$ of $u := f d\bar{z}^I$ is

$$\left(\bar{\partial}^*_{\omega_0}\bar{\partial} + \bar{\partial}\bar{\partial}^*_{\omega_0}\right) u = -\sum_i \frac{\partial^2 f}{\partial \bar{z}^i \partial z^i} d\bar{z}^I.$$

We compute the remaining terms of equation (3.17) in Lemma 3.1.

$$\begin{aligned}
&\bar{\partial}\left((\bar{\partial}\phi_0) \wedge^*_{\omega_0} u\right) + (\bar{\partial}\phi_0) \wedge^*_{\omega_0} \bar{\partial} u \\
=&\bar{\partial}\left(\sum_i \frac{\partial \phi_0}{\partial z^i} f d\bar{z}^i \wedge^*_{\omega_0} d\bar{z}^I\right) + (\bar{\partial}\phi_0) \wedge^*_{\omega_0} \sum_j \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j \wedge d\bar{z}^I \\
=&\sum_{i,j}\left(\frac{\partial^2 \phi_0}{\partial \bar{z}^j \partial z^i} f + \frac{\partial \phi_0}{\partial z^i}\frac{\partial f}{\partial \bar{z}^j}\right) d\bar{z}^j \wedge (d\bar{z}^i \wedge^*_{\omega_0}) d\bar{z}^I + \sum_{i,j} \frac{\partial \phi_0}{\partial z^i}\frac{\partial f}{\partial \bar{z}^j} (d\bar{z}^i \wedge^*_{\omega_0}) d\bar{z}^j \wedge d\bar{z}^I \\
=&\sum_{i\in I} \lambda_i f d\bar{z}^I + \sum_i \frac{\partial \phi_0}{\partial z^i}\frac{\partial f}{\partial \bar{z}^i} d\bar{z}^I.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\bar{\partial}^*_{\omega_0}\left((\bar{\partial}\phi_0) \wedge u\right) + (\bar{\partial}\phi_0) \wedge \bar{\partial}^*_{\omega_0} u \\
&\quad = \bar{\partial}^*_{\omega_0}\left(\sum_j \frac{\partial \phi_0}{\partial \bar{z}^j} f d\bar{z}^j \wedge d\bar{z}^I\right) + (\bar{\partial}\phi_0) \wedge \sum_i -\frac{\partial f}{\partial z^i} d\bar{z}^i \wedge^*_{\omega_0} d\bar{z}^I \\
&\quad = -\sum_{i\notin I} \lambda_i f d\bar{z}^I - \sum_i \frac{\partial \phi_0}{\partial \bar{z}^i}\frac{\partial f}{\partial z^i} d\bar{z}^I.
\end{aligned}$$

Applying Lemma 3.1, we have

$$\begin{aligned}
\square^{(q)}_{0,s} f d\bar{z}^I =& \sum_i \left(-\frac{\partial^2 f}{\partial \bar{z}^i \partial z^i} + \frac{\partial \phi_0}{\partial z^i}\frac{\partial f}{\partial \bar{z}^i} - \frac{\partial \phi_0}{\partial \bar{z}^i}\frac{\partial f}{\partial z^i}\right) d\bar{z}^I \\
&+ \left(\sum_{i\in I} \lambda_i - \sum_{i\notin I} \lambda_i + |\bar{\partial}\phi_0|^2_{\omega_0}\right) f d\bar{z}^I. \qquad \square
\end{aligned}$$

We now try to find the complete orthonormal system of the space $\mathrm{Ker}\,\square^{(q)}_{0,s}$. By Proposition 4.1, to consider the equation $\square^{(q)}_{0,s} u = 0$, we can assume u is of the form

$$u(z) := f_I(z) d\bar{z}^I \quad \text{where } d\bar{z}^I := d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q.$$

That is, we fix the multi-index $I := (1, \ldots, q) \in \mathcal{J}_{q,n}$ and let $f_I \in L^2_{dm}(\mathbb{C}^n) \cap$

$\mathcal{C}^\infty(\mathbb{C}^n)$. Note that $\Box^{(q)}_{0,s}u=0$ if and only if $\bar\partial^{(q)}_{0,s}u=0$ and $\bar\partial^{*,(q)}_{0,s}u=0$ which are equivalent to

$$\frac{\partial f_I}{\partial z^i}-\lambda_i\bar z^i f_I=0\ \ \forall\, i\in I \quad \text{and} \quad \frac{\partial f_I}{\partial \bar z^i}+\lambda_i z^i f_I=0\ \ \forall\, i\notin I.$$

Thus, $\Box^{(q)}_{0,s}u=0$ if and only if

$$F_I(z):=f_I(\bar{z^1},\dots,\bar{z^q},z^{q+1},\dots,z^n)e^{-\sum_{i=1}^q\lambda_i|z^i|^2+\sum_{i=q+1}^n\lambda_i|z^i|^2}$$

is a holomorphic function on $\mathbb{C}^n$. If we write $F_I(z)$ as the form $F_I(z)=\sum_{\alpha\in\mathbb{N}_0^n}a_\alpha z^\alpha$ for some coefficients $a_\alpha\in\mathbb{C}$, then

$$f_I(\bar{z^1},\dots,\bar{z^q},z^{q+1},\dots,z^n)=\sum_{\alpha\in\mathbb{N}_0^n}a_\alpha z^\alpha e^{\sum_{i=1}^q\lambda_i|z^i|^2-\sum_{i=q+1}^n\lambda_i|z^i|^2}.$$

We can apply Fubini's theorem and introduce polar coordinates by setting $z^i=r_ie^{\sqrt{-1}\theta_i}$ to compute that for all $\alpha,\alpha'\in\mathbb{N}_0^n$,

$$\begin{aligned}
&\left(z^\alpha e^{\sum_{i=1}^q\lambda_i|z^i|^2-\sum_{i=q+1}^n\lambda_i|z^i|^2}\ \middle|\ z^{\alpha'}e^{\sum_{i=1}^q\lambda_i|z^i|^2-\sum_{i=q+1}^n\lambda_i|z^i|^2}\right)_{\omega_0}\\
&=2^n\int_{\mathbb{C}^n}z^\alpha\bar z^{\alpha'}e^{2(\sum_{i=1}^q\lambda_i|z^i|^2-\sum_{i=q+1}^n\lambda_i|z^i|^2)}dm\\
&=2^n\left(\prod_{i=1}^n\int_0^{2\pi}e^{\sqrt{-1}(\alpha_i-\alpha_i')\theta_i}d\theta_i\right)\left(\prod_{i=1}^q\int_0^\infty r_i^{\alpha_i+\alpha_i'+1}e^{2\lambda_ir_i^2}dr_i\right)\\
&\quad\times\left(\prod_{i=q+1}^n\int_0^\infty r_i^{\alpha_i+\alpha_i'+1}e^{-2\lambda_ir_i^2}dr_i\right),
\end{aligned}$$

which is zero if $\alpha\neq\alpha'$. By the Parseval's identity, we can compute that:

$$\begin{aligned}
\|u\|^2_{\omega_0}&=2^n\int_{\mathbb{C}^n}|f_I|^2dm=2^n\sum_{\alpha\in\mathbb{N}_0^n}|a_\alpha|^2\int_{\mathbb{C}^n}|z^\alpha|^2e^{2(\sum_{i=1}^q\lambda_i|z^i|^2-\sum_{i=q+1}^n\lambda_i|z^i|^2)}dm\\
&=2^n(2\pi)^n\sum_{\alpha\in\mathbb{N}_0^n}|a_\alpha|^2\left(\prod_{i=1}^q\int_0^\infty r_i^{2\alpha_i+1}e^{2\lambda_ir_i^2}dr_i\cdot\prod_{i=q+1}^n\int_0^\infty r_i^{2\alpha_i+1}e^{-2\lambda_ir_i^2}dr_i\right).
\end{aligned}$$

By the assumption that $\|u\|^2_{\omega_0}$ is a finite number, we can conclude that if u is not identically zero, then $\lambda_i<0$ for all $i\in\{1,\dots,q\}$ and $\lambda_i>0$ for all $i\in\{q+1,\dots,n\}$. As a result, there exist nontrivial solutions of the equation $\square^{(q)}_{0,s}u = 0$ in $L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$ only when $p \in M(q)$. In other words, if $p \notin M(q)$, then

$$\operatorname{Ker}\square^{(q)}_{0,s} = \{0\} \quad \text{and} \quad B^{(q)}_{0,s}(z,w) \equiv 0.$$

Now, we focus on case $p \in M(q)$. Suppose $\lambda_i < 0$ for all $i \in \{1, \ldots, q\}$ and $\lambda_i > 0$ for all $i \in \{q+1, \ldots, n\}$. For brevity, we set

$$z^\alpha_q := (\bar{z}^1)^{\alpha_1} \cdots (\bar{z}^q)^{\alpha_q} (z^{q+1})^{\alpha_{q+1}} \cdots (z^n)^{\alpha_n}; \quad I := (1, \ldots, q) \in \mathcal{J}_{q,n}.$$

Observe that $f_I d\bar{z}^I$ is an element in $\operatorname{Ker}\square^{(q)}_{0,s}$ if and only if f_I is in the set

$$\{\tilde{F}_I(\bar{z^1}, \ldots, \bar{z^q}, z^{q+1}, \ldots, z^n) e^{-\sum_{i=1}^n |\lambda_i||z^i|^2}; \tilde{F}_I \text{ is a holomorphic function on } \mathbb{C}^n\}.$$

Therefore, we can see that the orthogonal basis of $\operatorname{Ker}\square^{(q)}_{0,s}$ is given by

$$\{z^\alpha_q e^{-\sum_{i=1}^n |\lambda_i||z^i|^2} d\bar{z}^I\}_{\alpha \in \mathbb{N}^n_0}.$$

Next, we denote

$$[\lambda] := (|\lambda_1|, \ldots, |\lambda_n|) \in \mathbb{R}^n.$$

We now compute the length of the orthogonal basis to normalize them. By Fubini's theorem and changing the variables by letting $z^i = r_i e^{\sqrt{-1}\theta_i}$ and $u_i = 2|\lambda_i| r_i^2$,

$$\begin{aligned}
&\|z^\alpha_q e^{-\sum_{i=1}^n |\lambda_i||z^i|^2} d\bar{z}^I\|^2_{\omega_0} \\
&\quad = 2^n \int_{\mathbb{C}^n} \prod_{i=1}^n |z^i|^{2\alpha_i} e^{-2\sum_{i=1}^n |\lambda_i||z^i|^2} dm = 2^n \prod_{i=1}^n (2\pi) \int_0^\infty r_i^{2\alpha_i+1} e^{-2|\lambda_i| r_i^2} dr_i \\
&\quad = 2^n \prod_{i=1}^n \frac{\pi}{(2|\lambda_i|)^{\alpha_i+1}} \int_0^\infty u_i^{\alpha_i} e^{-u_i} du_i = 2^n \prod_{i=1}^n \frac{\pi\,\Gamma(\alpha_i+1)}{(2|\lambda_i|)^{\alpha_i+1}} \\
&\quad = 2^n \prod_{i=1}^n \frac{\pi \alpha_i!}{(2|\lambda_i|)^{\alpha_i+1}} = \frac{\pi^n \alpha!}{2^{|\alpha|}[\lambda]^{\alpha+1}}.
\end{aligned}$$

Consequently,

$$\{\Psi_\alpha := \sqrt{\frac{2^{|\alpha|}[\lambda]^{\alpha+1}}{\pi^n \alpha!}} z^\alpha_q e^{-\sum_{i=1}^n |\lambda_i||z^i|^2} d\bar{z}^I\}_{\alpha \in \mathbb{N}^n_0}$$

is the orthonormal basis of $\operatorname{Ker}\Box_{0,s}^{(q)}$ and

$$\begin{aligned}
B_{0,s}^{(q)}(z,w) &= \sum_{\alpha\in\mathbb{N}_0^n} \Psi_\alpha(z)\otimes\Psi_\alpha^*(w)\\
&= \sum_{\alpha\in\mathbb{N}_0^n} \frac{2^{|\alpha|}[\lambda]^{\alpha+1}}{\pi^n\alpha!} z_q^\alpha \overline{w_q^\alpha} e^{-\sum_{i=1}^n|\lambda_i|(|z^i|^2+|w^i|^2)} d\bar{z}^I\otimes(\frac{\partial}{\partial\bar{w}})^I\\
&= \frac{|\lambda_1\cdots\lambda_n|}{\pi^n}\sum_{|\alpha|\in\mathbb{N}_0^n}\left(\frac{2^{|\alpha|}}{\alpha!}[\lambda]^\alpha z_q^\alpha\bar{w}_q^\alpha\right) e^{-\sum_{i=1}^n|\lambda_i|(|z^i|^2+|w^i|^2)} d\bar{z}^I\otimes(\frac{\partial}{\partial\bar{w}})^I\\
&= \frac{|\lambda_1\cdots\lambda_n|}{\pi^n} e^{2(\sum_{i=1}^q|\lambda_i|\bar{z}^iw^i+\sum_{i=q+1}^n|\lambda_i|z^i\bar{w}^i)-\sum_{i=1}^n|\lambda_i|(|z^i|^2+|w^i|^2)} d\bar{z}^I\otimes(\frac{\partial}{\partial\bar{w}})^I.
\end{aligned}$$

We summarize the results in the following theorem.

Theorem 4.2 (Bergman kernel for the model case)**.** *Consider the trivial vector bundle $T^{*,(0,q)}\mathbb{C}^n\otimes\mathbb{C}\to\mathbb{C}^n$ endowed with the standard Hermitian form ω_0 and the weight function ϕ_0 . In the case $p\in M(q)$, we assume $\lambda_i<0$ for all $i\le q$ and $\lambda_i>0$ for all $i>q$. The localized Bergman Kernel $B_{0,s}^{(q)}(z,w)$ for $(0,q)$-forms is given by*

$$\frac{|\lambda_1\cdots\lambda_n|}{\pi^n} e^{2(\sum_{i=1}^q|\lambda_i|\bar{z}^iw^i+\sum_{i=q+1}^n|\lambda_i|z^i\bar{w}^i)-\sum_{i=1}^n|\lambda_i|(|z^i|^2+|w^i|^2)}(d\bar{z}^1\wedge\cdots\wedge d\bar{z}^q)\otimes(\frac{\partial}{\partial\bar{w}^1}\wedge\cdots\wedge\frac{\partial}{\partial\bar{w}^q}).$$

Furthermore,

$$\{\Psi_\alpha := \sqrt{\frac{2^{|\alpha|}[\lambda]^{\alpha+1}}{\pi^n\alpha!}} z_q^\alpha e^{-\sum_{i=1}^n|\lambda_i||z^i|^2} d\bar{z}^1\wedge\cdots\wedge d\bar{z}^q\}_{\alpha\in\mathbb{N}_0^n}$$

is the orthonormal basis of $\operatorname{Ker}\Box_{0,s}^{(q)}\subset L^2_{\omega_0}(\mathbb{C}^n,T^{*,(0,q)}\mathbb{C}^n)$*. However, if $p\notin M(q)$, then*

$$\operatorname{Ker}\Box_{0,s}^{(q)}=\{0\}\quad\textit{and hence}\quad B_{0,s}^{(q)}(z,w)\equiv 0.$$

4.2. Mapping properties of the approximated integral operator

Returning to Assumption 4.1, the kernel section $B_s^{(q)}(z,w)$ is unknown to us so far. Our objective is to demonstrate that it must be precisely the Bergman kernel $B_{0,s}^{(q)}(z,w)$ in a model case established above. We embark on the proof by the following definition and lemma which helps us to translate $B_s^{(q)}(z,w)$ from an unknown kernel section to an operator on the Hilbert space $L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$.

Definition 4.1. Define the *approximated integral operator* as

$$\mathcal{B}_s^{(q)}u(z) := \int_{\mathbb{C}^n} B_s^{(q)}(z,w)u(w)dm(w) \quad \text{for all } u \in L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n).$$

Lemma 4.3 (Well-definition of the integral operator)**.** *For any $u \in L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$, the integral $\mathcal{B}_s^{(q)}u(z)$ converges for almost every $z \in \mathbb{C}^n$. Furthermore, the integral operator*

$$\mathcal{B}_s^{(q)} : L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \to L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$$

is a bounded linear map with its operator norm smaller than 1*.*

Proof. Let $u, v \in \Omega_c^{(0,q)}(\mathbb{C}^n)$ and observe that

$$\left(v \mid \mathcal{B}_s^{(q)}u\right)_{\omega_0} = \int_{\operatorname{supp} v}\int_{\operatorname{supp} u} \langle v(z)|B_s^{(q)}(z,w)u(w)\rangle_{\omega_0} 2^{2n} dm(w)dm(z).$$

Let $\varepsilon > 0$. By (3.22) and the fact that $\omega_{(k)} \to \omega_0$ and $P^{(q)}_{(k),c_k,s}(z,w) \to B_s^{(q)}(z,w)$ uniformly on $\operatorname{supp} v \times \operatorname{supp} u$, the above integral can be dominated as

$$\begin{aligned}&\left|\left(v \mid \mathcal{B}_s^{(q)}u\right)_{\omega_0}\right| \le (1+\varepsilon)\left|\left(v \mid \mathcal{P}^{(q)}_{(k),c_k,s}u\right)_{\omega_0}\right| \\ &\le (1+\varepsilon)^2\left|\left(v \mid \mathcal{P}^{(q)}_{(k),c_k,s}u\right)_{\omega_{(k)}}\right| \le (1+\varepsilon)^2\|v\|_{\omega_{(k)}}\|\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_{(k)}}\end{aligned}$$

for large enough k. We apply (3.22) and Lemma 3.3 to obtain

$$\|v\|_{\omega_{(k)}}\|\mathcal{P}^{(q)}_{(k),c_k}u\|_{\omega_{(k)}} \le (1+\varepsilon)^2\|v\|_{\omega_0}\|u\|_{\omega_0} \quad \text{for large enough } k.$$

Since $\varepsilon > 0$ is arbitrary, the estimates above mean $|\left(v \,|\, \mathcal{B}_s^{(q)}u\right)_{\omega_0}| \le \|v\|_{\omega_0}\|u\|_{\omega_0}$ which implies $\|\mathcal{B}_s^{(q)}u\|_{\omega_0} \le \|u\|_{\omega_0}$ because the test function v is arbitrary. We have completed the proof by density argument. ☐

Now, we state the key theorem of this section.

Theorem 4.4. *If* $\limsup_{k\to\infty} c_k/k = 0$*, then* $\mathcal{B}_s^{(q)}$ *is a bounded linear map*

$$\mathcal{B}_s^{(q)} : L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \to \operatorname{Ker} \square_{0,s}^{(q)}$$

with its operator norm smaller than 1.

Proof. By Lemma 4.3, it remains to show: **Claim.** *If* $\limsup_{k\to\infty} c_k/k = 0$*, then* $\square_{0,s}^{(q)}\mathcal{B}_s^{(q)}u = 0$ *for all* $u \in L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$

We may assume $u \in \Omega_c^{(0,q)}(\mathbb{C}^n)$ by density argument. Fix $\rho \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ to be a cut-off function. By assumption $P^{(q)}_{(k),c_k,s} \to B_s^{(q)}$ locally uniformly in $\mathcal{C}^\infty$,

$$\|\rho\square_{0,s}^{(q)}\mathcal{B}_s^{(q)}u\|_{\omega_0} \lesssim \|\rho\square_{0,s}^{(q)}\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_0}.$$

Recall Lemma 3.1 and the fact that $\omega_{(k)} \to \omega_0$ and $\phi_{(k)} \to \phi_0$ locally uniformly in $\mathcal{C}^\infty$. We can immediately conclude that the coefficients of $\square^{(q)}_{(k),s}$ converge to those of $\square^{(q)}_{0,s}$ locally uniformly on $\mathbb{C}^n$. By this fact,

$$\begin{aligned}\|\rho\square_{0,s}^{(q)}\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_0} &\lesssim \|\rho\square_{(k),s}^{(q)}\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_0}\\ &\lesssim \|\rho\square_{(k),s}^{(q)}\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_{(k)},B(\sqrt{k})} \leq \|\square_{(k),s}^{(q)}\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_0,B(\sqrt{k})},\end{aligned}$$

where the second inequality is from (3.21). By the relations (3.8) and (3.20),

$$\square_{(k),s}^{(q)}\mathcal{P}^{(q)}_{(k),c_k,s}u(\sqrt{k}z) = k^{-1}\square_{k,s}^{(q)}\mathcal{P}^{(q)}_{k,c_k,s}\left(u(\sqrt{k}z)\right).$$

By changing the variable,

$$\|\square_{(k),s}^{(q)}\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_{(k)},B(\sqrt{k})} = k^{-1}\|k^{n/2}\square_{k,s}^{(q)}\mathcal{P}^{(q)}_{k,c_k,s}\left(u(\sqrt{k}z)\right)\|_{\omega,B(1)}.$$

Define $u_k := k^{n/2}u(\sqrt{k}z)$ which is a section supported in $U \subset M$ for large enough k. Furthermore, by changing the variable, observe that $\|u_k\|_\omega = \|u\|_{\omega_{(k)}}$ and hence

$$\begin{aligned}\|\square_{(k),s}^{(q)}\mathcal{P}^{(q)}_{(k),c_k,s}u\|_{\omega_{(k)},B(\sqrt{k})} &= k^{-1}\|\square_{k,s}^{(q)}\mathcal{P}^{(q)}_{k,c_k,s}u_k\|_{\omega,B(1)} \leq \frac{c_k}{k}\|u_k\|_\omega\\ &= \frac{c_k}{k}\|u\|_{\omega_{(k)}} \lesssim \frac{c_k}{k}\|u\|_{\omega_0}.\end{aligned}$$

Then we apply the assumption $\limsup_{k\to\infty}\frac{c_k}{k}=0$ to conclude that $\|\rho\Box^{(q)}_{0,s}\mathcal{B}^{(q)}_s u\|_{\omega_0}=0$. Since ρ is arbitrary, we have $\Box^{(q)}_{0,s}\mathcal{B}^{(q)}_s u\equiv 0$. $\square$

Next, our main objective is to demonstrate that $\mathcal{B}^{(q)}_s : L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \to \operatorname{Ker}\Box^{(q)}_{0,s}$ is an orthogonal projection. By Theorem 4.4, it remains to show the following statement (cf. [30, theorem 3.1 in section 3.1]):

Statement 4.1. $\mathcal{B}^{(q)}_s u = u$ for all $u\in\operatorname{Ker}\Box^{(q)}_{0,s}$.

Remark 4.1. If the statement holds, we are able to complete the proof of the main theorems as follows.

Under Assumption 4.1, by Theorem 4.4 and Statement 4.1, we know that the operator $\mathcal{B}^{(q)}_s$ defined in Def.4.1 must be the Bergman projection $\mathcal{B}^{(q)}_{0,s}$ in the model case. By the uniqueness of the Schwartz kernel, we have $B^{(q)}_s(z,w)\equiv B^{(q)}_{0,s}(z,w)$.

According to Corollary 3.6, we know that each subsequence of $P^{(q),s}_{(k),c_k}(z,w)$ has a subsequence that converges locally uniformly to $B^{(q),s}_0(z,w) = e^{\phi_0(z)}B^{(q)}_{0,s}(z,w)e^{-\phi_0(w)}$ in $\mathcal{C}^\infty$. This means that $P^{(q),s}_{(k),c_k}(z,w)$ converges to $B^{(q),s}_0(z,w)$ locally uniformly in $\mathcal{C}^\infty$. Finally, by applying Theorem 4.2 and the relation (3.9), we complete the proof of the main theorem.

In the case $p\notin M(q)$, Theorem 4.2 tells us that $\operatorname{Ker}\Box^{(q)}_{0,s}=\{0\}$ and therefore $\mathcal{B}^{(q)}_s$ is a zero map. As a consequence, Statement 4.1 automatically holds, and hence we have the main theorem for the case $p\notin M(q)$.

Theorem 4.5 (main theorem for $p\notin M(q)$)**.** *If $p\notin M(q)$ and $\limsup_{k\to\infty}\frac{c_k}{k}=0$, then the scaled localized spectral (or Bergman if $c_k=0$) kernel $P^{(q)}_{(k),c_k,s}(z,w)\to 0$ locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$. Also, by (3.9), $P^{(q),s}_{(k),c_k}(z,w)\to 0$ locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$.*

In the remaining sections, we pay full attention to proving Statement 4.1 in case $p\in M(q)$.

4.3. Asymptotic of the function case

The discussion in Sections 4.1 and 4.2 is mainly in the context of localized spectral and Bergman kernels with localized Kodaira Laplacian. In this section, we would like to stay in the context of $P^{(q),s}_{(k),c_k}$ and $B^{(q),s}_{(k)}$ defined in Def. 1.4 rather than the localized kernels. First, we establish some notations. Define $\mathcal{P}^{(q)}_{(k),c_k} : L^2_{\omega_{(k)},\phi_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n \otimes L^{(k)}) \to L^2_{\omega_{(k)},\phi_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n \otimes L^{(k)})$ by

$$\left(\mathcal{P}^{(q)}_{(k),c_k} u\right)(\sqrt{k}z) = \mathcal{P}^{(q)}_{k,c_k}\left(u(\sqrt{k}w)\right).$$

Denote $\mathcal{B}^{(q)}_{(k)} := \mathcal{P}^{(q)}_{(k),0}$. Then, for $\eta \otimes s^{(k)} \in \Omega^{(0,q)}_c(B(\sqrt{k}), L^{(k)})$, we have

$$\mathcal{P}^{(q)}_{(k),c_k}(\eta \otimes s^{(k)})(z) = \int_{B(\sqrt{k})} P^{(q),s}_{(k),c_k}(z,w)\eta(w)dV_{\omega_{(k)}} \otimes s^{(k)}.$$

We now treat $s^{(k)}$ as the trivial section 1 of the trivial vector bundle $\mathbb{C} \to \mathbb{C}^n$ restricted on $B(\sqrt{k})$ and define $\mathcal{P}^{(q),s}_{(k),c_k} : L^2_{\omega_{(k)},\phi_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n) \to L^2_{\omega_{(k)},\phi_{(k)}}(B(\sqrt{k}), T^{*,(0,q)}\mathbb{C}^n)$ by

$$\mathcal{P}^{(q),s}_{(k),c_k} u := \int_{B(\sqrt{k})} P^{(q),s}_{(k),c_k}(z,w)u(w)dV_{\omega_{(k)}}.$$

For the case $c_k = 0$, denote $\mathcal{B}^{(q),s}_{(k)} := \mathcal{P}^{(q),s}_{(k),0}$. Recall that Assumption 4.1 means $P^{(q),s}_{(k),c_k}(z,w) \to B^{(q),s}(z,w) := e^{\phi_0(z)} B^{(q)}_s(z,w) e^{-\phi_0(w)}$ locally uniformly in $\mathcal{C}^\infty$. Next, define $\mathcal{B}^{(q),s} : L^2_{\omega_0,\phi_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \to L^2_{\omega_0,\phi_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$ by

$$\mathcal{B}^{(q),s}u(z) := \int_{\mathbb{C}^n} B^{(q),s}(z,w)u(w)dV_{\omega_0}.$$

By Theorem 4.3, we have $\mathcal{B}^{(q),s} : L^2_{\omega_0,\phi_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \to \operatorname{Ker}\Box^{(q)}_0$ is bounded with its operator norm smaller than 1. Moreover, Statement 4.1 is equivalent to following statement:

Statement 4.2.

$$\mathcal{B}^{(q),s}u = u \quad \text{for all } u \in \operatorname{Ker}\Box^{(q)}_0, \tag{4.1}$$

In this section, we focus on the case $q = 0$, $p \in M(0)$ and prove the Statement 4.2. Note that $\lambda_i > 0$ for all $i = 1, \dots, n$ under the assumption

$p \in M(0)$. We impose the conditions that $\limsup_{k\to\infty} \frac{c_k}{k} = 0$ and

$$\exists c < 1 \,\text{such that}\ \ \liminf e^{2c\min\lambda_i k^{1/2}} c_k > 0.$$

Let $\chi \in \mathcal{C}^\infty_c(B(1),[0,1])$ be a cut-off function such that $\chi\mid_{B(\sqrt{c'})} \equiv 1$ where c' is a number with $c < c' < 1$. Define

$$\chi_k := \chi(\frac{z}{k^{1/4}}).$$

We now embark on the proof of Statement 4.2. Given $u \in \operatorname{Ker}\Box^{(0)}_0$, our strategy is to construct a sequence $u_{(k)}$ converging to u such that

$$\mathcal{P}^{(0),s}_{(k),c_k} u_{(k)} - u_{(k)} \to 0 \quad \text{and} \quad \mathcal{P}^{(0),s}_{(k),s} u_{(k)} - \mathcal{B}^{(0),s}_s u \to 0.$$

Define

$$u_{(k)} := \chi_k u$$

which is clearly satisfies that $\|u_{(k)} - u\|_{\omega_0,\phi_0} \to 0$. By Theorem 4.2, we can see that

$$\operatorname{Ker}\Box^{(0)}_0 = \operatorname{span}\{z^\alpha\}_{\alpha\in\mathbb{N}^n_0}.$$

It is enough to check Statement 4.2 holds for the basis $\{z^\alpha\}_{\alpha\in\mathbb{N}^n_0}$ and hence we assume that u is of the form z^α. Now, we are going to show that $\mathcal{P}^{(0),s}_{(k),c_k} u_{(k)} - u_{(k)} \to 0$.

Theorem 4.6. *If $u = z^\alpha$ for some $\alpha \in \mathbb{N}^n_0$, we have*

$$\|(\mathcal{P}^{(0),s}_{(k),c_k} u_{(k)} - u_{(k)})\|_{\omega_{(k)},\phi_{(k)},B(\sqrt{k})} := \|(\mathcal{P}^{(0),s}_{(k),c_k} u_{(k)} - u_{(k)})e^{-\phi_{(k)}}\|_{\omega_{(k)},B(\sqrt{k})} \to 0.$$

As for the case $c_k = 0$ for all k, it also holds under the spectral gap condition 2 (Def. 1.3).

Proof. Define $u_k(z) := k^{n/2} u_{(k)}(\sqrt{k}z)$. Observe that $u_k \in \mathcal{C}^\infty_c(B(1)) \subset \mathcal{C}^\infty_c(M)$. By rescaling, we see

$$\|(\mathcal{P}^{(0),s}_{(k),c_k} u_{(k)} - u_{(k)})e^{-\phi_{(k)}}\|_{\omega_{(k)},B(\sqrt{k})} \le \|\mathcal{P}^{(0)}_{k,c_k}(u_k \otimes s^k) - u_k \otimes s^k\|_{\omega,k\phi}.$$

By the property of spectral kernel, we have

$$\begin{aligned}
&\|\mathcal{P}^{(0)}_{k,c_k}(u_k\otimes s^k)-u_k\otimes s^k\|^2_{\omega,k\phi}\le \frac{1}{c_k}\left(\Box^{(0)}_k u_k\otimes s^k|u_k\otimes s^k\right)_{\omega,k\phi}\\
&=\frac{1}{c_k}\|(\bar\partial u_k)\otimes s^k\|^2_{\omega,k\phi}=\frac{k}{c_k}\|(\bar\partial u_{(k)})e^{-\phi_{(k)}}\|^2_{\omega_{(k)}}.
\end{aligned}$$

Recalling the setting that $\phi(z)=\phi_0(z)+O(|z|^4)$, we get

$$|e^{\phi_0(z)-\phi_{(k)}(z)}-1|\lesssim|\phi_0(z)-\phi_{(k)}(z)|\lesssim\frac{|z|^4}{k}\quad\text{for all }\ |z|\le k^{1/4}.$$

Because $\operatorname{supp}\chi_k\subset B(k^{1/4})$, we can change the metrics $e^{-\phi_{(k)}}$ and $\omega_{(k)}$ by the estimate:

$$\frac{k}{c_k}\|(\bar\partial u_{(k)})e^{-\phi_{(k)}}\|^2_{\omega_{(k)}}\lesssim\frac{k}{c_k}\|(\bar\partial u_{(k)})e^{-\phi_0}\|^2_{\omega_{(k)}}\lesssim\frac{k}{c_k}\|(\bar\partial u_{(k)})e^{-\phi_0}\|^2_{\omega_0}.$$

The last inequality is by (3.21). By direct computation,

$$\begin{aligned}
\|(\bar\partial u_{(k)})e^{-\phi_0}\|^2_{\omega_0}&\lesssim\int_{\sqrt{c'}k^{1/4}<|z|<k^{1/4}}|\bar\partial\chi_k|^2|z^\alpha|^2e^{-2\phi_0}dm\\
&\lesssim k^Ne^{-c'\cdot 2\min\lambda_i\cdot k^{1/2}},
\end{aligned}\tag{4.2}$$

where N is an integer depends on α. Hence, by the condition $\liminf e^{2c\min\lambda_i\cdot k^{1/2}}c_k>0$, we obtain

$$\frac{k}{c_k}\|(\bar\partial u_{(k)})e^{-\phi_0}\|_{\omega_0}\lesssim k^{N+1}e^{2(c-c')\min\lambda_i\cdot k^{1/2}}\to 0\quad\text{since }\ c<c'.$$

For the Bergman kernel case $c_k=0$, by the spectral gap condition 2, we repeat that

$$\begin{aligned}
&\|(\mathcal{B}^{(0),s}_{(k)}u_{(k)}-u_{(k)})e^{-\phi_{(k)}}\|^2_{\omega_{(k)},B(\sqrt{k})}\le\|\mathcal{B}^{(0)}_k(u_k\otimes s^k)-(u_k\otimes s^k)\|^2_{\omega,k\phi}\\
&\lesssim e^{2c\min\lambda_i\cdot k^{1/2}}\left(\Box^{(0)}_k u_k\otimes s^k\mid u_k\otimes s^k\right)_{\omega,k\phi}\\
&\lesssim k^{N+1}e^{2(c-c')\min\lambda_i\cdot k^{1/2}}\to 0. \qquad\square
\end{aligned}$$

Before proving the main theorem for the function case, we need another lemma about convergence. The following lemma is in the context of scaled localized spectral or Bergman kernels and is applicable to the $(0,q)$-forms cases for all $q = 0, \ldots, n$.

Lemma 4.7. *Let $u = z_q^\alpha e^{-\sum|\lambda_i||z^i|^2} d\bar{z}^I$ for some $\alpha \in \mathbb{N}_0^n$ and $I \in \mathcal{J}_{q,n}$. For any $v \in \Omega_c^{(0,q)}(\mathbb{C}^n)$,*

$$\left(v \mid \mathcal{P}^{(q)}_{(k),c_k,s}\chi_k u - \mathcal{B}^{(q)}_s u\right)_{\omega_0} \to 0 \quad \textit{as } k \to \infty.$$

Proof. Let $v \in \Omega_c^{(0,q)}(\mathbb{C}^n)$. For any fixed positive integer $n_0 \in \mathbb{N}$, observe that for each $k \in \mathbb{N}$ with $k > n_0$, we can estimate that

$$\begin{aligned}&\left|\left(v \mid \mathcal{P}^{(q)}_{(k),c_k,s}\chi_k u - \mathcal{B}^{(q)}_s u\right)_{\omega_0}\right| \\ \leq &\left|\left(v \,|\, (\mathcal{P}^{(q)}_{(k),c_k,s}\chi_k - \mathcal{B}^{(q)}_s)\chi_{n_0} u\right)_{\omega_0}\right| + \|v\|_{\omega_0}\|(\mathcal{P}^{(q)}_{(k),c_k,s}\chi_k - \mathcal{B}^{(q)}_s)(\chi_{n_0} - 1)u\|_{\omega_0}.\end{aligned}$$

Moreover, by Lemma 3.3, $\mathcal{P}^{(q)}_{(k),c_k,s}\chi_k$ are uniformly bounded linear functionals on the space $L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$. For this reason, $\mathcal{P}^{(q)}_{(k),c_k,s}\chi_k - \mathcal{B}^{(q)}_s$ are also uniformly bounded linear functionals on $L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$.

Given an arbitrary number $\varepsilon > 0$, since $\chi_{n_0} u \to u$ in $L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$ as $n_0 \to \infty$, we can fix n_0 large enough such that

$$\|v\|_{\omega_0}\|(\mathcal{P}^{(q)}_{(k),c_k,s}\chi_k - \mathcal{B}^{(q)}_s)(\chi_{n_0} - 1)u\|_{\omega_0} < \varepsilon/2 \quad \text{for all}\, k \in \mathbb{N}.$$

Furthermore, by the assumption that $P^{(q)}_{(k),c_k,s}(z,w) \to B^{(q)}_s(z,w)$ locally uniformly,

$$\left|\left(v \mid (\mathcal{P}^{(q)}_{(k),c_k,s}\chi_k - \mathcal{B}^{(q)}_s)\chi_{n_0} u\right)_{\omega_0}\right| \to 0 \quad \text{as } k \to \infty.$$

Finally, combining the above estimates, we obtain

$$\left|\left(v|\mathcal{P}^{(q)}_{(k),c_k,s}\chi_k u - \mathcal{B}^{(q)}_s u\right)_{\omega_0}\right| < \varepsilon \text{ for large enough } k. \qquad \square$$

To apply the Lemma in the context of $\mathcal{P}^{(0),s}_{(k),c_k}$ and $\mathcal{B}^{(0),s}_{(k)}$, we simply deduce the following corollary by the relation (3.9) and the fact that $\phi_{(k)} \to \phi_0$ locally uniformly.

Corollary 4.8. *Let $u = z^\alpha$ for some $\alpha \in \mathbb{N}_0^n$. For any $v \in \mathcal{C}_c^\infty(\mathbb{C}^n)$,*

$$\left(v|\mathcal{P}^{(q),s}_{(k),c_k} u_{(k)} - \mathcal{B}^{(q),s} u\right)_{\omega_0} \to 0 \textit{ as } k \to 0.$$

Now, we are able to complete Statement 4.1 in Section 4.2 for the function case when $p \in M(0)$.

Theorem 4.9. *If c_k satisfies the conditions $\limsup_{k\to\infty} \frac{c_k}{k} = 0$ and*

$$\liminf_{k\to\infty} e^{2c \min \lambda_i \cdot k^{1/2}} c_k > 0$$

for some constant $c < 1$, then

$$B^{(0),s} u = u \quad \text{for all } \ u \in \operatorname{Ker} \Box_0^{(0),s}.$$

As for the Bergman kernel case $c_k = 0$ for all k, it also holds under the spectral gap condition of suitable exponential rate (cf. Def. 1.3).

Proof. Assume that u is of the form $u = z^\alpha$. To show $B_s^{(0)} u = u$, let $v \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ and observe that

$$\begin{aligned}\left(v \mid \mathcal{B}^{(0),s} u - u\right)_{\omega_0} = \left(v \mid \mathcal{B}^{(0),s} u - \mathcal{P}^{(0),s}_{(k),c_k} u_{(k)}\right)_{\omega_0} + \left(v \mid \mathcal{P}^{(0),s}_{(k),c_k} u_{(k)} - u_{(k)}\right)_{\omega_0} \\ + \left(v \mid u_{(k)} - u\right)_{\omega_0}.\end{aligned}$$

By Theorem 4.6, Corollary 4.8 and the fact that $\omega_{(k)} \to \omega_0$ and $\phi_{(k)} \to \phi_0$ locally uniformly, the right-hand side of the equality above tends to zero. This means $B^{(0),s} u \equiv u$ because v is arbitrary. $\square$

By Remark 4.1, we obtain the main theorem for the function case when $p \in M(0)$.

Theorem 4.10 (main theorem for function case)**.** *Suppose c_k is a sequence with*

$$\limsup_{k\to\infty} \frac{c_k}{k} = 0$$

and $p \in M(0)$. If there exists a constant $c < 1$ such that $\liminf_{k\to\infty} e^{2c \min \lambda_i \cdot k^{1/2}} c_k > 0$, then

$$P^{(q),s}_{(k)}(z, w) \to \frac{\lambda_1 \cdots \lambda_n}{\pi^n} \, e^{2(\sum_{i=1}^q \lambda_i \bar{z}^i w^i + \sum_{i=q+1}^n \lambda_i z^i \bar{w}^i - \sum_{i=1}^n \lambda_i |w^i|^2)}.$$

locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$. In the Bergman kernel case $c_k = 0$, the convergence also holds under the small spectral gap condition of suitable exponential rate in U (cf. Def. 1.3*).*

Remark 4.2. The proof in this section is not valid for the $(0,q)$-forms if $q \neq 0$. The reason is that for $|z| < c' k^{1/4}$ and $u \in \operatorname{Ker} \Box_0^{(q)}$, the equation

$$\bar{\partial}_k^* u(\sqrt{k} z) = 0$$

may not be true if $q \neq 0$. In the context of localized Kodaira Laplacian, we need to adjust u from the space $\operatorname{Ker} \Box_{0,s}^{(q)}$ to the space $\operatorname{Ker} \Box_{(k),s}^{(q)}$. It is natural to orthogonally project u from $\operatorname{Ker} \Box_{0,s}^{(0)}$ into space $\operatorname{Ker} \Box_{(k),s}^{(q)}$. However, we encounter a difficulty as we lack information about the Bergman projection corresponding to $\Box_{(k),s}^{(q)}$. One potential solution is to extend the Laplacian $\Box_{(k),s}^{(q)}$ to $\Box_{(k),s}^{(q)\sim}$ defined on the whole $\mathbb{C}^n$, where the Bergman kernel with respect to the extended Laplacian $\Box_{(k),s}^{(q)\sim}$ is tractable. This is the main idea of Section 4.4 and Section 4.5.

4.4. The spectral gap of the extended Laplacian on $\mathbb{C}^n$

In this section, we will extend the localized scaled Laplacian $\Box_{(k),s}^{(q)}$ which is defined on $B(\sqrt{k})$ to the whole $\mathbb{C}^n$. The extended localized Laplacian is identical to $\Box_{(k),s}^{(q)}$ in $B(k^\epsilon)$ where ϵ will be determined later in Section 4.5. From now on, we fix a cut-off function denoted by $\chi \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ such that its support is contained within the ball $B(2)$, and is identical to 1 on the ball $B(1)$. Let us choose a number ϵ such that $0 < \epsilon < 1/6$ and define the extended metric data on $\mathbb{C}^n$ by

$$\tilde{\phi}_{(k)}(z) := \chi(\frac{z}{k^\epsilon})\phi_{(k)}(z) + \left(1 - \chi(\frac{z}{k^\epsilon})\right)\phi_0$$

and the extended Hermitian form by

$$\tilde{\omega}_{(k)}(z) := \chi(\frac{z}{k^\epsilon})\omega_{(k)}(z) + \left(1 - \chi(\frac{z}{k^\epsilon})\right)\omega_0.$$

Recall the observations (3.3) and (3.4). Since $\epsilon < 1/6$, we have the uniform convergences

$$\|\tilde{\phi}_{(k)} - \phi_0\|_{\mathcal{C}^2} \to 0 \quad \text{and} \qquad \|\tilde{\omega}_{(k)} - \omega_0\|_{\mathcal{C}^2} \to 0.$$

Denote

$$\tilde{\bar{\partial}}_{(k),s}^{(q)} : \Omega^{(0,q)}(\mathbb{C}^n) \to \Omega^{(0,q+1)}(\mathbb{C}^n); \quad \tilde{\bar{\partial}}_{(k),s}^{(q)} : \Omega^{(0,q)}(\mathbb{C}^n) \to \Omega^{(0,q-1)}(\mathbb{C}^n)$$

to be the localized Cauchy-Riemann operator and its formal adjoint which are given by

$$\tilde{\bar{\partial}}^{(q)}_{(k),s} = \bar{\partial}^{(q)} + (\bar{\partial}\tilde{\phi}_{(k)}) \wedge \cdot; \qquad \tilde{\bar{\partial}}^{*,(q)}_{(k),s} = \bar{\partial}^{*,(q)}_{\tilde{\omega}_{(k)}} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge^{*}_{\tilde{\omega}_{(k)}},$$

respectively. Denote

$$\begin{aligned}\square^{(q)\sim}_{(k),s} =&\tilde{\bar{\partial}}^{*}_{(k),s}\tilde{\bar{\partial}}_{(k),s} + \tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^{*}_{(k),s} : \mathrm{Dom}\,\tilde{\square}^{(q)\sim}_{(k),s} \subset L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)\\ &\to L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)\end{aligned}$$

as the Gaffney extension of the localized Kodaira Laplacian with respect to the Hermitian form $\tilde{\omega}_{(k)}$ and the weight function $\tilde{\phi}_{(k)}$. It follows immediately from the constructions that $\bar{\partial}_{(k),s} \equiv \tilde{\bar{\partial}}_{(k),s}$, $\bar{\partial}^{*}_{(k),s} \equiv \tilde{\bar{\partial}}^{*}_{(k),s}$ and $\square^{(q)\sim}_{(k),s} \equiv \square^{(q)}_{(k),s}$ in $B(k^{\epsilon})$. Reasonably, we call the $\square^{(q)\sim}_{(k),s}$ *extended localized Laplacian.* We suppose $\lambda_i < 0$ for all $i = 1, \ldots, q_0$; $\lambda_i > 0$ for all $i = q_0 + 1, \ldots, n$. Then there exists a constant $c > 0$ such that for all $z \in \mathbb{C}^n$,

$$\frac{\partial^2\tilde{\phi}_{(k)}}{\partial z^i\partial\bar{z}^i}(z) < -c \quad \forall\, i = 1, \ldots, q_0 \quad \text{and} \quad \frac{\partial^2\tilde{\phi}_{(k)}}{\partial z^i\partial\bar{z}^i}(z) > c \quad \forall\, i = q_0 + 1, \ldots, n. \tag{4.3}$$

The following results tell us these estimates create a uniform lower bound of the first eigenvalues of $\square^{(q)\sim}_{(k),s}$.

Lemma 4.11. *For $q \neq q_0$, there is a constant $c > 0$ such that for all $u \in \mathrm{Dom}\,\square^{(q)\sim}_{(k),s}$,*

$$\left(\square^{(q)\sim}_{(k),s}u \mid u\right)_{\tilde{\omega}_{(k)}} = \|\tilde{\bar{\partial}}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}} + \|\tilde{\bar{\partial}}^{*}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}} > c\|u\|^2_{\tilde{\omega}_{(k)}}.$$

Therefore, $\|\square^{(q)\sim}_{(k),s}u\|_{\tilde{\omega}_{(k)}} > c\|u\|_{\tilde{\omega}_{(k)}}$.

Proof. Note that

$$\begin{aligned}\|\tilde{\bar{\partial}}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}} =&\| \left(\bar{\partial} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge\right) u\|^2_{\tilde{\omega}_{(k)}} \gtrsim \| \left(\bar{\partial} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge\right) u\|^2_{\omega_0};\\ \|\tilde{\bar{\partial}}^{*}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}} =&\| \left(\bar{\partial}^{*}_{\tilde{\omega}_{(k)}} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge^{*}_{\tilde{\omega}_{(k)}}\right) u\|^2_{\tilde{\omega}_{(k)}} \gtrsim \| \left(\bar{\partial}^{*}_{\omega_0} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge^{*}_{\omega_0}\right) u\|^2_{\omega_0}.\end{aligned} \tag{4.4}$$

Let $u = f d\bar{z}^I$ for some $f \in \mathcal{C}^{\infty}_c(\mathbb{C}^n)$ and $I \in \mathcal{J}_{q,n}$. Since $q \neq q_0$, there exists $i \in \{1, \ldots, n\}$ such that at least one of the following two cases holds:

- $i \notin I$ and $\lambda_i < 0$;
- $i \in I$ and $\lambda_i > 0$.

If the first case holds,

$$\begin{aligned}
&\| \left(\bar{\partial} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge\right) u\|^2_{\omega_0} \geq \int_{\mathbb{C}^n} | \frac{\partial f}{\partial \bar{z}^i} + \frac{\partial \tilde{\phi}_{(k)}}{\partial \bar{z}^i} f \,|^2 \, dm \\
&= \int_{\mathbb{C}^n} (\frac{\partial f}{\partial \bar{z}^i} + \frac{\partial \tilde{\phi}_{(k)}}{\partial \bar{z}^i} f)(\frac{\partial \bar{f}}{\partial z^i} + \frac{\partial \bar{\tilde{\phi}}_{(k)}}{\partial z^i} \bar{f}) dm \\
&= \int_{\mathbb{C}^n} |\frac{\partial f}{\partial \bar{z}^i}|^2 + \bar{f} \frac{\partial f}{\partial \bar{z}^i} \frac{\partial \bar{\tilde{\phi}}_{(k)}}{\partial z^i} + f \frac{\partial \bar{f}}{\partial z^i} \frac{\partial \tilde{\phi}_{(k)}}{\partial \bar{z}^i} + |\frac{\partial \tilde{\phi}_{(k)}}{\partial \bar{z}^i}|^2 |f|^2 dm. \qquad (4.5)
\end{aligned}$$

By integration by part, we compute that $\int_{\mathbb{C}^n} |\frac{\partial f}{\partial \bar{z}^i}|^2 dm = \int_{\mathbb{C}^n} |\frac{\partial f}{\partial z^i}|^2 dm$ and

$$\begin{aligned}
&\int_{\mathbb{C}^n} \bar{f} \frac{\partial f}{\partial \bar{z}^i} \frac{\partial \bar{\tilde{\phi}}_{(k)}}{\partial z^i} + f \frac{\partial \bar{f}}{\partial z^i} \frac{\partial \tilde{\phi}_{(k)}}{\partial \bar{z}^i} dm \\
&= \int_{\mathbb{C}^n} -2|f|^2 \frac{\partial^2 \tilde{\phi}_{(k)}}{\partial z^i \partial \bar{z}^i} - f \frac{\partial \bar{f}}{\partial \bar{z}^i} \frac{\partial \bar{\tilde{\phi}}_{(k)}}{\partial z^i} - \bar{f} \frac{\partial f}{\partial z^i} \frac{\partial \tilde{\phi}_{(k)}}{\partial \bar{z}^i} dm.
\end{aligned}$$

Applying these two equations and $|\frac{\partial f}{\partial z^i}|^2 + |\frac{\partial \tilde{\phi}_{(k)}}{\partial \bar{z}^i}|^2 |f|^2 - 2|f||\frac{\partial f}{\partial z^i}||\frac{\partial \tilde{\phi}_{(k)}}{\partial \bar{z}^i}| \geq 0$, we have

$$\| \left(\bar{\partial} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge\right) u\|^2_{\omega_0} \geq -2 \int_{\mathbb{C}^n} |f|^2 \frac{\partial^2 \tilde{\phi}_{(k)}}{\partial z^i \partial \bar{z}^i} dm \gtrsim -\inf \left(\frac{\partial^2 \tilde{\phi}_{(k)}}{\partial z^i \partial \bar{z}^i} \right) \|f\|^2_{\tilde{\omega}_{(k)}}. \qquad (4.6)$$

On the other hand, if the second case holds,

$$\begin{aligned}
&\| \left(\bar{\partial}^*_{\omega_0} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge^*_{\omega_0}\right) u\|^2_{\omega_0} \geq \int_{\mathbb{C}^n} (-\frac{\partial f}{\partial z^i} + \frac{\partial \tilde{\phi}_{(k)}}{\partial z^i} f)(-\frac{\partial \bar{f}}{\partial \bar{z}^i} + \frac{\partial \bar{\tilde{\phi}}_{(k)}}{\partial \bar{z}^i} \bar{f}) dm \\
&= \int_{\mathbb{C}^n} |\frac{\partial f}{\partial z^i}|^2 - \bar{f} \frac{\partial f}{\partial z^i} \frac{\partial \bar{\tilde{\phi}}_{(k)}}{\partial \bar{z}^i} - f \frac{\partial \bar{f}}{\partial \bar{z}^i} \frac{\partial \tilde{\phi}_{(k)}}{\partial z^i} + |\frac{\partial \tilde{\phi}_{(k)}}{\partial z^i}|^2 |f|^2 dm.
\end{aligned}$$

By integration by part again, we have $\int_{\mathbb{C}^n} |\frac{\partial f}{\partial z^i}|^2 dm = \int_{\mathbb{C}^n} |\frac{\partial f}{\partial \bar{z}^i}|^2 dm$ and

$$\begin{aligned}
&\int_{\mathbb{C}^n} -\bar{f} \frac{\partial f}{\partial z^i} \frac{\partial \bar{\tilde{\phi}}_{(k)}}{\partial \bar{z}^i} - f \frac{\partial \bar{f}}{\partial \bar{z}^i} \frac{\partial \tilde{\phi}_{(k)}}{\partial z^i} dm \\
&= \int_{\mathbb{C}^n} 2|f|^2 \frac{\partial^2 \tilde{\phi}_{(k)}}{\partial z^i \partial \bar{z}^i} + f \frac{\partial \bar{f}}{\partial z^i} \frac{\partial \bar{\tilde{\phi}}_{(k)}}{\partial \bar{z}^i} + \bar{f} \frac{\partial f}{\partial \bar{z}^i} \frac{\partial \tilde{\phi}_{(k)}}{\partial z^i} dm.
\end{aligned}$$

Combining equations above and $|\frac{\partial f}{\partial \bar{z}^i}|^2+|\frac{\partial\tilde{\phi}_{(k)}}{\partial z^i}|^2|f|^2-2|f||\frac{\partial f}{\partial \bar{z}^i}||\frac{\partial\tilde{\phi}_{(k)}}{\partial z^i}|\geq 0$,

$$\|\left(\bar{\partial}^*_{\omega_0}+(\bar{\partial}\tilde{\phi}_{(k)})\wedge^*_{\tilde{\omega}_{(k)}}\right)u\|^2_{\omega_0}\geq 2\int_{\mathbb{C}^n}|f|^2\frac{\partial^2\tilde{\phi}_{(k)}}{\partial z^i\partial\bar{z}^i}dm\gtrsim\inf\left(\frac{\partial^2\tilde{\phi}_{(k)}}{\partial z^i\partial\bar{z}^i}\right)\|f\|^2_{\tilde{\omega}_{(k)}}. \tag{4.7}$$

By (4.4),(4.6) and (4.7), we have completed the proof for the case $u\in\Omega^{(0,q)}_c(\mathbb{C}^n)$. Next, we are able to prove the lemma by density argument. The density argument here is somehow technical and based on the *Friedrich's Lemma* (cf.[12, Chapter 7, Lemma 3.3]). For the details of approximation, readers may consult [19, Lemma 5]. □

Corollary 4.12. *For $q\neq q_0$, the extended Laplacians $\square^{(q)\sim}_{(k),k}$ is bijective and has inverses*

$$N^q_k: L^2_{\tilde{\omega}_{(k)}}(\mathbb{C}^n,T^{*,(0,q)}\mathbb{C}^n)\to\operatorname{Dom}\square^{(q)\sim}_{(k),s}$$

which is a k-uniformly bounded operator.

Proof. According to Lemma 4.11, $\square^{(q)\sim}_{(k),s}$ is injective. To show the surjectivity, we choose an arbitrary $v\in L^2_{\tilde{\omega}_{(k)}}(\mathbb{C}^n,T^{*,(0,q)}\mathbb{C}^n)$ and consider the linear functional $\mathcal{T}_v$ on Rang $\square^{(q)\sim}_{(k),s}$ given by

$$\mathcal{T}_v(\square^{(q)\sim}_{(k),s}u)=(u\mid v)_{\tilde{\omega}_{(k)}}\quad\forall u\in\operatorname{Dom}\square^{(q)\sim}_{(k),s}.$$

Lemma 4.11 implies that $\|\mathcal{T}_v\|_{\tilde{\omega}_{(k)}}\leq\frac{\|v\|_{\tilde{\omega}_{(k)}}}{c}$ for a constant c independent of v and k. By the Hahn-Banach Theorem, the functional $\mathcal{T}_v$ can be extended to a bounded linear functional on $L^2_{\tilde{\omega}_{(k)}}(\mathbb{C}^n,T^{*,(0,q)}\mathbb{C}^n)$ with the same norm. By Riesz representation theorem, there exists a representative $\tilde{v}\in L^2_{\tilde{\omega}_{(k)}}(\mathbb{C}^n,T^{*,(0,q)}\mathbb{C}^n)$ such that

$$(u\mid v)_{\tilde{\omega}_{(k)}}=\mathcal{T}_v(\square^{(q)\sim}_{(k),s}u)=\left(\square^{(q)\sim}_{(k),s}u\mid\tilde{v}\right)_{\tilde{\omega}_{(k)}}\quad\forall u\in\operatorname{Dom}\square^{(q)\sim}_{(k),s}.$$

This means $\square^{(q)\sim}_{(k),s}\tilde{v}=v$ which proves the surjectivity. Define N^q_k such that $N^q_kv=\tilde{v}$. Lemma 4.11 implies $\|N^q_k\|_{\tilde{\omega}_{(k)}}\leq C$ for a constant C independent of k. □

We have shown that when $q\neq q_0$, the extended Laplacian $\square^{(q)\sim}_{(k),s}$ has a uniform spectral gap spec $\square^{(q)\sim}_{(k),s}\subset[c,\infty)$ for a positive constant c independent of k. Next, in the case $q = q_0$, we should prove that the uniform spectral gap also holds in the sense that spec $\Box^{(q)\sim}_{(k),s} \subset \{0\} \cup [c, \infty)$. Define

$$\tilde{\mathcal{B}}^{(q)}_{(k),s} : L^2_{\tilde{\omega}_{(k)}}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n) \to \operatorname{Ker} \Box^{(q)\sim}_{(k),s} \subset L^2_{\tilde{\omega}_{(k)}}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$$

to be the Bergman projection and $\tilde{B}^{(q)}_{(k),s}$ to be the Bergman kernel. The following representation of $\tilde{B}^{(q)}_{(k),s}$ is standard.

Theorem 4.13 (Hodge decomposition)**.** *We have the expression*

$$\tilde{\mathcal{B}}^{(q_0)}_{(k),s} = \mathrm{Id} - \tilde{\bar{\partial}}^{(q_0-1)}_{(k),s} N^{q_0-1}_k \tilde{\bar{\partial}}^{*,(q_0)}_{(k),s} - \tilde{\bar{\partial}}^{*,(q_0+1)}_{(k),s} N^{q_0+1}_{(k)} \tilde{\bar{\partial}}^{(q_0)}_{(k),s} \text{ on } \Omega^{(0,q)}_c(\mathbb{C}^n). \quad (4.8)$$

Here, N^q_k is the inverse of the Laplacian $\Box^{(q)\sim}_{k,s}$ established in Corollary 4.12*.*

Proof. Note that

$$\begin{aligned}
&\Box^{(q_0)\sim}_{(k),s} \left(\mathrm{Id} - \tilde{\bar{\partial}}^{(q_0-1)}_{(k),s} N^{q_0-1}_k \tilde{\bar{\partial}}^{*,(q_0)}_{(k),s} - \tilde{\bar{\partial}}^{*,(q_0+1)}_{(k),s} N^{q_0+1}_k \tilde{\bar{\partial}}^{(q_0)}_{(k),s}\right) \\
=&\tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s} + \tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s} - \tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s} N^{q_0-1}_k \tilde{\bar{\partial}}^*_{(k),s} \\
&- \tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s} N^{q_0+1}\tilde{\bar{\partial}}_{(k),s} \\
=&\tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s} + \tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s} - \tilde{\bar{\partial}}_{(k),s}\Box^{(q_0-1)\sim}_{(k),s} N^{q_0-1}_k \tilde{\bar{\partial}}^*_{(k),s} \\
&- \tilde{\bar{\partial}}^*_{(k),s}\Box^{(q_0+1)\sim}_{(k),s} N^{q_0+1}_k \tilde{\bar{\partial}}_{(k),s} \\
=&\tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s} + \tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s} - (\tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s} + \tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s}) = 0.
\end{aligned}$$

So the right-hand side of (4.8) has its image in $\operatorname{Ker}\Box^{(q)\sim}_{(k),s}$. It remains to show that Rang $\left(\tilde{\bar{\partial}}^{(q_0-1)}_{(k),s} N^{q_0-1}\tilde{\bar{\partial}}^{*,(q_0)}_{(k),s} - \tilde{\bar{\partial}}^{*,(q_0+1)}_{(k),s} N^{q_0+1}\tilde{\bar{\partial}}^{(q_0)}_{(k),s}\right) \perp \operatorname{Ker}\Box^{(q_0)\sim}_{(k),s}$. Given $u \in \Omega^{(0,q)}_c(\mathbb{C}^n)$ and $v \in \operatorname{Ker}\Box^{(q_0)\sim}_{(k),s}$, since $\tilde{\bar{\partial}}^*_{(k),s} v = \tilde{\bar{\partial}}_{(k),s} v = 0$,

$$\begin{aligned}
&\left((\tilde{\bar{\partial}}_{(k),s} N^{q_0-1}_k \tilde{\bar{\partial}}^*_{(k),s} - \bar{\partial}^* N^{q_0+1}_k \bar{\partial})u \mid v\right)_{\tilde{\omega}_{(k)}} \\
&= \left(N^{q_0-1}_k \tilde{\bar{\partial}}_{(k),s} u \mid \tilde{\bar{\partial}}^*_{(k),s} v\right)_{\tilde{\omega}_{(k)}} + \left(N^{q_0+1}_k \tilde{\bar{\partial}}_{(k),s} u \mid \tilde{\bar{\partial}}_{(k),s} v\right)_{\tilde{\omega}_{(k)}} = 0. \qquad \Box
\end{aligned}$$

We now deduce some identities which will be frequently utilized. Compute that

$$\|\tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s} N^{q_0-1}_k \tilde{\bar{\partial}}^*_{(k),s} u\|^2_{\tilde{\omega}_{(k)}}$$

$$= \left(\tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u \mid \tilde{\bar{\partial}}^*_{(k),s}N_{k,s}^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u\right)_{\tilde{\omega}_{(k)}}$$
$$= \left(\tilde{\bar{\partial}}^*_{(k),s}\square^{(q_0-1)\sim}_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u \mid \tilde{\bar{\partial}}^*_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u\right)_{\tilde{\omega}_{(k)}}$$
$$= \left(\tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}^*_{(k),s}u \mid \tilde{\bar{\partial}}^*_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u\right)_{\tilde{\omega}_{(k)}} = 0,$$

for all $u \in \Omega_c^{(0,q_0)}(\mathbb{C}^n)$. Similarly, we can compute that $\|\tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s}N_k^{q_0+1}\tilde{\bar{\partial}}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}} = 0$ for all $u \in \Omega_c^{(0,q_0)}(\mathbb{C}^n)$. Hence, we have

$$\tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s} = 0; \quad \tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s}N_k^{q_0+1}\tilde{\bar{\partial}}_{(k),s} = 0 \text{ on } \Omega_c^{(0,q_0)}(\mathbb{C}^n). \tag{4.9}$$

Moreover, we can apply the two equations above to see that

$$\tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s} = \tilde{\bar{\partial}}^*_{(k),s};\ \tilde{\bar{\partial}}_{(k),s}\tilde{\bar{\partial}}^*_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}_{(k),s} = \tilde{\bar{\partial}}_{(k),s} \text{ on } \Omega_c^{(0,q_0)}(\mathbb{C}^n). \tag{4.10}$$

Theorem 4.14 (uniform spectral gap for $\square^{(q_0)\sim}_{(k),s}$)**.** *There exists a constant c independent of k such that*

$$\|\tilde{\mathcal{B}}^{(q_0)}_{(k),s}u - u\|^2_{\tilde{\omega}_{(k)}} \le c\left(\|\tilde{\bar{\partial}}^{*,(q_0)}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}} + \|\tilde{\bar{\partial}}^{(q_0)}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}}\right) \quad \text{on } \Omega_c^{(0,q_0)}(\mathbb{C}^n).$$

Proof. By Lemma 4.13,

$$\tilde{\mathcal{B}}^{(q_0)}_{(k),s} - I = -\tilde{\bar{\partial}}^{(q_0-1)}_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^{*,(q_0)}_{(k),s} - \tilde{\bar{\partial}}^{*,(q_0+1)}_{(k),s}N_k^{q_0+1}\tilde{\bar{\partial}}^{(q_0)}_{(k),s} \quad \text{on } \Omega_c^{(0,q_0)}(\mathbb{C}^n).$$

Given $u \in \Omega_c^{(0,q_0)}(\mathbb{C}^n)$,

$$\begin{aligned}\|\tilde{\bar{\partial}}^{(q_0-1)}_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^{*,(q_0)}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}} &= \left(N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u \mid \tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u\right)_{\tilde{\omega}_{(k)}}\\ &\le \|N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u\|_{\tilde{\omega}_{(k)}}\|\tilde{\bar{\partial}}^*_{(k),s}\tilde{\bar{\partial}}_{(k),s}N_k^{q_0-1}\tilde{\bar{\partial}}^*_{(k),s}u\|_{\tilde{\omega}_{(k)}}\\ &\lesssim \|\tilde{\bar{\partial}}^*_{(k),s}u\|^2_{\tilde{\omega}_{(k)}}.\end{aligned}$$

The last inequality is from Corollary 4.12 and (4.10). Symmetrically, we can show that

$$\|\tilde{\bar{\partial}}^{*,(q_0+1)}_{(k),s}N_k^{q_0+1}\tilde{\bar{\partial}}^{(q_0)}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}} \le c\|\tilde{\bar{\partial}}^{(q_0)}_{(k),s}u\|^2_{\tilde{\omega}_{(k)}}.$$

The two estimates above imply the theorem. $\square$

4.5. Asymptotics of the general $(0,q)$-forms cases

In this section, we adopt Assumption 4.1 in Section 4.2 and consider the case $p \in M(q)$ where $q \in \{1, \dots, n\}$. We strengthen the condition of c_k by imposing $\limsup_{k\to\infty} \frac{c_k}{k} = 0$ and

$$\exists d \in \mathbb{N} \quad \text{such that} \quad \liminf_{k\to\infty} k^d c_k > 0.$$

The goal of this section is to show Statement 4.1 in Section 4.1. By rearrangement, we let $\lambda_i < 0$ for all $i = 1, \dots, q$ and $\lambda_i > 0$ for all $i = q+1, \dots, n$ for simplicity. Recall

$$z_q^\alpha := (\bar{z}^1)^{\alpha_1} \cdots (\bar{z}^q)^{\alpha_q} (z^{q+1})^{\alpha_{q+1}} \cdots (z^n)^{\alpha_n}; \quad I := (1, \dots, q) \in \mathcal{J}_{q,n}.$$

We now adopt the settings in Section 4.4. It is important to note that in the construction of $\tilde{\omega}_{(k)}$ and $\tilde{\phi}_{(k)}$, we impose the condition that $0 < \epsilon < 1/6$. Now, we require

$$0 < \epsilon < \min\{\frac{1}{2n+1}, \frac{1}{6}\}.$$

The reason is in the proof of Theorem 4.16.

We establish the notations of cut-off functions. Recall that $\chi \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ is the cut-off function which is fixed at the beginning of Section 4.4. Choose $\rho \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ as another cut-off function such that $\overline{\operatorname{supp}\rho} \subset \{z \in \mathbb{C}; 2/7 < |z| < 1\}$ and $\rho \equiv 1$ on $\{z \in \mathbb{C}; 3/7 < |z| < 6/7\}$. Construct a sequence of cut-off functions by

$$\chi_k(z) := \chi(\frac{7z}{k^\epsilon}); \quad \tilde{\chi}_k(z) := \chi(\frac{7z}{3k^\epsilon}); \quad \rho_k(z) := \rho(\frac{z}{k^\epsilon}). \tag{4.11}$$

Observe that $\operatorname{supp}\chi_k \subset \{z \in \mathbb{C};\ |z| < (2/7)k^\epsilon\}$ and $\operatorname{supp}\tilde{\chi}_k \subset \{z \in \mathbb{C};\ |z| < (6/7)k^\epsilon\}$. Moreover, the derivatives of $\tilde{\chi}_k$ are supported in the annuli $\{z \in \mathbb{C};\ (3/7)k^\epsilon < |z| < (6/7)k^\epsilon\}$ and the support of ρ_k are in the annuli $\{z \in \mathbb{C};\ (2/7)k^\epsilon < |z| < k^\epsilon\}$. Next, we define the following convention.

Definition 4.2. For any $u \in L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$, define

$$u_{(k)} := \tilde{\chi}_k\ \tilde{\mathcal{B}}^{(q)}_{(k),s}\ \chi_k u.$$

Now, the stage is set to demonstrate Statement 4.1. The strategy is similar to Section 4.3 except for the different constructions of $u_{(k)}$ resulting in the different estimates. First, our objective is to show the convergence $u_{(k)} \to u$ in $L^2_{\tilde{\omega}}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$ as $k \to \infty$ if $u \in \operatorname{Ker} \square_0^{(q)}$.

Lemma 4.15. *Let $u = z_q^\alpha e^{-\sum_{i=1}^n |\lambda_i||z^i|^2} d\bar{z}^I$ for some $\alpha \in \mathbb{N}_0^n$. There exists a constant C such that for large enough k,*

$$|\tilde{\bar{\partial}}^{(q_0)}_{(k),s} u|_{\omega_0} + |\tilde{\bar{\partial}}^{*,(q_0)}_{(k),s} u|_{\omega_0} \leq \frac{C}{\sqrt{k}} \tag{4.12}$$

for all $|z| < k^\epsilon$.

Proof. Denote $u =: f d\bar{z}^I$ where $f = z_q^\alpha e^{-\sum_{i=1}^n |\lambda_i||z^i|^2}$. By the formulas (3.14),(3.15) and (3.18), we can write down the expression of $(\tilde{\bar{\partial}}_{(k),s} - \bar{\partial}_{(k),s})u$ and $(\tilde{\bar{\partial}}^*_{(k),s} - \bar{\partial}^*_{(k),s})u$ as

$$\begin{aligned}
(\tilde{\bar{\partial}}_{(k),s} - \bar{\partial}_{0,s})u &= \left(\bar{\partial} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge\right) u - \left(\bar{\partial} - (\bar{\partial}\phi_0)\wedge\right) u = \left(\bar{\partial}(\tilde{\phi}_{(k)} - \phi_0)\right) \wedge u; \\
(\tilde{\bar{\partial}}^*_{(k),s} - \bar{\partial}^*_{0,s})u &= \left(\bar{\partial}^*_{\tilde{\omega}_{(k)}} + (\bar{\partial}\tilde{\phi}_{(k)})\wedge^*_{\tilde{\omega}_{(k)}}\right) u - \left(\bar{\partial}^*_{\omega_0} + (\bar{\partial}\phi_0)\wedge^*_{\omega_0} \cdot\right) u \\
&= \left(-\frac{\partial f}{\partial z^i} - f\frac{\partial \tilde{\varphi}_{(k)}}{\partial z^i}\right)(d\bar{z}^i)\wedge^*_{\tilde{\omega}_{(k)}} d\bar{z}^I \\
&\quad - \left(-\frac{\partial f}{\partial z^i} - f\frac{\partial \varphi_0}{\partial z^i}\right)(d\bar{z}^i)\wedge^*_{\omega_0} d\bar{z}^I \\
&\quad - f\left((d\bar{z}^i\wedge^*_{\tilde{\omega}_{(k)}})\overline{\theta}^*_{\partial/\partial\bar{z}^i,\tilde{\omega}_{(k)}} - (d\bar{z}^i\wedge^*_{\omega_0})\overline{\theta}^*_{\partial/\partial\bar{z}^i,\omega_0}\right) d\bar{z}^I \\
&\quad + f\left((\bar{\partial}\tilde{\phi}_{(k)})\wedge^*_{\tilde{\omega}_{(k)}} - (\bar{\partial}\phi_0)\wedge^*_{\omega_0}\right) d\bar{z}^I.
\end{aligned}$$

Denote $a_1(z)$ and $a_2(z)$ as the absolute maximum of the coefficients of the differential operators $\tilde{\bar{\partial}}_{(k),s} - \bar{\partial}_{0,s}$ and $\tilde{\bar{\partial}}^*_{(k),s} - \bar{\partial}^*_{0,s}$ at a point $z \in \mathbb{C}^n$, respectively. By (3.3) and (3.4),

$$|\tilde{\phi}_{(k)} - \phi_0|_{\mathcal{C}^2}(z) \lesssim \frac{|z|^3 + 1}{\sqrt{k}}; \quad |\tilde{\omega}_{(k)} - \omega_0|_{\mathcal{C}^2}(z) \lesssim \frac{|z| + 1}{\sqrt{k}} \quad \text{for all } |z| < 2k^\epsilon.$$

The coefficients of the differential operators $\tilde{\bar{\partial}}_{(k),s} - \bar{\partial}_{0,s}$ and $\tilde{\bar{\partial}}^*_{(k),s} - \bar{\partial}^*_{0,s}$ consist of the zero and first derivatives of $\phi_0 - \tilde{\phi}_{(k)}$. Moreover, the matrix of connection forms θ and the operator $\wedge^*_\cdot$ are smoothly depend on the zero and first derivatives of components of Hermitian forms. We can see that

$$|a_i(z)| \lesssim \frac{|z|^3+1}{\sqrt{k}} \quad \forall\, |z| < 2k^\epsilon \quad \text{and} \quad |a_i(z)| = 0 \quad \forall\, |z| > 2k^\epsilon.$$

Because any derivatives of u decay exponentially as $|z|$ goes to infinity, there is a constant $c > 0$ such that

$$|(\tilde{\bar{\partial}}_{(k),s} - \tilde{\bar{\partial}}_{0,s})u(z)|_{\omega_0} \lesssim \frac{|z|^3+1}{\sqrt{k}} e^{-c|z|^2} \text{ and } |(\tilde{\bar{\partial}}^*_{(k),s} - \tilde{\bar{\partial}}^*_{0,s})u(z)|_{\omega_0} \lesssim \frac{|z|^3+1}{\sqrt{k}} e^{-c|z|^2}$$

for all $z \in \mathbb{C}^n$. Since $|z|^3 e^{-c|z|^2}$ is a bounded function, we have completed the proof. $\square$

We can apply Lemma 4.15 to establish the following theorem which claims that $u_{(k)} \to u$.

Theorem 4.16. *If $u = z_q^\alpha e^{-\sum_{i=1}^n |\lambda_i||z^i|^2} d\bar{z}^I$ for some $\alpha \in \mathbb{N}_0^n$, then*

$$\|u_{(k)} - u\|_{\tilde{\omega}_{(k)}} \to 0 \quad \text{as } k \to \infty.$$

Proof. Note that $\|u_{(k)} - u\|_{\tilde{\omega}_{(k)}} \le \|u_{(k)} - \tilde{\chi}_k u\|_{\tilde{\omega}_{(k)}} + \|\tilde{\chi}_k u - u\|_{\tilde{\omega}_{(k)}}$. Clearly, the second term tends to zero by the decreasing of u as $z \to \infty$. For the first term,

$$\begin{aligned}\|u_{(k)} - \tilde{\chi}_k u\|_{\tilde{\omega}_{(k)}} =& \|\tilde{\chi}_k(\tilde{\mathcal{B}}^{(q)}_{(k),s}\chi_k u - u)\|_{\tilde{\omega}_{(k)}} \le \|\tilde{\mathcal{B}}^{(q)}_{(k),s}\chi_k u - u\|_{\tilde{\omega}_{(k)}} \\ \le& \|\tilde{\mathcal{B}}^{(q)}_{(k),s}\chi_k u - \chi_k u\|_{\tilde{\omega}_{(k)}} + \|\chi_k u - u\|_{\tilde{\omega}_{(k)}}.\end{aligned}$$

Since the second term of the right-hand side tends to zero, we only need to estimate $\|\tilde{\mathcal{B}}^{(q)}_{(k),s}\chi_k u - \chi_k u\|_{\tilde{\omega}_{(k)}}$. By Theorem 4.14,

$$\|\tilde{\mathcal{B}}^{(q)}_{(k),s}\chi_k u - \chi_k u\|^2_{\tilde{\omega}_{(k)}} \lesssim \|\tilde{\bar{\partial}}^*_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}} + \|\tilde{\bar{\partial}}_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}}.$$

It remains to claim $\|\tilde{\bar{\partial}}^*_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}} \to 0$ and $\|\tilde{\bar{\partial}}_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}} \to 0$.

For $\|\tilde{\bar{\partial}}_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}}$, we compute that $\tilde{\bar{\partial}}_{(k),s}\chi_k u = (\bar{\partial}\chi_k) \wedge u + \chi\tilde{\bar{\partial}}_{k,s} u$ and then

$$\|\tilde{\bar{\partial}}_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}} = \int_{\{|z|<k^\epsilon/7\}} |\tilde{\bar{\partial}}_{k,s} u|^2_{\tilde{\omega}_{(k)}} dV_{\tilde{\omega}_{(k)}}$$

$$+\int_{\{k^\epsilon/7<|z|<2k^\epsilon/7\}} |(\bar\partial\chi_k)\wedge u+\chi\tilde{\bar\partial}_{k,s}u|^2_{\tilde\omega_{(k)}}dV_{\tilde\omega_{(k)}}$$
$$\lesssim \int_{\{|z|<2k^\epsilon/7\}} |\tilde{\bar\partial}_{k,s}u|^2_{\omega_0}dm+\int_{\{k^\epsilon/7<|z|<2k^\epsilon/7\}}|u|^2_{\omega_0}dm.$$

Clearly, the second term $\int_{\{k^\epsilon/7<|z|<2k^\epsilon/7\}}|u|^2_{\omega_0}dm$ tends to zero by the decreasing of u as $z\to\infty$. By Lemma 4.15 and the setting $\epsilon<1/(2n)$, the first term can be dominated by

$$\int_{\{|z|<2k^\epsilon/7\}}|\tilde{\bar\partial}_{(k),s}u|^2_{\omega_0}dm\lesssim\frac{(k^\epsilon)^{2n}}{k}\to 0.$$

We have proven that $\|\tilde{\bar\partial}_{(k),s}\chi_k u\|^2_{\tilde\omega_{(k)}}\to 0$. Next, we will show $\|\tilde{\bar\partial}^*_{(k),s}\chi_k u\|_{\tilde\omega_{(k)}}$ $\to 0$ in a similar way. Compute $\tilde{\bar\partial}^*_{(k),s}\chi_k u_k=\sum_{i=1}^n\frac{\partial\chi_k}{\partial z^i}(d\bar z^i)\wedge^*_{\tilde\omega_{(k)}}u+\chi_k\,\tilde{\bar\partial}^*_{(k),s}u$ and repeat the above process to get

$$\|\tilde{\bar\partial}^*_{(k),s}\chi_k u\|^2_{\tilde\omega_{(k)}}\lesssim\int_{\{|z|<2k^\epsilon/7\}}|\tilde{\bar\partial}^*_{(k),s}u|^2_{\omega_0}dm+\int_{\{k^\epsilon/7<|z|<2k^\epsilon/7\}}|u|^2_{\omega_0}dm.$$

The second term clearly tends to zero and the first term also tends to zero by the fact that $\epsilon<\frac{1}{2n}$ and Lemma 4.15. We also have $\|\tilde{\bar\partial}^*_{k,s}\chi_k u\|_{\tilde\omega_{(k)}}\to 0$. $\square$

In the next step, we will display $\mathcal{P}^{(q)}_{(k),c_k,s}u_{(k)}-u_{(k)}\to 0$ in $L^2_{\tilde\omega_{(k)}}(\mathbb{C}^n,T^{*,(0,q)}\mathbb{C}^n)$. To do this, we need to estimate the decreasing rate of $\|\square^{(q)}_{(k),s}u_{(k)}\|_{\omega_{(k)}}$ as (4.2) in the proof of Theorem 4.6. Since $\left(\square^{(q)}_{(k),s}u_{(k)}\right)(z)=0$ for all $|z|<(1/7)k^\epsilon$, we only need to analyze $u_{(k)}$ on the annuli $\{(2/7)k^\epsilon<|z|<k^\epsilon\}$. The following lemma tells us that $u_{(k)}$ are *small* on the annuli. Notably, the proof effectively utilizes the property that $\operatorname{supp}\rho_k\cap\operatorname{supp}\tilde\chi_k=\emptyset$.

Lemma 4.17. *Consider the functional*

$$\rho_k\tilde{\mathcal{B}}^{(q)}_{(k),s}\chi_k: L^2_{\tilde\omega_{(k)}}(\mathbb{C}^n,T^{*,(0,q)}\mathbb{C}^n)\to L^2_{\tilde\omega_{(k)}}(\mathbb{C}^n,T^{*,(0,q)}\mathbb{C}^n).$$

For any $d\in\mathbb{N}$, there exists a constant C and $n_0\in\mathbb{N}$ such that the operator norm

$$\|\rho_k\tilde{\mathcal{B}}^{(q)}_{(k),s}\chi_k\|_{\tilde\omega_{(k)}}\le\frac{C}{k^d}\quad\text{for all }\ k\ge n_0.$$

Proof. For any $u \in \Omega_c^{(0,q)}(\mathbb{C}^n)$, by Theorem 4.13,

$$\begin{aligned}\rho_k \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u &= \rho_k \left(\mathrm{Id} - \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} - \bar{\tilde{\partial}}_{(k),s} N_k^{q-1} \bar{\tilde{\partial}}^*_{(k),s} \right) \chi_k u \\ &= -\rho_k \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u - \rho_k \bar{\tilde{\partial}}_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}^*_{(k),s} \chi_k u. \end{aligned} \tag{4.13}$$

Now, we aim to estimate $\|\rho_k \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}}$. Observe that

$$\begin{aligned}&\|\rho_k \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|^2_{\tilde{\omega}_{(k)}} \\ &\quad = \left(\rho_k \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u \mid \rho_k \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u \right)_{\tilde{\omega}_{(k)}} \\ &\quad = \left(N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u \mid \bar{\tilde{\partial}}_{(k),s} \rho_k^2 \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u \right)_{\tilde{\omega}_{(k)}} \\ &\quad = \left(\tilde{\rho}_k N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u \mid \bar{\tilde{\partial}}_{(k),s} \rho_k^2 \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u \right)_{\tilde{\omega}_{(k)}} \\ &\quad \le \|\tilde{\rho}_k N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}} \|\bar{\tilde{\partial}}_{(k),s} \rho_k^2 \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}},\end{aligned}$$

where $\tilde{\rho}_k \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ is another cut-off function such that $\operatorname{supp} \tilde{\rho}_k \supset \operatorname{supp} \rho_k$ and $\operatorname{supp} \tilde{\rho}_k \cap \operatorname{supp} \chi_k = \emptyset$. By direct computation,

$$\begin{aligned}&\bar{\tilde{\partial}}_{(k),s} \rho_k^2 \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u \\ &\quad = (\bar{\partial} \rho_k^2) \wedge \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u + \rho_k^2 \bar{\tilde{\partial}}_{(k),s} \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u \\ &\quad = (\bar{\partial} \rho_k^2) \wedge \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u + \rho_k^2 \bar{\tilde{\partial}}_{(k),s} \chi_k u \\ &\quad = (\bar{\partial} \rho_k^2) \wedge \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u,\end{aligned}$$

where the second equality is from (4.10) and the third is by the fact that $\operatorname{supp} \rho_k \cap \operatorname{supp} \chi_k = \emptyset$. We apply this computation to continue the previous estimate and get

$$\begin{aligned}&\|\rho_k \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|^2_{\tilde{\omega}_{(k)}} \\ &\quad \le \|\tilde{\rho}_k N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}} \|(\bar{\partial} \rho_k^2) \wedge \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}} \\ &\quad \lesssim k^{-\epsilon} \|\tilde{\rho}_k N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}} \|\tilde{\rho}_k \bar{\tilde{\partial}}^*_{(k),s} N_k^{q+1} \bar{\tilde{\partial}}_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}},\end{aligned} \tag{4.14}$$

where the term $k^{-\epsilon}$ arises during the computation of $\bar{\partial} \rho_k$ since $\sup_{|\alpha|=1} |\partial^\alpha \rho_k| \lesssim k^{-\epsilon}$. Moreover, the sequence $\tilde{\rho}_k$ can be taken to satisfy the condition $\sup_{|\alpha|=1} |\partial^\alpha \tilde{\rho}_k| \lesssim k^{-\epsilon}$. To iterate the preceding process , we show the following claim:

Claim. *There exists* $\tilde{\tilde{\rho}}_k \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ *with* $\operatorname{supp}\tilde{\tilde{\rho}}_k \supset \operatorname{supp}\tilde{\rho}_k$ *and* $\operatorname{supp}\tilde{\tilde{\rho}}_k \cap \operatorname{supp}\chi_k = \emptyset$ *such that* $\sup_{|\alpha|=1}|\partial^\alpha\tilde{\tilde{\rho}}_k| \lesssim k^{-\epsilon}$ *and*

$$\begin{aligned}&\|\tilde{\rho}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\|\tilde{\rho}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\\ &\lesssim k^{-\epsilon}\|\tilde{\tilde{\rho}}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\|\tilde{\tilde{\rho}}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}.\end{aligned}$$

To show the claim, by Lemma 4.11, we get

$$\begin{aligned}&\|\tilde{\rho}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\\ &\lesssim \|\bar{\tilde{\partial}}_{(k),s}\tilde{\rho}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}+\|\bar{\tilde{\partial}}^*_{(k),s}\tilde{\rho}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}.\end{aligned}$$

Moreover, we compute directly that

$$\begin{aligned}\bar{\tilde{\partial}}_{(k),s}\tilde{\rho}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u &= (\bar{\partial}\tilde{\rho}_k)\wedge N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u+\tilde{\rho}_k\bar{\tilde{\partial}}_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\\ &= (\bar{\partial}\tilde{\rho}_k)\wedge N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u;\\ \bar{\tilde{\partial}}^*_{(k),s}\tilde{\rho}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u &= -\sum_{i=1}^n\frac{\partial\tilde{\rho}_k}{\partial z^i}d\bar{z}^i\wedge^*_{\tilde{\omega}_{(k)}} N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\\ &\quad+\tilde{\rho}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u.\end{aligned}$$

Substitute these equations into the estimate and then dominate $\|\tilde{\rho}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}$ by

$$\begin{aligned}&\|(\bar{\partial}\tilde{\rho}_k)\wedge N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}+\|\sum_{i=1}^n\frac{\partial\tilde{\rho}_k}{\partial z^i}d\bar{z}^i\wedge^*_{\tilde{\omega}_{(k)}} N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\\ &\quad+\|\tilde{\rho}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\\ &\lesssim k^{-\epsilon}\|\tilde{\tilde{\rho}}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}+\|\tilde{\tilde{\rho}}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\end{aligned}$$

for some $\tilde{\tilde{\rho}}_k$ as described above. So,

$$\begin{aligned}&\|\tilde{\rho}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\|\tilde{\rho}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\\ &\lesssim k^{-\epsilon}\|\tilde{\tilde{\rho}}_k N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\|\tilde{\tilde{\rho}}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\\ &\quad+\|\tilde{\tilde{\rho}}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}}.\end{aligned}$$

For the last term of the right-hand side, we replace the ρ_k by $\tilde{\tilde{\rho}}_k$ in (4.14) and get

$$\begin{aligned}&\|\tilde{\tilde{\rho}}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}}\\ \lesssim&k^{-\epsilon}\|\tilde{\tilde{\tilde{\rho}}}_kN_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\|\tilde{\tilde{\tilde{\rho}}}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}.\end{aligned}$$

Combining the above estimates, we have completed the claim. Next, by (4.14) and iterating the claim, we can conclude that for any integer $N\in\mathbb{N}$, there exists a constant C and $\tilde{\rho}_k\in\mathcal{C}^\infty_c(\mathbb{C}^n)$ with $\operatorname{supp}\tilde{\rho}_k\supset\operatorname{supp}\rho_k$ and $\operatorname{supp}\tilde{\rho}_k\cap\operatorname{supp}\chi_k=\emptyset$ such that

$$\begin{aligned}&\|\rho_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|^2_{\tilde{\omega}_{(k)}}\\ \leq&Ck^{-N}\|\tilde{\rho}_kN_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}\|\tilde{\rho}_k\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}\chi_k u\|_{\tilde{\omega}_{(k)}}.\end{aligned}$$

Finally, we need to show the following fact:

Claim. *For all $v\in\Omega^{(0,q)}_c(\mathbb{C}^n)$*

$$\|\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|_{\tilde{\omega}_{(k)}}\leq\|v\|_{\tilde{\omega}_{(k)}};\quad \|N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|_{\tilde{\omega}_{(k)}}\lesssim\|v\|_{\tilde{\omega}_{(k)}}.$$

For the first term, by (4.10), we compute that

$$\begin{aligned}\|\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|^2_{\tilde{\omega}_{(k)}}&=\Big(N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\mid\bar{\tilde{\partial}}_{(k),s}\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\Big)_{\tilde{\omega}_{(k)}}\\ &=\Big(N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\mid\bar{\tilde{\partial}}_{(k),s}v\Big)_{\tilde{\omega}_{(k)}}=\Big(\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\mid v\Big)_{\tilde{\omega}_{(k)}}\\ &\leq\|\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|_{\tilde{\omega}_{(k)}}\|v\|_{\tilde{\omega}_{(k)}}.\end{aligned}$$

We get $\|\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|_{\tilde{\omega}_{(k)}}\leq\|v\|_{\tilde{\omega}_{(k)}}$. The second term follows by Lemma 4.11 that

$$\begin{aligned}\|N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|_{\tilde{\omega}_{(k)}}&\lesssim\|\bar{\tilde{\partial}}_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|_{\tilde{\omega}_{(k)}}+\|\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|_{\tilde{\omega}_{(k)}}\\ &=\|\bar{\tilde{\partial}}^*_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}v\|_{\tilde{\omega}_{(k)}}\leq\|v\|_{\tilde{\omega}_{(k)}},\end{aligned}$$

since $\bar{\tilde{\partial}}_{(k),s}N_k^{q+1}\bar{\tilde{\partial}}_{(k),s}=0$. We completed the proof of the second claim. After combining all the above results, we know that for any integer $N\in\mathbb{N}$, there exists a constant C such that

$$\|\rho_k \tilde{\bar{\partial}}^*_{(k),s} N_k^{q+1} \tilde{\bar{\partial}}_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}} \leq C k^{-N} \|u\|_{\tilde{\omega}_{(k)}}.$$

Symmetrically, we can literally repeat the process to show the analogous statement:

$$\|\rho_k \tilde{\bar{\partial}}_{(k),s} N_k^{q+1} \tilde{\bar{\partial}}^*_{(k),s} \chi_k u\|_{\tilde{\omega}_{(k)}} \lesssim C k^{-N} \|u\|_{\tilde{\omega}_{(k)}}.$$

Then the lemma follows by (4.13) and a density argument. □

Corollary 4.18. *For any* $u \in L^2_{\omega_0}(\mathbb{C}^n, T^{*,(0,q)}\mathbb{C}^n)$ *and* $d \in \mathbb{N}$,

$$k^d \left(\|\tilde{\bar{\partial}}^*_{(k),s} u_{(k)}\|^2_{\tilde{\omega}_{(k)}} + \|\tilde{\bar{\partial}}_{(k),s} u_{(k)}\|^2_{\tilde{\omega}_{(k)}} \right) \to 0.$$

Proof. Recall the fact that $\tilde{\bar{\partial}}_{(k),s} \tilde{\mathcal{B}}^{(q)}_{(k),s} = 0$ and $\tilde{\bar{\partial}}^*_{(k),s} \tilde{\mathcal{B}}^{(q)}_{(k),s} = 0$.

$$\begin{aligned}
\tilde{\bar{\partial}}_{(k),s} u_{(k)} &= (\bar{\partial} \tilde{\chi}_k) \wedge \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u + \tilde{\chi}_k \tilde{\bar{\partial}}_{(k),s} \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u = (\bar{\partial} \tilde{\chi}_k) \wedge \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u; \\
\tilde{\bar{\partial}}^*_{(k),s} u_{(k)} &= -\sum_{i=1}^n \frac{\partial \tilde{\chi}_k}{\partial z^i} (d\bar{z}^i) \wedge^*_{\tilde{\omega}_{(k)}} \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u + \tilde{\chi}_k \, \tilde{\bar{\partial}}^*_{(k),s} \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u \\
&= -\sum_{i=1}^n \frac{\partial \tilde{\chi}_k}{\partial z^i} (d\bar{z}^i) \wedge^*_{\tilde{\omega}_{(k)}} \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u.
\end{aligned}$$

Observe that derivatives of $\tilde{\chi}_k$ are supported in the annuli $\{3k^\epsilon/7 < |z| < 6k^\epsilon/7\}$ and $\rho_k \equiv 1$ on the annuli. We can see

$$\|\tilde{\bar{\partial}}_{(k),s} u_{(k)}\|^2_{\tilde{\omega}_{(k)}} \lesssim \|\rho_k \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u\|^2_{\tilde{\omega}_{(k)}}; \quad \|\tilde{\bar{\partial}}^*_{(k),s} u_{(k)}\|^2_{\tilde{\omega}_{(k)}} \lesssim \|\rho_k \tilde{\mathcal{B}}^{(q)}_{(k),s} \chi_k u\|^2_{\tilde{\omega}_{(k)}}.$$

By Lemma 4.17, we can immediately derive the corollary. □

Now, we show the theorem claiming that $\mathcal{P}^{(q)}_{(k),c_k,s} u_{(k)} - u_{(k)} \to 0$ which is similar to Lemma 4.6 in Section 4.3.

Theorem 4.19. *If there exists* $d \in \mathbb{R}$ *such that* $\liminf_{k\to\infty} k^d c_k > 0$, *then we have*

$$\|\mathcal{P}^{(q)}_{(k),c_k,s} u_{(k)} - u_{(k)}\|_{\omega_{(k)}} \to 0 \quad \text{as } k \to \infty.$$

In the case $c_k = 0$ *for all* k, *the convergence holds under the local small spectral gap condition of polynomial rate in* U *(cf. Def. 1.2).*

Proof. Define $u_k(z) := k^{n/2}u_{(k)}(\sqrt{k}z)$ which is a section with compact support in U. Then we have

$$\|\mathcal{P}^{(q)}_{(k),c_k,s}u_{(k)} - u_{(k)}\|_{\omega_{(k)},B(\sqrt{k})} = \|\mathcal{P}^{(q)}_{k,c_k,s}u_k - u_k\|_{\omega,B(1)}.$$

By the property of spectral kernel,

$$\|\mathcal{P}^{(q)}_{k,c_k,s}u_k - u_k\|^2_{\omega,B(1)} \leq \frac{1}{c_k}\left(\Box^{(q)}_{k,s}u_k \mid u_k\right)_\omega = \frac{1}{c_k}\left(\|\bar{\partial}^*_{k,s}u_k\|^2_\omega + \|\bar{\partial}_{k,s}u_k\|^2_\omega\right).$$

Moreover, note that

$$\left(\|\bar{\partial}^*_{k,s}u_k\|^2_\omega + \|\bar{\partial}_{k,s}u_k\|^2_\omega\right) = k\left(\|\tilde{\bar{\partial}}^*_{(k),s}u_{(k)}\|^2_{\tilde{\omega}_{(k)}} + \|\tilde{\bar{\partial}}_{(k),s}u_{(k)}\|^2_{\tilde{\omega}_{(k)}}\right)$$

by the relation (3.19). Combine them and get

$$\|\mathcal{P}^{(q)}_{(k),c_k,s}u_{(k)} - u_{(k)}\|_{\omega_{(k)},B(\sqrt{k})} \leq \frac{k}{c_k}\left(\|\tilde{\bar{\partial}}^*_{(k),s}u_{(k)}\|^2_{\tilde{\omega}_{(k)}} + \|\tilde{\bar{\partial}}_{(k),s}u_{(k)}\|^2_{\tilde{\omega}_{(k)}}\right).$$

By the assumption that $\liminf_{k\to\infty} k^N c_k > 0$ for some $N \in \mathbb{N}$ and Corollary 4.18, the right-hand sides of equations above must tend to zero.

In the Bergman kernel case $c_k = 0$, we apply the spectral gap condition 1 and get

$$\begin{aligned}\|\mathcal{B}^{(q)}_{(k),s}u_{(k)} - u_{(k)}\|_{\omega_{(k)}} &= \|\mathcal{B}^{(q)}_{k,s}u_k - u_k\|^2_{\omega,B(1)} \\ &\lesssim k^d\left(\Box^{(q)}_{k,s}u_k \mid u_k\right)_\omega = k^{d+1}\left(\|\tilde{\bar{\partial}}^*_{(k),s}u_k\|^2_{\tilde{\omega}_{(k)}} + \|\tilde{\bar{\partial}}_{(k),s}u_k\|^2_{\tilde{\omega}_{(k)}}\right).\end{aligned}$$

We apply Corollary 4.18 to complete the proof. ☐

Now, we are ready to overcome Statement 4.1 for the general cases of $(0,q)$-forms.

Theorem 4.20. *If there exists $d \in \mathbb{R}$ such that $\liminf_{k\to\infty} k^d c_k > 0$,*

$$\mathcal{B}^{(q)}_s u = u \quad \text{for all } u \in \operatorname{Ker}\Box^{(q)}_{0,s}.$$

As for the Bergman kernel case $c_k = 0$, it also holds under the local small spectral gap condition of polynomial rate in U (cf. Def. 1.2).

Proof. By Theorem 4.2, we may assume that u is of the form $u = z^\alpha_q e^{-\sum|\lambda_i||z^i|^2}d\bar{z}^I$ for some $\alpha \in \mathbb{N}^n_0$ by density argument. By Lemma

3.3, Lemma 4.16 and the decreasing of u,

$$\begin{aligned}\|\mathcal{P}^{(q)}_{(k),c_k,s}(\chi_k u - u_{(k)})\|_{\omega_0} &\lesssim \|\chi_k u - u_{(k)}\|_{\omega_0} \\ &\leq \|\chi_k u - u\|_{\omega_0} + \|u - u_{(k)}\|_{\omega_0} \to 0. \qquad (4.15)\end{aligned}$$

To show $\mathcal{B}^{(q)}_s u = u$, let $v \in \Omega^{(0,q)}_c(\mathbb{C}^n)$ and observe that

$$\begin{aligned}\left(v \mid \mathcal{B}^{(q)}_s u - u\right)_{\omega_0} &= \left(v \mid \mathcal{B}^{(q)}_s u - \mathcal{P}^{(q)}_{(k),c_k,s}\chi_k u\right)_{\omega_0} + \left(v \mid \mathcal{P}^{(q)}_{(k),c_k,s}(\chi_k u - u_{(k)})\right)_{\omega_0} \\ &\quad + \left(v \mid \mathcal{P}^{(q)}_{(k),c_k,s}u_{(k)} - u_{(k)}\right)_{\omega_0} + \left(v \mid u_{(k)} - u\right)_{\omega_0}.\end{aligned}$$

By Lemma 4.7 , Theorem 4.16, Theorem 4.19 and (4.15), the right-hand side of the above equation must tend to zero. $\square$

Eventually, we are able to complete the proof of the main theorem for the case $p \in M(q)$ by Remark 4.1.

Theorem 4.21. *Suppose c_k is a sequence such that*

$$\limsup_{k\to\infty} \frac{c_k}{k} = 0.$$

If $p \in M(q)$ and $\liminf_{k\to\infty} k^d c_k > 0$ for some $d \in \mathbb{N}$, then

$$\begin{aligned}&P^{(q),s}_{(k),c_k}(z,w) \to \\ &\quad \frac{|\lambda_1 \cdots \lambda_n|}{\pi^n} e^{2(\sum_{i=1}^{q}|\lambda_i|\bar{z}^i w^i + \sum_{i=q+1}^{n}|\lambda_i| z^i \bar{w}^i - \sum_{i=1}^{n}|\lambda_i||w^i|^2)} d\bar{z}^I \otimes (\frac{\partial}{\partial \bar{w}})^I\end{aligned}$$

locally uniformly in $\mathcal{C}^\infty$ on $\mathbb{C}^n$. In the case $c_k = 0$ for all $k \in \mathbb{N}$, the convergence also holds if $\Box^{(q)}_k$ satisfies the local small spectral gap condition of polynomial rate in U (cf. Def. 1.2).

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