Integral orbits over function fields

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Over the function field of a smooth projective curve over an algebraically closed field, we investigate the set of $S$-integral elements in a forward orbit under a rational function by establishing some analogues of the classical Siegel theorem.

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1. Introduction

Let $\varphi(z) \in F(z)$ be a rational function of degree $d \geq 2$ defined over a number field $F$ whose second iterate $\varphi^2(z)$ is not a polynomial, and let $\alpha \in F$. In [7, Sec. 3.7], Silverman proves that when $F = \mathbb{Q}$ the forward orbit $\{\varphi^n(\alpha) | n \geq 0\}$ contains only finitely many rational integers, and indicates that the proof can easily be adapted to the case of an arbitrary $F$ and its $S$-integers, where $S$ is a finite set of places of $F$ including all archimedean ones. (See [5] for stronger results.) The hypothesis $\varphi^2(z) \notin F[z]$ is equivalent to that no iterate of $\varphi$ is a polynomial, and is necessary for the finiteness of the number of $S$-integral points in a forward orbit. In [2], Hsia and Silverman obtained an explicit upper bound for the number of such integral points, even in a more general context. The purpose of this paper is to study analogous problems over algebraic function fields.

Let $k$ be an algebraically closed field, and $C/k$ be a smooth projective algebraic curve of genus $g_C$ with the function field $K := k(C)$, and $\overline{K}$ be the algebraic closure of $K$. In Sec. 2, we recall a height function $h$ on the field $K(z)$ of rational functions over $K$ such that $h$ restricts to the absolute logarithmic height on $K$ via the usual inclusion $K \subset K(z)$. For any finite subset $S' \subset C'(k)$ with a curve $C'/k$ of genus $g_{C'}$, we denote by $O_{S'}$ the ring of $S'$-integers in $k(C')$, by $O^*_S$, the group of its units,
put \( U_S := \{ f \in k(C') \mid f \in \mathcal{O}_{S'}, f - 1 \in \mathcal{O}_{S'} \} \) and \( \delta_C(|S'|) := 2g_{C'} - 2 + |S'| \). We fix a finite subset \( S \subset C(k) \) throughout this paper.

When the characteristic of \( K \) is zero, many statements in Diophantine approximation over number fields hold parallelly and even with effective results. For example, some effective versions of Roth’s theorem and Schmidt’s subspace theorem were established in 12, 13. The analogy between number fields and function fields goes particularly well when the consideration in the function field case is restricted to non-isotrivial objects, i.e. those which do not “lie” in the constant subfield under any automorphism of the ambient category. We identify a non-constant rational function \( \phi(z) \in K(z) \backslash K \) with a finite morphism \( \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) over \( K \); its degree \( \deg \phi \) (respectively, its separable degree \( \deg_s \phi \), inseparable degree \( \deg_i \phi \)) is defined to be the degree (respectively, separable degree, inseparable degree) of this morphism, which is the degree of the finite extension \( K(\mathbb{P}^1)/\phi^*(K(\mathbb{P}^1)) \).

**Definition.** For any rational function \( \phi(z) \in K(z) \) with degree at least 2, we say that \( \phi \) is *isotrivial* if there exists an isomorphism \( T : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) over \( K \) such that \( T \circ \phi \circ T^{-1} \in k(z) \), i.e. the morphism \( T \circ \phi \circ T^{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) is defined over \( K \). For any subset \( A \subset \mathbb{P}^1(K) \), we say that \( A \) is *isotrivial* if there exists an isomorphism \( T : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) over \( K \) such that \( T(A) \subset \mathbb{P}^1(k) \).

We recall that an isomorphism \( T : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K) \) is a Möbius transformation, which is identified with rational functions of the form \( T(z) = \frac{az + b}{cz + d} \) with \( (a, b, c, d) \in \text{PGL}_2(K) \). To ease the notation, we will simply denote this by \( T \in \text{PGL}_2(K) \). We also recall that the cross ratio \([a, b, c, d] \) of \( a, b, c, d \in \mathbb{P}^1(K) \), with \( a, b, c \) distinct, is defined to be \( T(d) \), where \( T \in \text{PGL}_2(K) \) such that \( T(a) = 0, T(b) = \infty \) and \( T(c) = 1 \).

In the case where the characteristic of \( K \) is zero and \( \phi(z) \in K(z) \) is not isotrivial, we will not only prove the exact analog of Silverman’s result on the finiteness of the number of \( S \)-integral elements in a forward orbit, but also give an effective bound for the height of such elements. In the extreme of the other case, i.e. when \( \phi(z) \in k(z) \), when ruling out the trivial exception where a forward orbit is contained in \( \mathbb{P}^1(k) \), we will also derive similar and actually stronger results. Our proof relies on the facts that the height (measured by \( h \)) of any element in \( U_S \) is no more than \( \delta_C(|S|) \), and that the set \( U_S \backslash k \) is finite (Mason’s ABC theorem for function fields, stated as Lemma 3 in Sec. 2).

**Theorem 1.** Suppose that the characteristic of \( K \) is zero. Let \( \phi(z) \in K(z) \) with \( d := \deg \phi \geq 2 \) and \( \phi^2(z) \notin K[z] \). Denote by \( j := \begin{cases} 4, & d = 2, \\ 3, & d \geq 3. \end{cases} \)

Recall that \( \delta_C(|S|) = 2g_C - 2 + |S| \).

(i) If \( \phi(z) \in k(z) \), then \( \phi^n(f) \notin \mathcal{O}_S \) for any \( f \in K \backslash k \) and \( n > j + \log_d \max \{1, \delta_C(|S|)\} \).
(ii) If \( \phi^n(f) \in \mathcal{O}_S \) with \( f \in \mathbb{K} \) and \( n \geq j \), then
\[
h(\phi^n(f)) \leq d^n \delta_C(|S|) + c_1 h(\phi),
\]
where \( c_1 = (4d^3 + 4d^2 + 5d + 1)(\frac{d}{d-1}) \).

(iii) The set \( \{ \phi^n(f) \in \mathcal{O}_S \mid f \in \mathbb{K}, n \geq j \) and \( h(\phi^n(f)) > c_2 h(\phi) \} \) is finite, where \( c_2 = (2d^3 + 1)(\frac{d}{d-1}) \).

(iv) If \( \phi \) is not isotrivial, then for each \( f \in \mathbb{K} \) the set \( \{ \phi^n(f) \in \mathcal{O}_S \mid n \geq 1 \} \) is finite.

Diophantine approximation has a very different story when the characteristic of \( \mathbb{K} \) is positive. For example, the statement of Roth’s theorem is false in positive characteristic, and the study of integral points usually needs to put on more conditions. (See [10] for more expositions.) Thus, it would be interesting to investigate whether we have similar conclusions of Theorem 1, especially whether (iv) still holds.

Inseparability is one of the reasons which makes things different in the positive-characteristic case. When \( \mathbb{K} \) has the positive characteristic \( p \), the set \( U_S \) is no longer finite. In fact, the \( p \)th power of every element in \( U_S \) is still in \( U_S \); fortunately, Mason’s ABC theorem says that this is the only reason for the infiniteness, and the same height bound \( \delta_C(|S|) \) works for elements in \( U_S \setminus \mathbb{K}^p \). Thus we classify elements in \( U_S \) by considering for each \( f \in \mathbb{K} \setminus \mathbb{K} \) the inseparable degree \( \deg_f \), which is by definition the inseparable degree of the associated finite morphism \( f : C \to \mathbb{P}^1 \) over \( k \), or equivalently, is defined to be \( \sup\{ p^n \mid f \in \mathbb{K}^p \} \). For convenience, we declare that \( \deg_f f = +\infty \) for all \( f \in \mathbb{k} \). Another basic difference appearing in the positive-characteristic case is that there are examples of \( \phi \in \mathbb{K}(z) \) with \( \deg_z \phi = 1 \) such that \( \phi^n(z) \in \mathbb{K}[z] \) for some \( n \geq 3 \) while \( \phi^i \notin \mathbb{K}[z] \) for any \( 0 < i < n \). (cf. [6])

Thus, it is necessary to assume that \( \deg_z \phi \geq 2 \) in the following.

**Theorem 2.** Suppose that \( \mathbb{K} \) has the positive characteristic \( p \). Let \( \phi(z) \in \mathbb{K}(z) \) with \( d := \deg \phi \), \( d := \deg_{\mathbb{K}} \phi \), \( d_z := \deg_{\mathbb{K}} \phi \geq 2 \) and \( \phi^2(z) \notin \mathbb{K}[z] \). Denote by
\[
j := \begin{cases} 4, & d_z = 2, \\ 3, & d_z \geq 3. \end{cases}
\]
Recall that \( \delta_C(|S|) = 2^q_{d-1} - 2 + |S| \).

(i) If \( \phi(z) \in \mathbb{K}(z) \) and \( \phi^n(f) \in \mathcal{O}_S \) with \( f \in \mathbb{K} \setminus \mathbb{k} \), then \( n \leq j + \log_{d_z} \max\{1, \delta_C \times (|S|)\} \); in this case, we have for any \( n \geq j \) that
\[
h(\phi^n(f)) \leq d^n d^{-j} \delta_C(|S|).
\]

(ii) Assume that \( \phi \) is not isotrivial and \( \infty \) is a periodic point of \( \phi \) with exact period \( m \). Denote by \( r := \min\{3m, m + j\} \). Then the set \( \{ \phi^n(f) \in \mathcal{O}_S \mid f \in \mathbb{K} \) and \( n \geq r \} \) is finite; moreover, for any element \( \phi^n(f) \) in this set, we have
\[
h(\phi^n(f)) \leq d^n p^f \delta_C(|S|) + c_3 h(\phi),
\]
where
\[
p^f = \min\{ \deg_{\mathbb{K}(\phi^{-r}(\infty))}[x, y, z, w] : x, y, z, w \in \phi^{-r}(\infty) \} < +\infty
\]
with $\deg_{s,K(\phi^{-r}(\infty))}$ denoting the inseparable degree of elements in the field $K(\phi^{-r}(\infty))$ obtained from $K$ by adjoining the pole set of $\phi^r$, and $c_3 = [p^3(4d^{3r} + 6d^{2r} + 4d^r) + 2d^{2r} + 3d^r + 1]^{1/d-1}$.

**Remark.** A similar (in fact, easier) proof would yield the following analog of Theorem 2(ii) in the case where $K$ has characteristic zero: Assume that $\phi$ is not isotrivial and $\infty$ is a periodic point of $\phi$ with exact period $m$. Denote by $d := \deg \phi$ and $r := \min\{3m, m + j\}$, where

$$j := \begin{cases} 4, & d = 2, \\ 3, & d \geq 3. \end{cases}$$

Then the set $\{\phi^n(f) \in O_S \mid f \in K$ and $n \geq r\}$ is finite; moreover, for any element $\phi^n(f)$ in this set, we have

$$h(\phi^n(f)) \leq d^n \delta_C(|S|) + (4d^{3r} + 4d^{2r} + 5d^r + 1)\frac{d^r - 1}{d - 1} h(\phi).$$

In the results for $\phi \in k(z)$, i.e. Theorem 1(i) and Theorem 2(i), our bounds for the number of $S$-integral elements in an orbit depend only on $\deg_s \phi$. The existence of bounds of this sort may be concluded from Theorem 12(iii) and is predicted in the dynamical analogue of a conjecture of Lang (cf. [7, Conjecture 3.47]). In the context of Nevanlinna theory over the field $\mathbb{C}$ of complex numbers, Ru and Yi [4] also prove the following analog: Let $\phi \in \mathbb{C}(z)$ be a rational function of degree $d \geq 2$ with $\phi^2 \notin \mathbb{C}[z]$. Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ be a holomorphic map. If the image of $\phi^n(f)$ omits $\infty$ for some integer $n \geq 4$, then $f$ must be constant. On the other hand, Theorem 1.5 in Towsley’s paper [9] is similar to the finiteness conclusion in our Theorem 2(ii); however, his result is formulated for global function fields and relies on the Northcott property, which fails in our setting.

Toward proving our theorems, the most important intermediate step is to derive from Mason’s ABC theorem the analogy of the classical Siegel’s theorem over function fields. This step is done in Sec. 3 after the classical version is recalled. Each version of Siegel’s theorems is a statement about the preimage of $O_S$ in $K$ under a rational function $\varphi \in K(z)$. The classical version over number fields says that this preimage is a finite set, provided that $\varphi$ has at least three poles. In our function-field case, we show that the condition that $\varphi^{-1}(\infty)$ is non-isotrivial ensures the same finiteness conclusion. (Remark 13 in Sec. 3 suggests that a non-isotrivial $\varphi$ is not enough, for this preimage may contain the constant subfield $k$.) Also, in the case where $\varphi \in k(z)$ (or more generally, when $|\varphi^{-1}(\infty) \cap \mathbb{P}^1(k)| \geq 3$), we show that this preimage still has some finiteness property in the sense that it contains only finitely many non-constant elements with a fixed inseparable degree. Our Siegel’s theorem for function fields also provides a height bound for the elements in question; such a bound is still not known in the classical case for number fields.

Applying those Siegel’s theorems, we prove our main theorems in Sec. 4. We are not very sure whether the assumption in Theorem 2(ii) is indispensable. In Sec. 5,
we study the isotriviality of quadratic rational functions, and the pole set of their second iterates.

2. Preliminaries

We define a height function on the field $\bar{K}(z)$ of rational function over $\bar{K}$ such that it restricts to the canonical height function on $\bar{K}$ over $k$. Let $L$ be a finite extension of $\bar{K}$, and $C_L/k$ be a smooth projective curve with function field $L$. For each $q \in C_L(k)$, define $v_q^L : L \rightarrow \mathbb{Z} \cup \{\infty\}$ by letting $v_q^L(f)$ be the order of the rational function $f : C_L \rightarrow k$ at $q$. (By convention, $v_q^L(0) := \infty$.) For any $(\alpha_0, \ldots, \alpha_n) \in L^{n+1}$ with $n \geq 1$, we define

$$h_L(\alpha_0, \ldots, \alpha_n) := \sum_{q \in C_L(k)} -\min\{v_q^L(\alpha_0), \ldots, v_q^L(\alpha_n)\}.$$  

More generally, for any $(\alpha_0, \ldots, \alpha_n) \in \bar{K}^{n+1}$ with $n \geq 1$, we define $h(\alpha_0, \ldots, \alpha_n) := \prod_{E} h_E(\alpha_0, \ldots, \alpha_n)$, where $E$ is a finite extension of $\bar{K}$ containing all the $\alpha_i$’s. It is a standard fact that $h(\alpha_0, \ldots, \alpha_n)$ is well defined (that is, independent of choices of $E$), and depends only on the element in $\mathbb{P}^n(\bar{K})$ represented by $(\alpha_0, \ldots, \alpha_n)$. Finally, for any rational function

$$\phi = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0} \in \bar{K}(z)$$

with $a_i$’s, $b_j$’s in $\bar{K}$ and $a_n b_m \neq 0$ such that the polynomials in numerator and denominator have no common zero in $\bar{K}$, we define the height of $\phi$ by

$$h(\phi) := h(a_0, \ldots, a_n, b_0, \ldots, b_m).$$

Finally, for any $c \in \bar{K}$, we set $h(c) := h(c, 1)$; for any $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{P}^1(\bar{K}))^n \setminus \{(*, \ldots, *)\}$, we define $h(\alpha_1, \ldots, \alpha_n)$ by ignoring all the $i$th entries with $\alpha_i = \infty$. (For example, $h(1, \infty, \alpha_3) := h(\alpha_3)$ for any $\alpha_3 \in \bar{K}$.)

Our main tool in this paper is the following version of the ABC theorem for function fields. Recall that $S \subset C(k)$ a finite subset, and $U_S = \{ f \in \bar{K} | f \in \mathcal{O}_S, f - 1 \in \mathcal{O}_S^\times \}$.

**Lemma 3 (ABC theorem).** For any $f \in U_S$, we have the following height inequality

$$h(f) \leq (\deg_i f) \delta_C(|S|).$$  

Moreover, for each $q \in \mathbb{N}$ the set $U_S(q) := \{ f \in U_S | \deg_q f = q \}$ is finite.

**Proof.** By the standard ABC theorem (cf. [3, Chap. VI, Lemma 10 and Lemma 11]), $U_S(1)$ is a finite set, and any $f_1 \in U_S(1)$ satisfies $h(f_1) \leq \delta_C(|S|)$. Write any $f \in U_S$ as $f = f_1^{\deg_i f}$, where $f_1 \in \bar{K}$ and $\deg_i f_1 = 1$. Note that $f \in U_S$ implies $f_1 \in U_S(1)$ since $f - 1 = (f_1 - 1)^{\deg_i f}$, and that $h(f) = (\deg_i f) h(f_1)$. This completes the proof. □
Proposition 5.

Remark. By the sum formula for valuations of a function field, we have $U_{\mathcal{S}} \neq \emptyset \Rightarrow |S| \geq 2 \Rightarrow \delta_{C}(|S|) = 2g_{C} - 2 + |S| \geq 0$ since $g_{C} \geq 0$.

For $p \in \mathcal{O}(k)$ and $f \in k$, we let $v_{p}^{+}(f) := \max\{0, v_{p}(f)\}$, which counts the multiplicity of zero of $f$ at $p$. The following is a more general version of the ABC theorem in one-dimensional case. It is a simple consequence of the truncated second main theorem in [11].

Theorem 4. Let $a_{1}, \ldots, a_{n}$ be $n$ distinct elements in $k$. Then for any $f \in k \setminus k$, we have

$$(n - 1)h(f) \leq \sum_{\mathfrak{p} \not\mid S} \left( \sum_{i=1}^{n} \min\{\deg v_{\mathfrak{p}}(f - a_{i})\} + \min\{\deg v_{\mathfrak{p}}(f^{-1})\} \right) + \deg f \cdot \max\{0, \delta_{C}(|S|)\}.$$

Proposition 5. Let $C_{L}/k$ be a smooth projective curve with genus $g_{L}$ and its function field $L := k(C_{L})$ being a finite extension of $k$. Assume that $\{\alpha_{1}, \ldots, \alpha_{m}\}$ is a basis of $L/k$ such that all $\alpha_{i} \in L(D) := \{\alpha \in L : v_{q}^{L}(\alpha) + m_{q} \geq 0 \text{ for each } q \in C_{L}\}$ for some divisor $D = \sum_{q \in C_{L}} m_{q}q \in \text{Div}(C_{L})$. Then

$$(g_{L} - 1) \leq [L : k] \cdot (g_{C} - 1) + \deg D.$$

Consequently,

\[
\frac{1}{[L : k]} \cdot (g_{L} - 1) \leq (g_{C} - 1) + h(1, \alpha_{1}, \ldots, \alpha_{m}).
\]

Proof. The first assertion is Proposition 3.11.1 in [8], and then the second one follows from the first easily by taking

$$D = - \sum_{q \in C_{L}} \min\{0, v_{q}^{L}(\alpha_{1}), \ldots, v_{q}^{L}(\alpha_{m})\}q.$$ 

The next two results follow easily from the definition of $h$.

Proposition 6. For any $\phi, \psi \in k(z)$, we have

$$h(\phi \circ \psi) \leq h(\phi) + \deg \phi \cdot h(\psi).$$

In particular, we see that $h(\phi^{n}) \leq (\deg \phi)^{n-1} \cdot h(\phi)$.

Lemma 7. For any $T(z) := [\alpha_{1}, \alpha_{2}, \alpha_{3}, z] \in \text{PGL}_{2}(k)$, we have $h(T) = h(T^{-1}) \leq 2h(1, \alpha_{1}, \alpha_{2}, \alpha_{3})$.

Proposition 8. For any polynomial $Q(z) = \sum_{i=0}^{n} a_{i}z^{i}$ of degree $n$ with $a_{i} \in k$, we have $h(1, \alpha_{1}, \ldots, \alpha_{m}) \leq h(a_{0}, \ldots, a_{n})$, where $\alpha_{1}, \ldots, \alpha_{m}$ are zeros of $Q$.

Proof. We may assume that $a_{n} = 1$. Let $\alpha_{1}, \ldots, \alpha_{n}$ be all zeros of $Q$ listed with multiplicities, $L := k(\alpha_{1}, \ldots, \alpha_{n})$ and $C_{L}/k$ be a smooth projective curve with function field $L$. For each $q \in C_{L}(k)$, we extend $v_{q}^{L} : L \to \mathbb{Z} \cup \{\infty\}$ to $L[z]$ by $v_{q}^{L}(c_{m}z^{m} + c_{m-1}z^{m-1} + \cdots + c_{0}) := \min_{0 \leq i \leq m} v_{q}^{L}(c_{i})$. A form of the standard
Lemma 10. The Gauss lemma shows that $v_Q^L(P_1P_2) = v_Q^L(P_1) + v_Q^L(P_2)$ for any $P_1, P_2 \in L[z]$. Hence we have

$$v_Q^L(Q) = \sum_{i=1}^{n} v_Q^L(z - \alpha_i) = \sum_{i=1}^{n} \min\{0, v_Q^L(\alpha_i)\} \leq \min\{0, v_Q^L(\alpha_1), \ldots, v_Q^L(\alpha_n)\}.$$  

Now the desired conclusion follows from the definition of $h$. \qed

We recall briefly the definition and basic facts about the canonical height attached to a rational function over $K$. (See [1, 7] for details.)

**Definition.** To each $\phi \in K(z)$ with $d := \deg \phi \geq 2$, we attach a canonical height function $\hat{h}_\phi : K^1(K) \to \mathbb{R}_{\geq 0}$ by letting

$$\hat{h}_\phi(P) := \lim_{n \to \infty} \frac{1}{d^n} h(\phi^n(P))$$

for each $P \in K^1(K)$.

**Theorem 9 (Baker [1, Theorem 1.6]).** Let $\phi \in K(z)$ with $\deg \phi \geq 2$. If $\phi$ is not isotrivial, then there exists $\epsilon > 0$, depending on $K$ and $\phi$, such that the set $\{P \in K^1(K) : \hat{h}_\phi(P) \leq \epsilon\}$ is finite.

**Lemma 10.** Let $\phi \in K(z)$ with $\deg \phi \geq 2$. If $\phi$ is not isotrivial, then $\phi^m$ is not isotrivial for each natural number $m$.

**Proof.** Suppose that $\phi^m$ is isotrivial for some natural number $m$, i.e. $\varphi := T \circ \phi^m \circ T^{-1} \in k(z)$ for some $T \in \text{PGL}_2(K)$. For all natural numbers $n$, we have $h(\phi^mn(T^{-1}(P))) = h(T^{-1}(\phi^n(P))) \leq h(T^{-1}) + h(\phi^n(P))$ for all $P \in K^1(K)$ by Proposition 6. Hence for all $Q \in K^2(k)$ and all natural numbers $n$, we have $h(\phi^mn(T^{-1}(Q))) \leq h(T^{-1})$; this shows that $\hat{h}_\phi(T^{-1}(Q)) = 0$. Since $K^1(k)$ is infinite, Theorem 9 concludes that $\phi$ is isotrivial. \qed

The first two conclusions of the following lemma is the function-field analog of [7, Proposition 3.44], which treats the number field case.

**Lemma 11.** For any $\phi \in K(z)$ with $\phi^2(z) \notin K[z]$ and $\deg \phi \geq 2$, we have $|\phi^{-4}(\infty)| \geq 3$; if $\deg \phi \geq 3$, then we have $|\phi^{-3}(\infty)| \geq 3$; if $\phi(\infty) = \infty$, then we have $|\phi^{-2}(\infty)| \geq 3$.

**Proof.** The last conclusion just follows from the following chain

$$\{\infty\} \subset \phi^{-1}(\infty) \subset \phi^{-2}(\infty) \subset \phi^{-3}(\infty) \subset \cdots,$$

where each inclusion is proper by the assumption $\phi^2(z) \notin K[z]$, which implies $\phi(z) \notin K[z]$.

It is left to prove the first two conclusions. In the case where the characteristic of $K$ is zero, these conclusions can be obtained by following verbatim the proof
of \[7, \text{Proposition 3.44}\]. The only characteristic-depending ingredient in this proof is the following weak Riemann–Hurwitz formula in the zero-characteristic case

\[2 \deg \phi - 2 = \sum_{P \in \mathbb{P}^1} (\deg \phi - \#(\phi^{-1}(P)))\]
as well as the fact that each summand on the right-hand side is non-negative. In the case where the characteristic of $K$ is positive, we have the following inequality

\[2 \deg_s \phi - 2 \geq \sum_{P \in \mathbb{P}^1} (\deg_s \phi - \#(\phi^{-1}(P))),\] (2.2)

where each summand on the right-hand side is non-negative. This suffices for a proof of the first two conclusions in the positive-characteristic case. We remark that (2.2) follows from the following standard result for the separable $\psi$

\[2 \deg \psi - 2 \geq \sum_{P \in \mathbb{P}^1} (\deg \psi - \#(\psi^{-1}(P))),\]

where each summand on the right-hand side is non-negative, as well as the bijectivity of the Frobenius map.

3. Siegel’s Theorem for Rational Functions Over Function Fields

The classical Siegel’s theorem is stated as follows (cf. [7, Theorem 3.42]).

**Theorem (Siegel).** Let $F$ be a number field and $S$ be a finite set of places of $F$ containing all the archimedean ones. Let $\phi(z) \in F(z)$ with at least three distinct poles in $\mathbb{P}^1$. Then there are only finitely many $\alpha \in F$ such that $\phi(\alpha)$ is an $S$-integer.

We note that it is possible to give an explicit bound on the number of such $\alpha$; however there is still no effective bound on the height of such $\alpha$. In contrast, we will deduce an effective analog in the function field case from our main tool “ABC theorem” (i.e. Lemma 3), which does provide an explicit height bound.

For any $\varphi(z) \in K(z)$, we set

\[I_\varphi := \{ f \in K : \varphi(f) \in O_S \},\]
\[I_\varphi(q) := \{ f \in I_\varphi : \deg f = q \} \quad \text{for each } q \in \mathbb{N}.\]

Note that $I_\varphi(d) = \emptyset$ unless either $q = 1$ or $q$ is a power of the characteristic of $K$.

**Theorem 12.** Let $\varphi(z) \in K(z)$ and $d := \deg \varphi$. Suppose that $|\varphi^{-1}(\infty)| \geq 3$. Denote by $L := K(\varphi^{-1}(\infty))$ the field obtained from $K$ by adjoining the set $\varphi^{-1}(\infty)$. Then the following statements hold.

(i) If $K$ has characteristic zero, then we have for any $f \in I_\varphi$ that

\[h(f) \leq \delta_C(|S|) + (4d^2 + 4d + 5)h(\varphi);\]

moreover, there are only finitely many such $f$ with $[a_1, a_2, a_3, f] \notin k$ for some subset $\{a_1, a_2, a_3\} \subseteq \varphi^{-1}(\infty)$ with cardinality three.
(ii) If \( \varphi^{-1}(\infty) \cap \mathbb{P}^1(k) \geq 3 \), then for each \( q \in \mathbb{N} \) the set \( \mathcal{I}_\varphi(q) \) is finite, and for any \( f \in \mathcal{I}_\varphi \) we have that
\[
h(f) \leq q[L : K]_i \delta_\mathcal{C}(|S|) + q[L : K]_i (4d^2 + 4d + 3)h(\varphi),
\]
where \([L : K]_i\) denotes the inseparable degree of \( L \) over \( K \).

(iii) If \( \varphi(z) \in k(z) \) and \( |\varphi^{-1}(\infty)| \geq \delta_\mathcal{C}(|S|) + 3 \), then \( \mathcal{I}_\varphi = k \).

(iv) If \( \varphi^{-1}(\infty) \) is not isotrivial, then \( \mathcal{I}_\varphi \) is a finite set, and for any \( f \in \mathcal{I}_\varphi \) we have that
\[
h(f) \leq d_i \delta_\mathcal{C}(|S|) + [d_i(4d^2 + 6d + 4) + 2d + 3]h(\varphi),
\]
where
\[
d_i = \min \{ \deg_{i,L}[\alpha_1, \alpha_2, \alpha_3, \alpha_4] : \alpha_i \in \varphi^{-1}(\infty) \text{ for each } i \} < +\infty \quad (3.1)
\]
with \( \deg_{i,L} \) denoting the inseparable degree of elements in \( L \).

**Remark 13.** The assumption in (iv) that \( \varphi^{-1}(\infty) \) is not isotrivial guarantees that \( d_i < +\infty \). The conclusion that \( \mathcal{I}_\varphi \) is finite may fail if we only assume that \( \varphi \) is not isotrivial. For example, if \( \varphi(z) = P(z)/Q(z) \) with \( Q(z) \in k[z] \) and \( P(z) \in \mathcal{O}_S[z] \), then \( k \subset \mathcal{I}_\varphi \).

**Proof of Theorem 12.** Let \( S_L := \pi^{-1}(S) \), where \( \pi : C_L \to C \) is the morphism from the curve \( C_L/k \) with \( L = k(C_L) \). Because \( |\varphi^{-1}(\infty)| \geq 3 \), there exist some \( T \in \text{PGL}_2(L) \) and some \( \{\alpha_1, \alpha_2, \alpha_3\} \subset \varphi^{-1}(\infty) \) such that \( T(\{\alpha_1, \alpha_2, \alpha_3\}) = \{0, 1, \infty\} \).

Letting \( \psi = \varphi \circ T^{-1} \), we may write \( \psi = \frac{P(z)}{Q(z)} \) in \( L(z) \) with \( P \) and \( Q \) sharing no common zeros in \( K \) such that \( \deg P > \deg Q \) and \( Q(0) = Q(1) = 0 \). Let \( C \subset L \) be the set of all coefficients of \( P(z) \) and \( Q(z) \). Let \( R = \text{Res}(P, Q) \) be the resultant of \( P \) and \( Q \). Recall that \( d = \deg \varphi = \deg \psi \). It follows easily from the definition of the resultant that
\[
h_L(R) \leq (\deg P + \deg Q) h_L(\psi) \leq (2d - 1) h_L(\psi).
\]
Furthermore, there exist \( A, B \in \mathbb{Z}[C][z] \) such that
\[
A(z)P(z) + B(z)Q(z) = R. \quad (3.2)
\]
Indeed, \( A \) and \( B \) can be constructed explicitly (cf. [7, Sec. 2.4]).

Let \( \tilde{S}_L \) be the minimal subset of \( C_L(k) \) containing \( S_L \) such that \( C \subset \mathcal{O}_{\tilde{S}_L} \) and \( \{R, b\} \subset \mathcal{O}_{\tilde{S}_L} \), where \( b \) is the leading coefficient of \( Q \). Then \( Q^{-1}(0) \subset \mathcal{O}_{\tilde{S}_L} \) and
\[
|\tilde{S}_L| \leq |S_L| + h_L(\psi) + h_L(R) + h_L(b) \leq [L : K]|S| + (2d + 1) h_L(\psi). \quad (3.3)
\]
Since \( \{\alpha_1, \alpha_2, \alpha_3\} \subset \varphi^{-1}(\infty) \), Proposition 8 implies that
\[
h(1, \alpha_1, \alpha_2, \alpha_3) \leq h(\varphi). \quad (3.4)
\]
Since \( T(\{\alpha_1, \alpha_2, \alpha_3\}) = \{0, 1, \infty\} \), we have \( h(T^{-1}) = h(T) \leq 2h(1, \alpha_1, \alpha_2, \alpha_3) \leq 2h(\varphi) \) by Lemma 7 and from (3.4), hence Proposition 6 gives
\[
h(\psi) = h(\varphi \circ T^{-1}) \leq (2d + 1) h(\varphi). \quad (3.5)
\]
Thus (3.3) becomes

\[ |\tilde{S}_L| \leq |L : K|(|S| + (2d + 1)^2 h(\varphi)). \] (3.6)

We also have

\[ g_L - 1 \leq |L : K|((g_C - 1) + h(1, \alpha_1, \ldots, \alpha_j)) \leq |L : K|((g_C - 1) + h(\varphi)), \] (3.7)

where \( \{\alpha_1, \ldots, \alpha_j\} = \varphi^{-1}(\infty) \), and the first inequality follows from Proposition 5, while the second one is due to Proposition 8. Thus it follows from (3.6) and (3.7) that

\[ \delta_{C_L}(|\tilde{S}_L|) = 2(g_L - 1) + |\tilde{S}_L| \leq |L : K|(\delta_C(|S|) + (4d^2 + 4d + 3)h(\varphi)). \] (3.8)

Note that for any \( f \in K \), we have

\[ h(f) \leq h(T^{-1}) + h(T(f)) \leq 2h(\varphi) + h(T(f)). \] (3.9)

Now let \( f \in \mathcal{I}_\varphi \) and \( \beta := T(f) \in L \), i.e. \( \varphi(f) \in \mathcal{O}_S, \beta = [\alpha_1, \alpha_2, \alpha_3, f] \). Then \( \psi(\beta) \in \mathcal{O}_{S_L} \subseteq \mathcal{O}_{\tilde{S}_L} \). We assert that \( \beta - \gamma \in \mathcal{O}_{\tilde{S}_L} \) for every \( \gamma \in Q^{-1}(0) \). To prove this, we fix \( q \in C_L \setminus \tilde{S}_L \). First note that \( v_q(P(\beta)) \geq v_q(Q(\beta)) \); also, the fact \( \psi(\infty) = \infty \) shows \( v_q(\beta) \geq 0 \). From (3.2) and the choice of \( \tilde{S}_L \), we have

\[ 0 = v_q(R) \geq \min\{v_q(A(\beta)), v_q(B(\beta))\} + \min\{v_q(P(\beta)), v_q(Q(\beta))\} \geq v_q(Q(\beta)), \]

where the last inequality also uses the fact \( A, B \in \mathbb{Z}[C[z] \subset \mathcal{O}_{\tilde{S}_L}[z] \). This shows that \( Q(\beta)^{-1} \in \mathcal{O}_{\tilde{S}_L} \). From the fact that both \( \beta \) and the coefficients of \( Q \) are in \( \mathcal{O}_{\tilde{S}_L} \), it follows that \( Q(\beta) \in \mathcal{O}_{\tilde{S}_L}^\times \). Since \( Q^{-1}(0) \subset \mathcal{O}_{\tilde{S}_L} \), we now have

\[ Q(\beta) = \prod_{\gamma \in Q^{-1}(0)} (\beta - \gamma), \]

where the left-hand side \( Q(\beta) \) is in \( \mathcal{O}_{\tilde{S}_L} \), while each factor on the right-hand side is in \( \mathcal{O}_{\tilde{S}_L} \). This proves our assertion. Since \( 0 \in Q^{-1}(0) \), it follows that for each \( \gamma \in Q^{-1}(0) \) we have \( \tilde{S}_{L,\gamma} = \tilde{S}_L \cup \{q \in C_L(k) \mid v_q^f(\gamma) > 0\} \).

(3.10)

Because this holds for arbitrary \( \beta \in T(\mathcal{I}_\varphi) \), we have for every \( \alpha \in \varphi^{-1}(\infty) \setminus T^{-1}(0, \infty) \) that

\[ T(\mathcal{I}_\varphi) \subset T(\alpha)U_{\tilde{S}_L,T(\alpha)}. \] (3.11)

To prove (i), we recall that \( f \in \mathcal{I}_\varphi \) and put \( \alpha = T^{-1}(1) \) in (3.11). Since the characteristic of \( K \) is zero, Lemma 3 then yields \( h(T(f)) \leq \frac{1}{K} \delta_{C_L}(|\tilde{S}_L|) \). Now from (3.9) and (3.8), we get that

\[ h(f) \leq 2h(\varphi) + \frac{1}{|L : K|} \delta_{C_L}(|\tilde{S}_L|) \leq \delta_C(|S|) + (4d^2 + 4d + 5)h(\varphi) \] (3.12)
as desired. Lemma 3 also says that there are only finitely many possibilities for \( f \) such that \([\alpha_1, \alpha_2, \alpha_3, f] = T(f) \notin k\). Being chosen from the finite set \( \phi^{-1}(\infty) \), there are also only finitely many possibilities for \( \alpha_1, \alpha_2, \alpha_3 \); this completes the proof of (i).

To prove (ii), we note that since \( |\phi^{-1}(\infty) \cap \mathbb{P}^1(k)| \geq 3 \), we may assume that \( \{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{P}^1(k) \) and hence \( T \in \text{PGL}_2(k) \), which implies that \( h(f) \leq h(T(f)) \) by the first inequality in (3.9), and that \( \deg_{i,L}(T(f)) = \deg_{i,L} f = [L : k], \deg_i f \), where the last equality follows since the inseparable degree is multiplicative over the tower \( L \supset K \supset k(f) \). Then by (3.11), we see for each \( q \in \mathbb{N} \) that

\[
T(\mathcal{I}_\varphi(q)) \subset U_{\tilde{S}_{L,1}}([L : k],q),
\]

which shows that \( \mathcal{I}_\varphi(q) \) is finite by Lemma 3. Now we conclude from (3.8) that

\[
h(f) \leq h(T(f)) \leq \frac{1}{[L : k]} \deg_{i,L}(T(f)) \delta_{C_L}(|\tilde{S}_L|)
\]

\[
\leq [L : k] \deg_i \delta_C(|S|) + [L : k] \deg_i f (4d^2 + 4d + 3) h(\varphi)
\]

(3.13)

as desired.

To prove (iii), we first note that \( k \subset \mathcal{I}_\varphi \) and \( \phi^{-1}(\infty) \subset \mathbb{P}^1(k) \) since \( \varphi(z) \in k(z) \). Thus we have \( L = K, \tilde{S}_{L,1} = \tilde{S}_L = S \), and \( T \in \text{PGL}_2(k) \). Let \( n := |\phi^{-1}(\infty)| \). Assuming that \( T(\alpha_1) = \infty, T(\alpha_2) = 0 \), we see from (3.11) that \( \{T(f) - T(\alpha_2), \ldots, T(f) - T(\alpha_n)\} \subset C, i.e. v_p(T(f) - T(\alpha_i)) = 0 \) and \( v_p(T(f)^{-1}) = v_p(T(f) - T(\alpha_2)) = 0 \) for each \( p \in C(k) \setminus S \) and \( i \geq 2 \). If \( f \in \mathcal{I}_\varphi \setminus k \), then \( T(f) \notin k \) and thus Theorem 4 implies that

\[
(n - 2) \deg_i(T(f)) \leq (n - 2) h(T(f)) \leq (\deg_i(T(f))) \max\{0, \delta_C(|S|)\},
\]

whence \( n - 2 \leq \delta_C(|S|) \). This proves (iii).

To prove (iv), we suppose that \( \phi^{-1}(\infty) \) is not isotrivial. Then there exists \( \alpha_4 \in \phi^{-1}(\infty) \) such that we may assume that \([\alpha_1, \alpha_2, \alpha_3, \alpha_4] = T(\alpha_4) \notin k \) and that the integer \( d_4 \) defined by (3.1) is exactly \( \deg_{i,L} T(\alpha_4) \), which is finite. Noting that \( d_4 \geq \deg_{i,L} T(\alpha_4) \geq \min(\deg_{i,L} T(f), \deg_{i,L} \frac{T(f)}{T(\alpha_4)} \right) \) and that \( T(f) \in U_{\tilde{S}_{L,1}} \cap T(\alpha_4) \times U_{\tilde{S}_{L,T(\alpha_4)}} \) by (3.11), we conclude that

\[
T(f) \in \left( \bigcup_{j=1}^{d_4} U_{\tilde{S}_{L,1}}(j) \right) \cup T(\alpha_4) \left( \bigcup_{j=1}^{d_4} U_{\tilde{S}_{L,T(\alpha_4)}}(j) \right).
\]

(3.14)

Since \( f \in \mathcal{I}_\varphi \) is arbitrary, this shows that \( \mathcal{I}_\varphi \) is finite by Lemma 3. For the height estimate, first we note from (3.10) that \( \tilde{S}_{L,1} = \tilde{S}_L \) and \( |\tilde{S}_{L,T(\alpha_4)}| \leq |\tilde{S}_L| + h_L(T(\alpha_4)) \), hence

\[
\delta_{C_L}(|\tilde{S}_{L,1}|) = \delta_{C_L}(|\tilde{S}_L|), \quad \delta_{C_L}(|\tilde{S}_{L,T(\alpha_4)}|) \leq \delta_{C_L}(|\tilde{S}_L|) + h_L(T(\alpha_4)).
\]

(3.15)
Now (3.14) and Lemma 3 yields
\[
h(T(f)) = \frac{1}{[L:K]} h_L(T(f))
\leq \frac{1}{[L:K]} \max\{d_i \delta_{CL}(|\tilde{S}_L|), h_L(T(\alpha_4)) + d_i \delta_{CL}(|\tilde{S}_{L,T}(\alpha_4)|)\}
\leq \frac{1}{[L:K]} d_i \delta_{CL}(|\tilde{S}_L|) + (d_i + 1) \frac{1}{[L:K]} h_L(T(\alpha_4))
\leq d_i \delta_C(|S|) + d_i (4d^2 + 4d + 3) h(\varphi) + (d_i + 1) h(T(\alpha_4)),
\]
where we also use (3.15) and (3.8). Proposition 8 and (3.5) give \( h(T(\alpha_4)) \leq h(\psi) \leq (2d + 1) h(\varphi) \). Together with (3.9), we finally get
\[
h(f) \leq (2d + 3) h(\varphi) + d_i (\delta_C(|S|) + (4d^2 + 6d + 4) h(\varphi)),
\]
which proves (iv).

\section{Proof of the Main Theorems}

\subsection{The case where \( \phi \in k(z) \)}

The most important properties which allow us to obtain the conclusions depending only on degrees of \( \phi \) in the case where \( \phi \in k(z) \) are that for any \( f \in K \setminus k \), we have
\[
\deg_\tau(\phi(f)) = (\deg_\tau(\phi))(\deg_\tau f), \quad h(\phi(f)) = (\deg \phi) h(f).
\]
Noting that \( \deg_\tau \phi = \deg \phi \) and \( \deg_\tau \phi = 1 \) in the zero-characteristic case, we unify the proof of our main results in the case where \( \phi \in k(z) \) as follows.

Proof of (i) in Theorem 1 and (i) in Theorem 2. The definition for \( j \) ensures that \( |\phi^{-j}(\infty)| \geq 3 \) by Lemma 11. Letting \( f \in K \setminus k \) and \( \varphi = \phi^j \), we have \( \phi^n(f) = \varphi(\phi^{n-j}(f)) \) for any \( n \geq j \). Suppose that \( \phi^n(f) \in \mathcal{O}_S \). Since \( \deg_\tau(\phi^{n-j}(f)) = (\deg_\tau \phi)^{n-j}(\deg_\tau f) \), it follows that \( \phi^{n-j}(f) \in \mathcal{I}_\psi((\deg_\tau \phi)^{n-j}(\deg_\tau f)) \) and thus Theorem 12(ii) gives \( h(\phi^{n-j}(f)) \leq (\deg_\tau \phi)^{n-j}(\deg_\tau f) \delta_C(|S|) \). Since \( h(\phi^{n-j}(f)) = (\deg \phi)^{n-j} h(f) \) and \( h(f) \geq \deg_\tau f \) because \( f \notin k \), we get \( (\deg \phi)^{n-j} (\deg_\tau f) \leq (\deg_\tau \phi)^{n-j} \delta_C(|S|) \), whence \( (\deg_\tau \phi)^{n-j} \leq \delta(|S|) \), i.e. \( n \leq j + \log_4 \max \{1, \delta_C(|S|)\} \). This proves (i) in Theorem 1 and the first part of (i) in Theorem 2. For the second part of (i) in Theorem 2, we just note that
\[
h(\phi^n(f)) = (\deg \phi)^j h(\phi^{n-j}(f)) \leq (\deg \phi)^j (\deg_\tau \phi)^{n-j} (\deg_\tau f) \delta_C(|S|).
\]

\subsection{The case where \( K \) has characteristic zero}

Proof of (ii), (iii) and (iv) in Theorem 1. The definition for \( j \) ensures that \( |\phi^{-j}(\infty)| \geq 3 \) by Lemma 11. Letting \( f \in K \) and \( \varphi = \phi^j \), we have \( \phi^n(f) = \varphi(\phi^{n-j}(f)) \) for any \( n \geq j \). Suppose that \( \phi^n(f) \in \mathcal{O}_S \). Then we see that \( \phi^{n-j}(f) \in \mathcal{I}_\psi \) and
Proof.

Proposition in (iv).

Lemma 15.

Proposition 14.

It remains to prove Theorem 4.3.

where

Thus

\[ h(\phi^n(f)) \leq d^i h(\phi^{n-i}(f)) + h(\phi^i) \]

\[ \leq d^i \delta_C(|S|) + (4d^{i+1} + 4d^i + 5d^i + 1)h(\phi^i) \]

\[ \leq d^i \delta_C(|S|) + (4d^{i+1} + 4d^i + 5d^i + 1) \left( \frac{d^i - 1}{d - 1} \right) h(\phi), \]

where the first and the third inequalities follow from Proposition 6 and the second one is an application of Theorem 12(i). This proves (ii). Also, Theorem 12(ii) says that there are only finitely many such \( f \) with \( [a_1,a_2,a_3,\phi^{n-j}(f)] \notin K \) for some subset \( \{a_1,a_2,a_3\} \subseteq \varphi^{-1}(\infty) = \varphi^{-j}(\infty) \) with cardinality three. By Lemma 7 and Proposition 8, for all the remaining \( f \), i.e., those with \( [a_1,a_2,a_3,\phi^{n-j}(f)] \in K \) for every subset \( \{a_1,a_2,a_3\} \subseteq \varphi^{-1}(\infty) = \varphi^{-j}(\infty) \) with cardinality three, we have that

\[ h(\phi^{n-j}(f)) \leq h(T^{-1}) \leq 2h(1,a_1,a_2,a_3) \leq 2h(\phi^j) \leq 2 \left( \frac{d^i - 1}{d - 1} \right) h(\phi), \]

where \( T(z) := [a_1,a_2,a_3,z] \in \text{PGL}_2(K) \). It follows that \( h(\phi^n(f)) \leq (2d^i + 1) \left( \frac{d^i - 1}{d - 1} \right) h(\phi) \), and this proves (iii). Finally, to prove (iv), we assume that for some \( f \in K \) the set \( \{\phi^n(f) \in O_S|n \geq 1\} \) is infinite; in particular, \( f \) is not preperiodic under \( \phi \). Then (ii) implies that \( h(\phi^n(f)) \) has an infinite subsequence which is bounded by a constant independent of \( n \); thus \( \hat{h}_f(\phi) = 0 \) and hence \( \hat{h}_f(\phi^n(f)) = 0 \) for each \( n \geq 1 \). Now Theorem 9 yields that \( \phi \) is isotrivial, contradicting the assumption in (iv).

\[ \square \]

4.3. The case where \( K \) has positive characteristic \( p \)

It remains to prove Theorem 2(ii). To do this, we first derive some relationship between the isotriviality of a rational function and that of its pole set.

Proposition 14. Let \( \varphi \in K(z) \), and \( E \subset \mathbb{P}^1(k) \) with cardinality at least three. Suppose that \( \varphi^{-1}(E) \) is a subset of \( \mathbb{P}^1(k) \). Then \( \varphi \in k(z) \).

Proof. Let \( E = \{a,b,c\} \subset \mathbb{P}^1(k) \). We first note that we may assume that \( E = \{0,1,\infty\} \) as we may replace \( \varphi \) by \( T \circ \varphi \) with \( T \in \text{PGL}_2(k) \) satisfying \( T(a) = 0 \), \( T(b) = 1 \) and \( T(c) = \infty \). Because both \( \varphi^{-1}(0) \) and \( \varphi^{-1}(\infty) \) are in \( \mathbb{P}^1(k) \), we have \( \varphi(z) = f \cdot \varphi_0(z) \) where \( f \in K \) and \( \varphi_0(z) \in k(z) \). As \( \varphi^{-1}(1) \) is in \( \mathbb{P}^1(k) \), we may choose \( z_0 \in \mathbb{P}^1(k) \) such that \( \varphi(z_0) = 1 \). Consequently, we have \( 1 = f \cdot \varphi_0(z_0) \). Therefore, \( f \in k \). This shows that \( \varphi \in k(z) \).

\[ \square \]

Lemma 15. Let \( \varphi \in K(z) \) and \( E \subset \mathbb{P}^1(\bar{K}) \) with cardinality at least three. Suppose that \( E \cup \varphi^{-1}(E) \) is isotrivial. Then \( \varphi \) is isotrivial.

Proof. Since \( E \cup \varphi^{-1}(E) \) is isotrivial, there is a \( T \in \text{PGL}_2(\bar{K}) \) such that \( T(E \cup \varphi^{-1}(E)) \subset \mathbb{P}^1(k) \), and hence

\[ (T \circ \varphi \circ T^{-1})^{-1}(T(E)) = (T \circ \varphi^{-1} \circ T^{-1})(T(E)) = T(\varphi^{-1}(E)) \subset \mathbb{P}^1(k). \]
Noting that $T(E) \subset \mathbb{P}^1(k)$ with cardinality at least three, Proposition 14 implies that $T \circ \varphi \circ T^{-1} \in k(z)$ as desired.

**Proof of Theorem 2(ii).** Let

$$E := \begin{cases} 
\phi^{-2} (\infty), & m = 1, \\
\phi^{-3} (\infty), & m \geq 2,
\end{cases}$$

where we recall that $m$ is the exact period of $\infty$ under $\phi$ and

$$j := \begin{cases} 
4, & d_s = 2, \\
3, & d_s \geq 3.
\end{cases}$$

Then $|E| \geq 3$ by Lemma 11. Since $\phi$ is not isotrivial, Lemma 10 shows that $\phi^m$ is not isotrivial, then Lemma 15 implies that $\phi^{-r}(\infty) = E \cup (\phi^m)^{-1}(E)$ is not isotrivial by the definition of $r$. Letting $f \in K \setminus k$ and $\varphi = \phi^r$, we have $\phi^n(f) = \phi(\phi^{n-r}(f))$ for any $n \geq r$. Suppose that $\phi^n(f) \in \mathcal{O}_S$. Then we see that $\phi^{n-r}(f) \in \mathcal{I}_S$. Since $\mathcal{I}_S$ is finite by Theorem 12(iv), this shows that the set $\{\phi^n(f) \in \mathcal{O}_S \mid f \in K \text{ and } n \geq r\}$ is finite. For any element $\phi^n(f)$ in this set, Theorem 12(iv) gives $h(\phi^{n-r}(f)) \leq \delta_C(|S|) + [d_r(4d^2 + 6d^2 + 4) + 2d^2 + 3]\|\phi^r\|$, where we note that the definition of $d_r$ in Theorem 12(iv) is exactly the same as that of $p^f$ in Theorem 2(ii); hence we conclude that

$$h(\phi^n(f)) \leq p^f h(\phi^{n-r}(f)) + h(\phi^r) \leq d_r p^f \delta_C(|S|) + [p^f(4d^3 + 6d^2 + 4d^2) + 2d^2 + 3d^2 + 1]h(\phi^r) \leq d_r p^f \delta_C(|S|) + [p^f(4d^3 + 6d^2 + 4d^2) + 2d^2 + 3d^2 + 1] \left( \frac{d^r - 1}{d - 1} \right) h(\phi).$$

\[ \square \]

**5. Some Remarks About Isotriviality of Rational Functions**

For simplicity of notation, we denote by

$$\phi^T := T \circ \phi \circ T^{-1}$$

for any $T \in \text{PGL}_2(K)$ and $\phi \in K(z)$. We give the following algorithms, which check whether a rational function $\phi \in K(z)$ is isotrivial.

**Proposition 16.** Let $\phi \in K(z)$.

(i) Assume that $\phi$ has at least two distinct fixed points. Take $T_1 \in \text{PGL}_2(K)$ such that $\phi^{T_1}(0) = 0$ and $\phi^{T_1}(\infty) = \infty$. Then $\phi$ is isotrivial if and only if there exists $c \in K^*$ such that $c \cdot \phi^{T_1}(c^{-1}z)$ is in $k(z)$.

(ii) Assume that $\phi$ has at least three distinct fixed points. Take $T_2 \in \text{PGL}_2(K)$ such that $0, 1$ and $\infty$ are fixed points of $\phi^{T_2}$. Then $\phi$ is isotrivial if and only if $\phi^{T_2}(z)$ is in $k(z)$. 
(iii) Assume that preimage of some fixed point of $\phi$ contains at least two distinct points. Take $T_3 \in \text{PGL}_2(\mathbf{K})$ such that $\phi^{T_3}(0) = \phi^{T_3}(\infty) = \infty$. Then $\phi$ is isotrivial if and only if there exists $c \in \bar{\mathbf{K}}^*$ such that $c \cdot \phi^{T_3}(c^{-1}z)$ is in $k(z)$.

(iv) Assume that preimage of some fixed point of $\phi$ contains at least three distinct points. Take $T_4 \in \text{PGL}_2(\mathbf{K})$ such that $\phi^{T_4}(0) = \phi^{T_4}(1) = \phi^{T_4}(\infty) = \infty$. Then $\phi$ is isotrivial if and only if $\phi^{T_4}(z)$ is in $k(z)$.

Remark. (i) For any $T \in \text{PGL}_2(\mathbf{K})$, $\phi \in \mathcal{K}(z)$, and $\alpha \in \mathbb{P}^1(\bar{\mathbf{K}})$, we have $\phi^T((T \times (\alpha)) = T(\phi(\alpha))$. Also, for any $E \subset \mathbb{P}^1(\bar{\mathbf{K}})$ with its cardinality at most 3, there always exists some $T \in \text{PGL}_2(\mathbf{K})$ such that $T(E) \subset \{0, 1, \infty\}$. This justifies the existence of $T_i, i \in \{1, 2, 3, 4\}$, in the statement.

(ii) The assumption in the statement (i) always holds except the case where $\phi^T(z) = z + b$ for some $T \in \text{PGL}_2(\mathbf{K})$ and $b \neq 0$.

(iii) An interesting case where the assumption in the statement (iii) holds is when $\phi$ is not conjugate to a polynomial.

Proof of Proposition 16. For each statement, we only have to prove the “only if” part. Hence, let us suppose that $\phi$ is isotrivial, i.e. $A \circ \phi^{T_i} \circ A^{-1}(z) := \psi(z) \in k(z)$ for some $A \in \text{PGL}_2(\mathbf{K})$, where $i \in \{1, 2, 3, 4\}$. Note that all the fixed points of $\psi$ are in $\mathbb{P}^1(k)$, and that $A$ gives a bijection from the fixed points of $\phi^{T_i}$ to those of $\psi$.

Under the assumptions of either (i) or (ii), both $A(0)$ and $A(\infty)$ are fixed points of $\psi$, hence they are in $\mathbb{P}^1(k)$. We choose $B \in \text{PGL}_2(k)$ such that $B(A(0)) = 0$ and $B(A(\infty)) = \infty$. Hence there exists $c \in \bar{\mathbf{K}}^*$ such that $B \circ A(z) = cz$. Then we see that $(\phi^{T_i})^{B \circ A}(z) = \psi^B(z) \in k(z)$, where $i \in \{1, 2\}$. This proves (i). For (ii), we note that $c$ is a fixed point of $(\phi^{T_i})^{B \circ A}$ and hence lies in $k^*$, i.e. $B \circ A(z) \in \text{PGL}_2(k)$. This shows $\phi^{T_i}(z) \in k(z)$ as desired.

On the other hand, the assumptions of either (iii) or (iv) imply $A(\infty) \in \mathbb{P}^1(k)$, and the argument in the last paragraph yields $B \in \text{PGL}_2(k)$ such that $B(A(\infty)) = \infty$ and $(\phi^{T_i})^{B \circ A}(z) \in k(z)$, where $i \in \{3, 4\}$. In particular, $B \circ A(z) = cz + \alpha$ for some $c, \alpha \in \bar{\mathbf{K}}$. Note that $(\phi^{T_i})^{B \circ A}(\alpha) = B \circ A \circ \phi^{T_i}(0) = \infty$, which shows that $\alpha \in \mathbb{P}^1(k)$. Taking $D(z) = z - \alpha \in \text{PGL}_2(k)$, we see that $c \cdot \phi^{T_i}(c^{-1}z) = [(\phi^{T_i})^{B \circ A}]^D \times (z) \in k(z)$. This proves (iii). For (iv), we observe that $[(\phi^{T_4})^{B \circ A}]^D(c) = \phi^{T_4}(1) = \infty$, whence $c \in k$. This shows $\phi^{T_4}(z) \in k(z)$ as desired. \hfill \square

Based on the Normal Forms Lemma for rational functions with degree 2 (cf. [7, Lemma 4.59]), we classify the isotriviality of these functions.

Proposition 17. Suppose that the characteristic of $\mathbf{K}$ is not 2. Let $\phi \in \mathcal{K}(z)$ with deg $\phi = 2$, and $\lambda_1, \lambda_2, \lambda_3$ be the multipliers of its fixed points.

(i) If $\lambda_1 \lambda_2 \neq 0$, then there is a $T \in \text{PGL}_2(\mathbf{K})$ such that

$$\phi^T(z) = \frac{z^2 + \lambda_1 z}{\lambda_2 z + 1}.$$ 

In this case, $\phi$ is isotrivial if and only if both $\lambda_1$ and $\lambda_2$ are in $k$. 


(ii) If $\lambda_1 \lambda_2 = 1$, then there is a $T \in \text{PGL}_2(\mathbf{k})$ such that

$$\phi^T(z) = z + \frac{1}{z} + \sqrt{1 - \lambda_3}. $$

In this case, $\phi$ is isotrivial if and only if $\lambda_3$ is in $\mathbf{k}$.

**Proof.** For both parts, the existence of $T$ follows from the Normal Forms Lemma, and it remains to prove the necessary condition for the isotriviality. Suppose that $\phi$ is isotrivial. In case (i), Proposition 16(i) implies that there is a $c \in \bar{\mathbf{k}}^*$ such that

$$c \cdot \phi^T(c^{-1}z) = \frac{z^2 + c \lambda_1 z}{\lambda_2 z} + c := \psi(z) \in \mathbf{k}(z).$$

Hence $\lambda_1, \lambda_2 \in \mathbf{k}$. In case (ii), Proposition 16(iii) implies that there is a $c \in \bar{\mathbf{k}}^*$ such that

$$c \cdot \phi^T(c^{-1}z) = z + \frac{c^2}{z} + c\sqrt{1 - \lambda_3} \in \mathbf{k}(z).$$

Hence, $c$ and $\lambda_3$ are in $\mathbf{k}$. □

To deduce Theorem 2(ii) via Siegel’s theorem, our task is to find some condition under which the pole set of some iterate of $\phi$ is not isotrivial. The next two results give further investigation in the case where $\deg \phi = 2$.

**Proposition 18.** Suppose that the characteristic of $\mathbf{K}$ is not 2. Let $\phi(z) = z + a + \frac{b}{z} \in \mathbf{K}(z)$, where $a \in \mathbf{K}$. Then the following are equivalent:

(i) $a$ is in $\mathbf{k}$.

(ii) $\phi$ is isotrivial.

(iii) $\phi^{-2}(\infty)$ is isotrivial.

**Proof.** The equivalence of (i) and (ii) is by Proposition 17. We only need to show (iii) implies (i). We first compute that $\phi^{-2}(\infty) = \{0, \infty, \beta_1, \beta_2\}$, where $\beta_1$ and $\beta_2$ are roots of $z^2 + az + 1 = 0$ in $\mathbf{K}$. Let $T(z) := z/\beta_1$. Suppose that $\phi^{-2}(\infty)$ is isotrivial. Then $\frac{\beta_1}{\beta_2} = T(\beta_2) \in \mathbf{k}$ since $T(\{0, \infty, \beta_1\}) \subset \mathbb{P}^1(\mathbf{k})$. As $\beta_1 \beta_2 = 1$, we have

$$\frac{\beta_2}{\beta_1} = \frac{\beta_2}{\beta_2} = -1 - a \beta_2.$$ 

Then $a \beta_2$ is in $\mathbf{k}$. By symmetry, $a \beta_1$ is also in $\mathbf{k}$. Therefore, we obtain $a \in \mathbf{k}$. □

**Proposition 19.** Suppose that the characteristic of $\mathbf{K}$ is $p \neq 2 > 0$. Let $\phi(z) = \frac{z^2 + abz}{az + 1}$ where $a, b \in \mathbf{K}$ and $ab \neq 1$. Let $\triangle := (1 + a)^2 - \frac{1}{b}$ and $R := \sqrt[p]{1 - ab} \neq 0$. Then $\frac{\beta_2}{\beta_1}$ is in $\mathbb{P}^1(k)$ if and only if $\phi^{-2}(\infty)$ is isotrivial.

**Proof.** We first note that $\phi^{-2}(\infty) = \{\infty, -\frac{1}{b}, \alpha_1, \alpha_2\}$, where $\alpha_1$ and $\alpha_2$ are zeros of

$$A(z) := z^2 + (1 + a)z + \frac{1}{b}.$$
Furthermore, since the resultant of $A(z)$ and $z + \frac{1}{b}$ equals

$$
\left( -\frac{1}{b} - \alpha_1 \right) \left( -\frac{1}{b} - \alpha_2 \right) = \frac{1}{b^2} (1 - ab) = R \neq 0,
$$

the rational function $T(z) = \frac{z + \frac{1}{b}}{z - \alpha_1}$ is indeed in $\text{PGL}_2(K)$. As $T(\{\infty, -\frac{1}{b}, \alpha_1\}) = \{1, 0, \infty\} \subset \mathbb{P}^2(k)$, we see that $\phi^{-2}(\infty)$ is isotrivial if and only if $T(\alpha_2)$ is in $k$. Then the desired conclusion follows because

$$
T(\alpha_2) = \frac{\alpha_2 + \frac{1}{b}}{\alpha_2 - \alpha_1} = \frac{\alpha_1 + \frac{1}{b}}{\alpha_2 - \alpha_1} + 1,
$$

and

$$
(T(\alpha_2) - 1)T(\alpha_2) = \frac{\alpha_1 + \frac{1}{b}}{\alpha_2 - \alpha_1} \cdot \frac{\alpha_2 + \frac{1}{b}}{\alpha_2 - \alpha_1} = R, \quad \square
$$

In Theorem 2(ii), the positive integer $r$ is chosen so that we can show that $\phi^{-r}(\infty)$ is not isotrivial under the assumption that $\phi$ is not isotrivial; in the case where $\phi(\infty) = \infty$, our choice is $r = 3$. The following example shows that this choice is best possible.

**Example.** Let $t \in K \setminus k$, $a = -\frac{a_2 + a_3 + 3}{t^2 + 2}$, and $b = \frac{t + 2}{t}$. Take $\phi(z) = \frac{z^3 + az + 1}{z - 1}$. By Proposition 17, $\phi$ is not isotrivial. On the other hand, one compute that

$$
\phi^{-2}(\infty) = \left\{ \infty, t, -\frac{t^2}{t + 2}, \frac{t}{t + 2} \right\}.
$$

Let

$$
T(z) = \frac{z + \frac{1}{t^2}}{z - t}.
$$

Then we see that $T(\phi^{-2}(\infty)) = \{1, \infty, 0, -1\}$ and that $\phi^{-2}(\infty)$ is isotrivial. In fact,

$$
\Delta := (1 + a)^2 - \frac{4}{b} = \frac{t^2(t + 1)^2}{(t + 2)^2},
$$

$$
R := \frac{1}{b^2} (1 - ab) = \frac{2t^2(t + 1)^2}{(t + 2)^2},
$$

and $\frac{R}{\Delta} = 2 \in \mathbb{P}^1(k)$ as predicted by Proposition 19.

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**References**