Local–Global Principle of Affine Varieties Over a Subgroup of Units in a Function Field

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Over a large class of function fields, we show that for many linear varieties in an affine space, the set of their points over the topological closure of a certain subgroup of the group of units of the function field is exactly the topological closure of the set of their points over this subgroup. This provides some evidence on the split-algebraic-torus analog of a conjecture for abelian varieties by Poonen and Voloch, as well as the function-field analog of an old conjecture by Skolem.

1 Introduction

Let $K/k$ be a function field, that is, $K$ is finitely generated over $k$ with transcendence degree 1 such that $k$ is relatively algebraically closed in $K$. We denote by $\bar{k}$ the algebraic closure of $k$ inside a fixed algebraic closure $\bar{K}$ of $K$. Let $\Sigma_{K/k}$ be the set of all places of $K/k$. For each finite subset $S$ of $\Sigma_{K/k}$, we denote by $O_S$ the ring of $S$-integers in $K$. We fix a cofinite subset $\Sigma \subset \Sigma_{K/k}$ and endow $\prod_{v \in \Sigma} K_v^*$ with the natural product topology. Let $M$ be a natural number, and $\mathbb{A}^M$ be the affine $M$-space, whose coordinate is denoted by $X = (X_1, \ldots, X_M)$. We also fix a closed $K$-variety $W$ in $\mathbb{A}^M$. We say that $W$ is transversal if it does not contain any translate of some coordinate axis, and that $W$ is homogeneous if it is defined by homogeneous polynomials. For each $i \in \{1, \ldots, M\}$, we let $\phi_i: \mathbb{A}^M \to \mathbb{A}^{M-1}$ be the dehomogenization (rational) map $(X_1, \ldots, X_M) \mapsto \left(\frac{X_1}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots, \frac{X_M}{X_i}\right)$ with respect to $X_i$. We give special notation for $K$-hyperplanes as follows. For any
ring $R$ and any two elements $a = (a_1, \ldots, a_M)$ and $b = (b_1, \ldots, b_M)$ in $\mathbb{A}^M(R)$, we define $ab = (a_1b_1, \ldots, a_Mb_M) \in \mathbb{A}^M(R)$ and $a \cdot b = \sum_{i=1}^M a_ib_i \in R$; we also write $b \cdot X$ for the function $\sum_{i=1}^M b_iX_i$ on $\mathbb{A}^M$. For any $b \in \mathbb{A}^M(K) \setminus \{(0, \ldots, 0)\}$, we denote by $H_b$ (respectively, $H'_b$) the hyperplane in $\mathbb{A}^M$ defined by $b \cdot X = 0$ (respectively, $b \cdot X + 1 = 0$). We note that if $b = (b_1, \ldots, b_M) \in \mathbb{A}^M(K)$ and $b_i \neq 0$, then $\phi_b(H_b)$ is exactly the nonhomogenous hyperplane $H'_{\phi_b(b)}$ in $\mathbb{A}^{M-1}$. By a linear $K$-variety (or linear variety defined over $K$) in $\mathbb{A}^M$, we mean an intersection of $K$-hyperplanes. For any subset $\Theta$ of some ring, let $W(\Theta)$ denote the set of points on $W$ with each coordinate in $\Theta$. Let $\Gamma$ be a subgroup of $K^*$. Via the diagonal embedding, we identify $W(\Gamma)$ with its image in $W(\prod_{\gamma \in \Sigma} K_{\gamma}^*)$ and denote by $\overline{W(\Gamma)}$ its topological closure. We naturally identify $\Gamma$ with $\mathbb{A}^1(\Gamma)$, and write $\overline{\Gamma}$ for $\mathbb{A}^1(\overline{\Gamma})$. In fact, both $\Gamma$ and $\overline{\Gamma}$ are not only subsets of $\prod_{\gamma \in \Sigma} K_{\gamma}^*$, but also topological subgroups. Moreover, for each $v \in \Sigma_{K/k}$, we denote by $\Gamma_v$ the topological closure of the image of the natural inclusion $\Gamma \subset K_v^*$, being continuous, this inclusion induces a subtopology of $\Gamma_v$ which will be referred to as $v$-adic subtopology. Note that $\Gamma_v$ is a topological subgroup of $K_v^*$.

In this paper, we investigate the circumstances where the equality

$$W(\overline{\Gamma}) = \overline{W(\Gamma)} \tag{1}$$

holds. In fact, (1) is almost the analog for split algebraic tori to Conjecture C in [8] for abelian varieties. It does not qualify as an exact analog because there are easy counterexamples to (1). In fact, any point in $(\overline{\Gamma} \cap K^*) \setminus \Gamma$ corresponds to a zero-dimensional $K$-variety $W$ in $\mathbb{A}^1$ for which (1) fails, and Example 0 in [11] shows that such points may really exist. However, in the case of zero characteristic, this phenomenon disappears under some restriction on the largeness of $\Gamma$, which is automatically satisfied in the abelian-variety counterpart by properness. More precisely, we prove the following result in this article.

**Theorem 1.** Suppose that $k$ has characteristic 0, and that $\Gamma$ is contained in $O^*_S$ for some finite $S \subset \Sigma_{K/k}$. Then for any $v \in \Sigma_{K/k}$, the $v$-adic subtopology of $\Gamma$ is discrete. In particular, $\overline{\Gamma} = \Gamma$. □

The story in the case of positive characteristic is completely different. Example 1 in [11] shows that, even if $\Gamma$ is cyclic, we may easily have $\overline{\Gamma} \neq \Gamma$; in particular, one cannot expect $W(\overline{\Gamma}) = W(\Gamma)$ if $W$ is homogeneous, since then $(r_1, \ldots, r_M) \in W(\Gamma)$ implies $(r_1\gamma, \ldots, r_M\gamma) \in W(\overline{\Gamma}) \setminus W(\Gamma)$ for any $\gamma \in \overline{\Gamma} \setminus \Gamma$. Even worse, Example 1 shows that this is not the only situation in which a homogeneous $W$ fails to satisfy $W(\overline{\Gamma}) = W(\Gamma)$. 


However, once we assume that $\Gamma$ is contained in $O_S^*$ for some finite $S \subset \Sigma_{K/k}$ as in the case of zero characteristic, a special case of the main result in Sun [11] shows that in case where $k$ is finite, (1) holds for any zero-dimensional $W$. This special case is established by investigating the topology of $\Gamma$. While mimicking this investigation for an infinite $k$, we have to consider two additional hypotheses as follows:

(SG) $K$ is separably generated over $k$,

(SH) each separable Hilbert subset of $k$ is infinite.

Definitions of some terms from above are briefly recalled as follows. We say that $K$ is separably generated over $k$ if there exists $t \in K$ such that $K$ is finite separable over $k(t)$. For any separable irreducible polynomials $f_1(T, X), \ldots, f_m(T, X)$ in $k(T)[X]$ and any nonzero polynomial $g(T)$ in $k[T]$, the subset $H_k(f_1, \ldots, f_m; g)$ of $k$ consisting of those $a \in k$ such that $g(a) \neq 0$ and that each $f_i(a, X)$ is defined and irreducible in $k[X]$ is called a separable Hilbert subset of $k$ [3].

The condition (SG) just rules out weird examples of function fields which one usually would not like to consider; if $k$ is perfect, then (SG) always holds ([7], Corollary 4.4 of Chapter VIII). On the other hand, (SH) holds if $k$ is the field of rational function in one variable over any field $\ell$ ([3], Theorem 13.3.5 for the case where $\ell$ is finite, Proposition 13.2.1 for the other case), and is preserved under any algebraic extension with a finite separable degree ([3], Proposition 12.3.3). This gives a large class of function fields $K/k$ satisfying both (SG) and (SH).

Following [8, 11], we try to prove (1) by reducing it to established zero-dimensional cases. More explicitly, we make $\bar{\Gamma}$-valued points of $W$ descend to its proper subvariety; see Lemma 11 in Section 4. Proposition 4 of [11] achieves this descent under some nonisotriviality hypothesis on $W$, which always fails for hyperplanes; see Section 4.1 for more discussion. However, a more refined notion of isotriviality, first introduced in Voloch [12], enables us accomplish the desired descent works under a more reasonable hypothesis. Following [2], for any subgroup $\Delta$ of $K^*$, we declare that $W$ is $\Delta$-isotrivial if for some $d \in A_M(\Delta)$ the translate $dW$ is defined by polynomials over $k$. Modifying the arguments in Derksen and Masser [2] which let $\Gamma$-valued points of a finite union of linear $K$-varieties descend to its proper subvariety, we achieve a similar descent for $(\prod_{v \in \Sigma} \Gamma_v)$-valued points, and derive the following result, where $\rho(\Gamma) = \bigcap_{m \geq 0} (kK^{p^m})^* \Gamma$.

**Theorem 2.** Suppose that $k$ has positive characteristic $p$, that either $k$ is finite or both (SG) and (SH) hold, that $\Gamma$ is contained in $O_S^*$ for some finite $S \subset \Sigma_{K/k}$, that $W$ is a
union of homogeneous linear $K$-varieties with no $\rho(\Gamma)$-isotrivial homogeneous linear $K$-subvariety whose dimension is larger than one. Then we have $W(\bar{\Gamma}) = \overline{W(\Gamma)}$. Moreover, if we also assume that each one-dimensional homogeneous linear $K$-subvariety of $W$ is not $\Gamma$-isotrivial, then we have $W(\bar{\Gamma}) = \emptyset$. □

In case where $\Gamma \subset \mathcal{O}_S^*$ for some finite $S \subset \Sigma_{K/k}$ and $W = H_b$ for some $b \in \mathbb{A}^M(\mathcal{O}_S^*)$, the assertion (1) implies the function field analog of an old conjecture raised by Skolem [10]. In Remark 2.4 of [5], this old conjecture is rephrased as

$$\mathcal{G}_L(F, M, b, S, \Gamma) \Leftrightarrow \mathcal{G}_G(F, M, b, S, \Sigma, \Gamma),$$

where $F$ is a number field, $S$ is a set of places of $F$ which contains all Archimedean ones, and

- $\mathcal{G}_L(F, M, b, S, \Gamma)$ stands for For each nonzero ideal $I \subset \mathcal{O}_S$ there exists $x \in \mathbb{A}^M(\Gamma)$ such that $b \cdot x \in I$.
- $\mathcal{G}_G(F, M, b, S, \Gamma)$ stands for There exists $x \in \mathbb{A}^M(\Gamma)$ such that $b \cdot x = 0$.

Although Skolem gave a proof for the case where $M = 2$ [10], his conjecture seems to have been largely ignored in the recent literature until Harari and Voloch [5] noticed its connection with (1). In fact, our main theorems may be translated into results on the function field analog of Skolem’s conjecture via the following equivalences:

$$\mathcal{G}_L(K/k, M, b, \Sigma_{K/k} \setminus \Sigma, \Gamma) \Leftrightarrow H_b(\bar{\Gamma}) \neq \emptyset,$$
$$\mathcal{G}_G(K/k, M, b, \Sigma_{K/k} \setminus \Sigma, \Gamma) \Leftrightarrow H_b(\Gamma) \neq \emptyset.$$

For instance, if either $k$ is finite or both (SG) and (SH) hold, then we always have $\mathcal{G}_L(K/k, 2, b, S, \Gamma) \Leftrightarrow \mathcal{G}_G(K/k, 2, b, S, \Gamma)$.

This paper is organized as follows. In Section 2, we give some easy observations which reduces the proof of (1) to easier cases, and review some relevant properties of function fields with positive characteristic. Section 3 is devoted to the investigation and its applications of the topological properties of $\Gamma$; in particular, we prove Theorem 1 as an easy consequence. Finally, we prove Theorem 2 in Section 4 by developing results on the descent of $(\prod_{v \in \Sigma} \Gamma_v)$-valued points of a hyperplane and then of a finite union of homogeneous linear $K$-varieties to its proper subvarieties.
Example 1. Let $K = \mathbb{F}_p(t)$ be a purely transcendental extension of $\mathbb{F}_p$, and $\Sigma$ be a cofinite subset of $\Sigma_{K/\mathbb{F}_p}$ such that $t$ is an unit of the valuation ring associated to each $v \in \Sigma$. Then the sequence $(t^{p^n})_{n \geq 1}$ in the cyclic subgroup $\langle t \rangle$ of $K^*$ generated by $t$ converges to $\alpha \in \langle t \rangle \setminus K^*$ [11, Example 1], and thus $-(1 + \alpha) \in \langle -(1 + t) \rangle \setminus K^*$. Consider the subgroup $\Gamma = \langle t, -(1 + t) \rangle$ of $K^*$ generated by $\{t, -(1 + t)\}$. We see that $(1, \alpha, -(1 + \alpha))$ lies in $H_{(1,1,1)}(\Gamma) \setminus H_{(1,1,1)}(\Gamma)$, but is not equal to $(r_1\gamma, r_2\gamma, r_3\gamma)$ for any $(r_1, r_2, r_3) \in H_{(1,1,1)}(\Gamma)$ and any $\gamma \in \tilde{\Gamma} \setminus \Gamma$. □

2 Preliminaries

2.1 A reduction lemma

The following lemma lists some easy observations that reduce (1) to simpler cases. Among them, only assertion (c) will be used in the proof of our main results (more precisely, Theorem 2).

 Lemma 1. 

(a) Suppose that $W = W_1 \times W_2$ for some $K$-variety $W_j$ in $\mathbb{A}^{M_j}$ such that $W_j(\tilde{\Gamma}) = \overline{W_j(\Gamma)}$ for $j \in \{1, 2\}$. Then $W(\tilde{\Gamma}) = \overline{W(\Gamma)}$.

(b) Suppose that $W$ is homogeneous. Then for each $i \in \{1, \ldots, M\}$, we have $\phi_i(W)(\tilde{\Gamma}) = \phi_i(W(\tilde{\Gamma}))$ and $\phi_i(W(\Gamma)) = \phi_i(W(\Gamma))$.

(c) Suppose that $W$ is homogeneous and $\phi_i(W)(\tilde{\Gamma}) = \phi_i(W(\Gamma))$ for some $i \in \{1, \ldots, M\}$. Then $W(\tilde{\Gamma}) = \overline{W(\Gamma)}$. □

Proof. Both (a) and (b) are clear. For (c), let $(\gamma_1, \ldots, \gamma_M) \in W(\tilde{\Gamma})$. Then $(\frac{n_1}{\gamma_1}, \ldots, \frac{n_{M-1}}{\gamma_{M-1}}, \frac{n_M}{\gamma_M}) \in \phi_i(W)(\tilde{\Gamma}) = \phi_i(W(\Gamma))$. Thus for each $j \in \{1, \ldots, M\}$ there exists some $r_j \in \Gamma$ such that $\gamma_j = r_j \gamma_i$. Then $(r_1, \ldots, r_M) \in W(\Gamma)$. Choosing a sequence $(x_\ell)_{\ell \geq 1}$ in $\Gamma$ which converges to $\gamma_i$ in $\prod_{v \in \Sigma} K_v^*$, we see that $((r_1 x_\ell, \ldots, r_M x_\ell))_{\ell \geq 1}$ is a sequence in $W(\Gamma)$ which converges to $(\gamma_1, \ldots, \gamma_M)$ in $W(\prod_{v \in \Sigma} K_v^*)$. Therefore $(\gamma_1, \ldots, \gamma_M) \in \overline{W(\Gamma)}$ as desired. ■

Assertion (b) says the homogeneity assumption on $W$ in Theorem 2 is not really a restriction. On the other hand, combined with assertion (a), we conclude by Theorem 2 that any direct product $W$ of one-dimensional homogeneous linear $K$-varieties (i.e., $W$ is an irreducible coset over $K$ in the terminology used by Derksen and Masser [2]) satisfies (1), even though such $W$ does not necessarily satisfy the hypothesis of Theorem 2.
2.2 Properties of function fields with positive characteristic

In this section, we assume that $k$ has positive characteristic $p$.

Lemma 2. $\bigcap_{n \geq 0} kK^{p^n} = k$. □

Proof. We only have to show that $kK^{p^n} \subseteq kK^{p^m}$ for each $m > n \geq 0$, for then $K$ is transcendental over $\bigcap_{n \geq 0} kK^{p^n}$ and this yields the desired conclusion. Suppose that $kK^{p^m} = kK^{p^n}$ for some $m > n \geq 0$. Then $kK^{p^n}$ is finitely generated over $k$ and $(kK^{p^n})^{p^{m-n}} k = kK^{p^n}$. Applying Proposition 4.9 in Chapter VIII of Lang [7], we get to a contradiction that $kK^{p^n}$ is algebraic over $k$. ■

Lemma 3. For each natural number $n$, we have $K^{p^n} \cap \bar{K} = K^{p^n}$. □

Proof. By Poonen and Voloch [8, Lemma 3.2], $K_v \cap \bar{K}$ is a separable extension over $K$, and thus $K^{p^n} \cap \bar{K}$ is a separable extension over $K^{p^n}$. Hence, being both separable and purely inseparable, the extension $K^{p^n} \cap K$ over $K^{p^n}$ is trivial. ■

Recall that $\rho(\Gamma) = \bigcap_{m \geq 0} (kK^{p^m})^s \Gamma$. In the case where $k$ is finite, we have $\rho(\Gamma) \subset \sqrt{\Gamma}$, where $\sqrt{\Gamma} = \{x \in K^*: x^n \in \Gamma$ for some nonzero integer $n\}$ (see, e.g., [12, Lemma 3]). Although $\rho(\Gamma) \subset \sqrt{\Gamma}$ fails in general (by Lemma 2, the assertion $k = \rho(1) \subset \sqrt{1}$ would force $k$ to be contained in $\overline{\mathbb{F}_p}$), the following lemma give some control on $\rho(\Gamma)$ in terms of $\Gamma$, provided that $k$ is perfect.

Lemma 4. Suppose that $k$ is perfect, and that $\Gamma$ is contained in $O_S^s$ for some $S \subset \Sigma_{K/k}$. Then $\rho(\Gamma)$ is also contained in $O_S^s$. □

Proof. Any $v \in \Sigma_{K/k}$ gives a discrete valuation $v : K^* \to \mathbb{Z}$ such that $v(s) = 1$ for some $s \in K^*$ and that $v((K^{p^n})^s) \subset p^n \mathbb{Z}$. For each $v \notin S$, we have $v(\rho(\Gamma)) = v(\bigcap_{n}(K^{p^n})^s \Gamma) = v(\bigcap_{n}(K^{p^n})^s) \subset \bigcap_{n} p^n \mathbb{Z} = \{0\}$. This completes the proof. ■

Lemma 5. Suppose that (SG) holds. Then for any $n > 0$, we have $K \cap \bar{k}K^{p^n} = kK^{p^n}$. □

Proof. Denote by $k^{sep}$ the separable closure of $k$ in $\bar{k}$. As $K \cap k^{sep} K^{p^n}$ is both separable and purely inseparable over $kK^{p^n}$, they are equal to each other; hence, it remains to show that $K \cap \bar{k}K^{p^n} \subset k^{sep} K^{p^n}$. Let $K = k(t, y)$ with $y$ separable over $k(t)$. Then $K = k(t, y^{p^n})$
and thus \( K \cap \kappa K^p = k(t, y^p) \cap \kappa(t^p, y^p) \subset k^{sep}(t, y^p) \cap \kappa(t^p, y^p) \). The irreducible polynomial of \( y^p \) over \( k^{sep}(t) \) is still irreducible over \( \kappa(t) \), and therefore \( k^{sep}(t, y^p) \cap \kappa(t^p, y^p) = k^{sep}(t^p, y^p) \) since \( k^{sep}(t) \cap \kappa(t^p) = k^{sep}(t^p) \). This completes the proof. \( \square \)

**Lemma 6.** Suppose that \( k \) is perfect. Then for each \( v \in \Sigma_{K/k} \) and each \( n > 0 \), any \( K^p \)-linear map \( \phi : K \to K \) is continuous with respect to the \( v \)-adic topology. \( \square \)

**Proof.** Any \( v \in \Sigma_{K/k} \) gives a discrete valuation \( v : K^* \to \mathbb{Z} \) such that \( v(s) = 1 \) for some \( s \in K^* \) and that \( v((K^{p^n})^*) \subset p^n\mathbb{Z} \). It follows that \( \{s^j\}_{0 \leq j \leq p-1} \) is \( K^p \)-linearly independent. Since \( [K : K^p] = p^n \), we conclude that \( \{s^j\}_{0 \leq j \leq p-1} \) is a \( K^p \)-linear basis for \( K \). To prove this lemma, it is enough to show the continuity of \( \phi \) at 0. Let \( x = \sum_{j=0}^{p^n-1} c_j s^j \neq 0 \) with all \( c_j \in K^p \). Then \( v(x) = \min_{c_j \neq 0} (v(c_j) + j) \) and \( \phi(x) = \sum_{j=0}^{p^n-1} c_j \phi(s^j) \). Thus, we have \( v(\phi(x)) \geq \min_{c_j \neq 0} (v(c_j) + v(\phi(s^j))) \geq v(x) + \min_{0 \leq j \leq p^n-1} (v(\phi(s^j)) - j) \). This finishes the proof since

\[
\min_{0 \leq j \leq p^n-1} (v(\phi(s^j) - j))
\]

is independent of \( x \). \( \square \)

An iterative derivation on a field \( L \) is a sequence \( \{D^{(i)}\}_{i \geq 0} \) of elements in the \( L \)-algebra of additive endomorphisms on \( L \) such that

1. \( D^{(0)} \) is the identity operator.
2. \( D^{(i)}(xy) = \sum_{j=0}^{i} D^{(j)}(x)D^{(i-j)}(y) \), for \( i \geq 0 \) and \( x, y \in L \).
3. \( D^{(i)}D^{(j)} = \binom{i+j}{i} D^{(i+j)} \) for \( i, j \geq 0 \), where \( D^{(i)}D^{(j)} \) denotes the composition of \( D^{(i)} \) and \( D^{(j)} \), and the rational integer \( \binom{i+j}{i} \) is the binomial coefficient.

**Proposition 1.** Suppose that (SG) holds. Then there exists an iterative derivation \( \{D^{(i)}_K\}_{i \geq 0} \) on \( K \) such that for each \( m \geq 1 \) we have \( \{x \in K : D^{(i)}_K(x) = 0 \} \) for any \( 1 \leq \ell < p^m \) = \( kK^p \) and that each \( D^{(i)}_K \) is continuous with respect to the \( v \)-adic topology for each \( v \in \Sigma_{K/k} \). \( \square \)

**Proof.** Choose \( t \in \kappa K \) such that \( \kappa(t) \subset \kappa K \subset k((t)) \). By García and Voloch [4, Remark 1], there exists an iterative derivation \( \{D^{(i)}_{KK}\}_{i \geq 0} \) on \( \kappa K \) such that \( \kappa K^p = \{x \in \kappa K : D^{(i)}_{KK}(x) = 0 \} \) for any \( 1 \leq \ell < p^m \) for \( m \geq 1 \). For each \( i \geq 0 \), let \( D^{(i)}_K = D^{(i)}_{KK}|_K \). Then Lemma 5 shows \( \{x \in K : D^{(i)}_K(x) = 0 \} \) for any \( 1 \leq \ell < p^m \) = \( K \cap \kappa K^p = kK^p \). By Lemma 6, each \( D^{(i)}_{KK} \) is continuous with respect to the \( w \)-adic topology for each \( w \in \Sigma_{KK/k} \). Since each element in
Σ_K/k extends to an element in Σ_{kK/k}, this shows that each D^{(i)}_K is continuous with respect to the v-adic topology for each v ∈ Σ_K/k. ■

3 Topological Properties of Γ

The difficult Chow–Lang–Neron theorem for abelian varieties [6] has the following easy analogy for the multiplicative group. For any finite subset S ⊂ Σ_K/k, the quotient group O^*_S/k^* is finitely generated (Corollary 1 of Proposition 14.1. in [9]). This induces the next result, which reduces our investigation of the topology of Γ to one of its finitely generated subgroup.

Lemma 7. Suppose that Γ is contained in O^*_S for some finite S ⊂ Σ_K/k. Then for any v ∈ Σ_K/k, the subgroup G_v = Γ ∩ (1 + m_v) is finitely generated and is open in the v-adic subtopology Γ.

Proof. G_v is open in the v-adic subtopology of Γ since 1 + m_v is an open subgroup of K^*_v. The inclusion Γ ⊂ O^*_S induces the map G_v → O^*_S/k^*, which is injective because k^* ∩ (1 + m_v) is trivial. Then G_v is finitely generated since O^*_S/k^* is so. ■

Since we always need to make use of Lemma 7, we assume, from now on, that Γ is contained in O^*_S for some finite S ⊂ Σ_K/k.

Proof of Theorem 1. Fix v ∈ Σ_K/k and let U_n = Γ ∩ (1 + m_v^n) for n ≥ 1. Then U_n is open in the v-adic subtopology of Γ. Since k has characteristic zero, the quotient groups U_n/U_{n+1} are torsion-free for all n. Lemma 7 shows that U_1 is finitely generated, hence U_n is trivial for some n.

The following lemma is due to a conversation with Ming-Lun Hsieh.

Lemma 8. Suppose that both (SG) and (SH) hold. Then for any finite separable extension L of K, there are infinitely many v ∈ Σ_K/k which extend uniquely and unramifidedly to a place of L.

Proof. By (SG), K is a finite separable extension of the purely transcendental extension k(t) over k, and, therefore, it is enough to assume that K = k(t). In this case, the desired conclusion follows from the fact, which is guaranteed by (SH), that the separable Hilbert
subset $H_k(f; 1)$ is infinite, where $L = K(y)$ with $y$ being a root of the separable irreducible polynomial $f \in k(t)[X]$. ■

From now on, we further assume that $k$ has positive characteristic $p$. Motivated by the proof of Theorem 1 of [1], we give the following result:

Lemma 9. Suppose that either $k$ is finite or both (SG) and (SH) hold. Then for any integer $m$ prime to $p$, the subgroup $\Gamma^m$ is open in $\Gamma$. □

Proof. The case where $k$ is finite is proved in Sun [11, Lemma 12]. Thus, we assume that $k$ is infinite. Fix an integer $m$ prime to $p$. By Lemma 7, we may assume that $\Gamma$ is finitely generated, and hence so is its $m$-division group $\Gamma^m = \{x \in \Gamma : x^m \in \Gamma\}$ of $\Gamma$ in $K^*$. Then by Sun [11, Lemma 5], we may further assume that $\Gamma = \Gamma^m \cap K^*$. Let $L$ be the finite Galois extension of $K$ obtained by adjoining all the $m$th roots in $\bar{K}$ of every element in $\Gamma$. For each field $E$ such that $K \subset E \subset L$, Lemma 8 yields a place $v_E \in \Sigma$, which extends to a unique place $v_E$ of $E$ such that $[E_{v_E} : K_{v_E}] = [E : K]$. Let $S$ be the finite set consisting of those $v_E$ such that there is no proper intermediate field between $K$ and $E$. We shall complete the proof by showing that $\Gamma \cap U_S \subset \Gamma^m$, where $U_S = \prod_{v \in S} 1 + m_v$ is an open subgroup of $\prod_{v \in S} K_v^*$. Since $\Gamma = \Gamma^m \cap K^*$, we have $\Gamma \cap (K^*)^m = \Gamma^m$ and thus only need to show that $\Gamma \cap U_S \subset (K^*)^m$. Assume $x \in \Gamma \cap U_S \setminus (K^*)^m$ and let $F$ be the extension of $K$ obtained by adjoining an $m$th root of $x$. Then we have $K \subset E \subset F \subset L$ for some $E$ such that $v_E \in S$. Since $[E_{v_E} : K_{v_E}] = [E : K] \neq 1$, it follows that $x$ has no $m$th root in $K_{v_E}$, which contradicts the assumption $x \in U_S$ by Hensel’s lemma. ■

Lemma 10. For every $v \in \Sigma_{K/k}$, any subgroup of $\Gamma$ containing $\Gamma^p^n$ for some a nature number $n$ is open in the $v$-adic subtopology of $\Gamma$. □

Proof. It suffices to show that those subgroups $\Gamma^p^n$ are open in the $v$-adic subtopology of $\Gamma$. Similar to the proof of Lemma 9, we fix a nature number $n$, and assume that $\Gamma = \{x \in K^* : x^p^n \in \Gamma\}$ is finitely generated. From Lemma 3, it follows that $(K_v^*)^p^n \cap \Gamma \leq (K_v^*)^p^n \cap \Gamma = \Gamma^p^n$. Then it suffices to show that $(K_v^*)^p^n \cap \Gamma$ is open in the $v$-adic subtopology of $\Gamma$. Note that $(K_v^*)^p^n$ is closed in $K_v^*$ and consequently $K_v^*/(K_v^*)^p^n$ is Hausdorff. Consider the map $\Gamma \to K_v^*/(K_v^*)^p^n$ induced from the inclusion $\Gamma \subset K_v^*$, which is continuous with respect to the $v$-adic subtopology of $\Gamma$. Since this map factors through $\Gamma/\Gamma^p^n$, its image is finite, whence discrete. This completes our proof. ■
Corollary 1. For every \( v \in \Sigma_{K/k} \) and every subgroup \( \Delta \) of \( \Gamma \) containing \( \Gamma^{p^n} \) for some \( n \in \mathbb{N} \), the homomorphism
\[
\Gamma/\Delta \to \Gamma_v/\Delta_v
\]
is bijective. \( \square \)

Proof. By Lemma 10, every such subgroup \( \Delta \) is open and closed in the \( v \)-adic subtopology of \( \Gamma \). Then the desired conclusion follows (cf. [11, Lemma 8]). \( \square \)

Corollary 2. Suppose that either \( k \) is finite or both (SG) and (SH) hold. Then any subgroup of \( \Gamma \) containing \( \Gamma^m \) for some \( m \in \mathbb{N} \) is open. \( \square \)

Proof. Combine Lemmas 10 and 9. \( \square \)

Corollary 3. Suppose that either \( k \) is finite or both (SG) and (SH) hold. Then for any subgroup \( \Delta \) of \( \Gamma \) containing \( \Gamma^m \) for some \( m \in \mathbb{N} \), the homomorphism
\[
\Gamma/\Delta \to \tilde{\Gamma}/\tilde{\Delta}
\]
is bijective. \( \square \)

Proof. Parallel to the proof of Corollary 1, this desired conclusion follows from Corollary 2. \( \square \)

Recall that \( K^* \) is regarded as a topological subgroup of \( \prod_{v \in \Sigma} K_v^* \).

Corollary 4. Suppose that either \( k \) is finite or both (SG) and (SH) hold. Then \( \Gamma \) is closed in \( K^* \). \( \square \)

Proof. Let \( P \in \tilde{\Gamma} \cap K^* \). Lemma 7 shows that \( P \in \tilde{\Gamma}_0 \) for some finitely generated subgroup \( \Gamma_0 \subset \Gamma \) such that \( \Gamma_0 \) is contained in a finitely generated closed subgroup \( \Gamma_S \) of \( O^*_S \) for some finite \( S \subset \Sigma_{K/k} \). By enlarging \( S \), we may assume \( S \cup \Sigma = \Sigma_{K/k} \) and hence both \( O^*_S \) and \( \Gamma_S \) are closed in \( K^* \). By Corollary 2, every subgroup of \( \Gamma_S \) with a finite index is open; being an intersection of subgroups of \( \Gamma_S \) with finite indices, \( \Gamma_0 \) is closed in \( \Gamma_S \), and thus in \( K^* \). This shows \( P \in \Gamma_0 \subset \Gamma \) and finishes our proof. \( \square \)

Corollary 5. Suppose that either \( k \) is finite or both (SG) and (SH) hold. Then every torsion element of \( \tilde{\Gamma} \) lies in \( \Gamma \). \( \square \)
Proof. By Lemma 7, for every element of $\tilde{\Gamma}$ there exists some finitely generated subgroup $\Gamma_0$ of $\Gamma$ such that this element lies in $\tilde{\Gamma}_0$. By Corollary 2, for any integer $m$ the subgroup $\Gamma_0^m$ of $\Gamma_0$ is open, hence [11, Lemma 7] implies that every torsion element of $\tilde{\Gamma}_0$ lies in $\Gamma_0$. This completes our proof.

Being analogous to [8, Proposition 3.7], the following result shows that (1) holds for any zero-dimensional variety $W$. Since the proofs are also parallel, we omit some details, which can also be found in the proof of another analogous result, the special case of [11, Theorem 1] where $\dim X = 0$, proved in Sun [11, Section 3.3].

Proposition 2. Suppose that either $k$ is finite or both (SG) and (SH) hold. Then for any zero-dimensional variety $Z$ in $A^M$ defined over $K$, we have $Z(\tilde{\Gamma}) = Z(\Gamma)$. □

Proof. Replacing $K$ by a finite extension if necessary, we may assume that the restriction $i_v|Z(K)$ of the natural map $i_v : A^M(K) \to A^M(K_v)$ is a bijection. We only have to show that $Z(\tilde{\Gamma}) \subset A^M(\Gamma)$. Let us start with an arbitrary $Q = (Q_v)_{v \in \Sigma} \in Z(\tilde{\Gamma}) \subset \prod_{v \in \Sigma} Z(K^*_v)$, with each $Q_v \in Z(K^*_v)$. Then there is a sequence $\{P_n\}_{n \geq 1}$ in $A^M(\Gamma)$ such that at each $v \in \Sigma$, the sequence $\{i_v(P_n)\}_{n \geq 1}$ has $Q_v$ as its limit point in $A^M(K_v)$. Write $Q_{(v)} = i_v^{-1}(Q_v) \in Z(K^*_v)$. By Lemma 7, there is a finitely generated subgroup $\Gamma_0$ of $\Gamma$ such that the sequence $\{P_n\}_{n \geq 1}$ is contained in $A^M(\Gamma_0)$. From Lemma 10, it follows that for every pair $v, w \in \Sigma$, the quotient $Q_{(v)}Q_{(w)}^{-1}$ is a torsion point. By the finiteness of the set $\{Q_{(v)}\}_{v \in \Sigma}$, we conclude that $QQ_{(w)}^{-1}(Q_{(w)})_{v \in \Sigma}$ is a torsion element in $A^M(\tilde{\Gamma})$, and hence, by Corollary 5, it actually lies in $A^M(\Gamma)$. In particular, we see that $Q \in A^M(\Gamma)$ as desired. ■

4 Descent Result for $\tilde{\Gamma}$-valued Points of a Finite Union of Linear $K$-varieties

Recall that we are in the case where $k$ has positive characteristic $p$, and have defined that $\rho(\Gamma) = \cap_{m \geq 0}(kK^p)^*\Gamma$. For each $i \in \{1, \ldots, M\}$, we denote by $\psi_i$ the self-map on $A^M$ which replaces the $i$th component of a given element by 1 and keeps the others unchanged.

The relevance of this section to our main goal in this paper, that is, proving (1), is captured in the following easy observation.

Lemma 11. Suppose that $W$ has a $K$-subvariety $V$ satisfying $W(\tilde{\Gamma}) = V(\tilde{\Gamma}) = \overline{V(\Gamma)}$. Then $W(\tilde{\Gamma}) = \overline{W(\Gamma)}$. □

Proof. $W(\tilde{\Gamma}) = V(\tilde{\Gamma}) = \overline{V(\Gamma)} \subset \overline{W(\Gamma)} \subset W(\tilde{\Gamma})$. ■
The idea in Lemma 11 is first implemented in Poonen and Voloch [8, Lemma 3.13], where the ambient space is an abelian varieties (as to a split algebraic torus in the present setting). Proposition 4 of [11] provides another implementation for some subvarieties of semiabelian varieties. Specialized to the current setting, this result implies that, in the case where \( k \) is finite, if \( \Gamma \) is finitely generated and all the largest dimensional irreducible components of each translate \( cW \) of \( W \) by \( c \in \mathbb{A}^M(\bar{K}^*) \) is not defined over a finite field, then \( W \) has a closed \( K \)-subvariety \( V \) satisfying \( W(\bar{\Gamma}) = V(\bar{\Gamma}) \). However, this assumption on \( W \) always fails for hyperplanes. In fact, for every \( b = (b_1, \ldots, b_M) \in \mathbb{A}^M(K) \setminus \{(0, \ldots, 0)\} \), if \( \{i_1, \ldots, i_n\} = \{i : 1 \leq i \leq M, b_i = 0\} \) and \( \{j_1, \ldots, j_m\} = \{1, \ldots, M\} \setminus \{i_1, \ldots, i_n\} \), then \( \psi_{i_1} \circ \cdots \circ \psi_{i_n}(b)H_b \) and \( \psi_{j_1} \circ \cdots \circ \psi_{j_m}(b)H'_b \) are, respectively, the hyperplanes \( H_{\psi_{i_1} \circ \cdots \circ \psi_{i_n}(0, \ldots, 0)} \) and \( H'_{\psi_{j_1} \circ \cdots \circ \psi_{j_m}(0, \ldots, 0)} \), both of which are clearly defined over \( k \).

To have \( \bar{\Gamma} \)-valued points of \( W \) descend to its proper subvariety, the key observation is \( \bar{\Gamma} \subset \prod_{v \in \Sigma} \Gamma_v \subset \prod_{v \in \Sigma} K_v^* \). In Section 4.1, we derive a descent result for \((\prod_{v \in \Sigma} \Gamma_v)\)-valued points of transversal \( K \)-hyperplanes. Although this result is not so realistic, one of its simplest cases plays a key role in proving the major result in Section 4.2, where the descent for \((\prod_{v \in \Sigma} \Gamma_v)\)-valued points of a finite union of homogeneous linear \( K \)-varieties is treated.

In order to have an iterative derivation as our main tool, we further assume throughout this section that \((SG)\) holds. We fix \( \{D^{(j)}_K\}_{j \geq 0} \) on \( K \) with the properties stated in Proposition 1. By continuity, \( \{D^{(j)}_K\}_{j \geq 0} \) extends to an iterative derivation \( \{D^{(j)}_v\}_{j \geq 0} \) on \( K_v \) for each \( v \in \Sigma_{K/k} \). For each \( v \in \Sigma_{K/k} \) and each natural number \( m \), denote by \( (kK^{P^m})_v \) the smallest subfield of \( K_v \) containing \( kK^{P^m} \); note that \( D^{(j)}_v|_{(kK^{P^m})} \) is the zero map for any \( 1 \leq j < p^m \).

**Lemma 12.** For each \( v \in \Sigma_{K/k} \) and each natural number \( m \), we have \( (kK^{P^m})_v \cap K = kK^{P^m} \).

**Proof.** Let \( x \in (kK^{P^m})_v \cap K \). Then if \( 1 \leq \ell < p^m \), we have \( D^{(\ell)}_v(x) = D^{(\ell)}_K(x) = 0 \). This shows that \( x \in kK^{P^m} \). \( \blacksquare \)

### 4.1 Hyperplanes

In this subsection, we derive a descent result for \((\prod_{v \in \Sigma} \Gamma_v)\)-valued points of \( K \)-hyperplanes by using the common idea in the proof of Lemma 3.13 in [8] and of...
Proposition 4 in [11]. We describe this idea as follows. The goal is to divide $\tilde{\Gamma}$ into cosets of $\tilde{\Delta}$ for some subgroup $\Delta$ of $\Gamma$ such that

(a) There exists a finite subset $U$ of $A^M(K^*)$ such that $\tilde{\Gamma} = \bigcup_{u \in U} u\tilde{\Delta}$.

(b) There exists a zero-dimensional $K$-subvariety $Z$ of $W$ such that $W(\tilde{\Delta}) = Z(\tilde{\Delta})$.

From (a) and (b), it follows that

$$W(\tilde{\Gamma}) = \bigcup_{u \in U} uW(\tilde{\Delta}) = \bigcup_{u \in U} uZ(\tilde{\Delta}) = \left(\bigcup_{u \in U} uZ\right)(\tilde{\Delta}).$$

Since $(\bigcup_{u \in U} uZ)\tilde{\Delta} = \left(\bigcup_{u \in U} uZ\right)(\Delta)$ by Proposition 2, this shows that $W(\tilde{\Gamma}) \subset A^M(K^*)$ and hence that $W(\tilde{\Gamma}) = W(\Gamma)$ by Corollary 4.

We usually achieve (b) by choosing a sufficiently small $\Delta$ such that those points of $W(\tilde{\Delta})$, as a subset of $W(\prod_{v \in \Sigma} K_v^*)$, satisfy sufficiently many equations. In the following proposition, we treat the case where $W$ is a transversal hyperplane, and prove that, under some devised hypotheses, the choice $\Delta = \Gamma \cap (kK^m)^*$ for a fixed $m$ works. By linearity of $W$, we actually find an irreducible $Z$ satisfying (b).

**Proposition 3.** Let $b \in A^M(K^*)$, and $m$ be a natural number.

(a) Suppose that the components of $b$ are linearly independent over $kK^m$. Then we have

$$H_b((kK^m)_v^*) = \emptyset$$

for all $v \in \Sigma_{K/k}$.

(b) Suppose that for some $j$ the components of $\psi_j(b)$ are linearly independent over $kK^m$. Then for some $P \in H'_b(K)$ we have

$$\prod_{v \in \Sigma_{K/k}} H'_b((kK^m)_v^*) \subset \{P\}.$$ 

If, moreover, the components of either $b$ or some $\psi_\ell(b)$ are linearly dependent over $kK^m$, then we have

$$H'_b((kK^m)_v^*) = \emptyset$$

for all $v \in \Sigma_{K/k}$. $\square$
Proof. Fix some \( v \in \Sigma_{K/k} \) and some \( c = (c_1, \ldots, c_M) \in H_b((kK^p)^* \cup H'_b((kK^p)^*)) \). Then \( \sum_{j=1}^N b_j c_j = e \), where \( e \in \{0, 1\} \). For any \( 0 \leq i < p^m \), because \( D^{(i)}_K(c_j) = 0 \), we have that
\[
\sum_{j=1}^N D^{(i)}_K(b_j) c_j = D^{(i)}_K(e). \tag{2}
\]

Denote by \( c \) (respectively, \( e \)) the \( M \)-by-1 matrix with the \( j \)th component \( c_j \) (respectively, \( D^{(j)}_K(e) \)). For any set \( I = \{i_1, \ldots, i_M\} \) of \( M \) nonnegative integers such that \( 0 = i_1 < i_2 < \cdots < i_M < p^m \), let \( T_{b,I} \) be the \( M \)-by-\( M \) matrix with the entry \( D^{(i_j)}_K(b_j) \) being at the \( l \)th row and \( j \)th column. From (2), we have \( T_{b,I} c = e \), which implies
\[
(\det T_{b,I}) c = T_{b,I}^* e. \tag{3}
\]

where \( T_{b,I}^* \) denotes the adjugate matrix of \( T_{b,I} \). If \( e = 0 \) and the components of \( b \) are linearly independent over \( kK^p \), then by García and Voloch [4, Theorem 1], \( \det T_{b,I} \neq 0 \) for some \( I \), which contradicts \( c \neq 0 \). This proves (a).

To prove (b), we consider the case where \( e = 1 \). Note that the \( j \)th component of \( T_{b,I}^* e \) is exactly \( \det T_{\psi_j(b),I} \). Under the assumption in the first part, García and Voloch [4, Theorem 1] implies that \( \det T_{\psi_j(b),I} \neq 0 \) for some \( j \) and \( I \). Hence, there is at most one choice for \( c \) satisfying (3), and this choice gives \( P \in W'_b(K) \). If the additional hypothesis also holds, then either \( \det T_{b,I} \) or some component of \( T_{b,I}^* e \) is zero, and (3) is impossible. \( \blacksquare \)

Proposition 4. Let \( b \in \mathbb{A}^M(K^*) \). For each natural number \( m \), let \( R_m \subset \Gamma \) be a complete set of representatives of \( \Gamma / (\Gamma \cap (kK^p)^*) \).

(a) For each \( m \), we have that \( R_m \) is a finite set.

(b) Suppose that there is some \( m \) such that for every \( r \in \mathbb{A}^M(R_m) \) the components of \( br \) are linearly independent over \( kK^p \). Then for all \( v \in \Sigma_{K/k} \), we have \( H_b(\Gamma_v) = \emptyset \).

(c) Suppose that there is some \( m \) such that for every \( r \in \mathbb{A}^M(R_m) \) the components of some \( \psi_j(br) \) are linearly independent over \( kK^p \). Then there exists a finite union \( Z \) of irreducible zero-dimensional \( K \)-subvarieties of \( H'_b \) such that \( \prod_{v \in \Sigma_{K/k}} H'_b(\Gamma_v) \subset Z(K) \), and that the number of irreducible components of \( Z \) is exactly the number of \( r \in \mathbb{A}^M(R_m) \) such that the components of \( br \) and those of each \( \psi_j(br) \) are linearly independent over \( kK^p \). \( \square \)
Proof. First, note that for each \( m \) the kernel of the natural map

\[
\Gamma \to (O^*_{\mathbb{A}^n}(k)/O^*_{\mathbb{A}^n}(k))^m
\]

is contained in \((kK^m)^*\). This proves (a) since \( O^*_{\mathbb{A}^n}(k)/O^*_{\mathbb{A}^n}(k) \) is finitely generated.

Fix a place \( v \in \Sigma_{K/k} \). For any \( m \), let \( \Delta^{(m)} = \Gamma \cap (kK^m)^* \). Then we have

\[
\Gamma_v = \bigcup_{\gamma \in R_m} \gamma \Delta^{(m)}_v \subset \bigcup_{\gamma \in R_m} \gamma (kK^m)_v^*,
\]

where the first equality follows from Corollary 1. This gives

\[
H_b(\Gamma_v) \subset H_b \left( \bigcup_{\gamma \in R_m} \gamma (kK^m)_v^* \right) = \bigcup_{r \in A^M(R_m)} r H_{br}( (kK^m)_v^* ).
\]

To prove (b), note that the assumption is devised such that for some \( m \) Proposition 3(a) shows that \( H_{br}( (kK^m)_u^* ) = \emptyset \) for each \( r \in A^M(R_m) \), yielding the desired conclusion.

The proof for (c) is similar. We have

\[
H'_b(\Gamma_v) \subset \bigcup_{r \in A^M(R_m)} r H'_{br}( (kK^m)_v^* ).
\]

Let \( U \subset A^M(R_m) \) be the subset consisting of those \( r \) such that the components of \( br \) and those of each \( \psi_j(br) \) are linearly independent over \( kK^m \). By Proposition 3(b), for each \( u \in U \), there exists some \( P_u \in H'_b(K) \) such that \( \prod_{v \in \Sigma_{K/k}} H'_b((kK^m)_v^*) \subset \{ P_u \} \), while for each \( r \in A^M(R_m) \setminus U \), we have \( H'_{br}( (kK^m)_v^* ) = \emptyset \) for all \( v \in \Sigma_{K/k} \). By (a), we let \( Z \) be the finite union of irreducible zero-dimensional \( K \)-subvarieties of \( H'_b \), each corresponding the \( K \)-rational point \( uP_u \) on \( H'_b \) for some \( u \in U \). Then we see that the number of irreducible components of \( Z \) is as claimed, and that \( H'_b(\Gamma_v) \subset Z(K) \) as desired. ■

As an immediate corollary, we give an artificial result on (1) for transversal hyperplanes.

Corollary 6. Under additional hypothesis that either \( k \) is finite or (SH) holds, the assumption in (b) of Proposition 4 implies \( H_b(\tilde{\Gamma}) = H_b(\Gamma) = \emptyset \), and the assumption in (c) implies \( H'_b(\tilde{\Gamma}) = H'_b(\Gamma) \).

\( \square \)
Proof. In view of Proposition 4 and the fact \( W(\bar{\Gamma}) \subset \prod_{v \in \Sigma} W(\Gamma_v) \) as well as Lemma 11, we only have to show that \( \bar{\Gamma} \cap K^* = \Gamma \), which is concluded by Corollary 4.

In Proposition 4, the assumption (c) is vacuous in the case where \( M = 1 \); the corresponding conclusion in Corollary 6 follows solely from the fact \( \Gamma \) is closed in \( K^* \), which is established in Corollary 4. The number-field analog of this fact is essentially [1, Theorem 1].

In case where \( M = 2 \), the assumption in (b) (respectively in (c)) of Proposition 4 is equivalent to that \( H_b \) (respectively, \( H'_b \)) is not \( \rho(\Gamma) \)-isotrivial. This observation yields the following result, which will play a role in the next subsection, where we treat the descent of \((\prod_{v \in \Sigma} \Gamma_v)\)-valued points on a finite union of homogeneous linear \( K \)-varieties.

Corollary 7. For each \( v \in \Sigma_{K/k} \), we have \( \Gamma_v \cap K^* \subset \rho(\Gamma) \).

Proof. Let \( b \in \Gamma_v \cap K^* \) and consider the line \( H_{(b,-1)} \) in \( \mathbb{A}^2 \). Since \( (1, b) \in H_{(b,-1)}(\Gamma_v) \), Proposition 4(b) shows that for each natural number \( m \) there exists some \( r = (r_1, r_2) \in \mathbb{A}^2(\Gamma) \) such that \( br_1 \) and \( -r_2 \) are linearly dependent over \( kK^{p^m} \). This shows that \( b \in \bigcap_m (kK^{p^m})^* \Gamma = \rho(\Gamma) \).

4.2 Finite unions of homogeneous linear \( K \)-varieties

Lemma 13. For any positive integers \( e, n \) and any subgroup \( \Delta \subset K^* \) contained in \( O^*_{S} \) for some finite \( S \subset \Sigma_{K/k} \), there exists a finite subset \( Z(\Delta, n, e) \subset K \) with the following property: For any \( v \in \Sigma_{K} \) and \( c \in \mathbb{A}^n((kK^{p^e})_v) \) and \( r \in \mathbb{A}^n(\Delta_v) \) satisfying that \( c \cdot r = 1 \) and that the components of \( r \) are linearly independent over \( (kK^{p^e})_v \), we have \( cr \in \mathbb{A}^n(Z(\Delta, n, e)) \).

Proof. Fix \( v \in \Sigma_{K} \) and \( c = (c_1, \ldots, c_n) \in \mathbb{A}^n((kK^{p^e})_v) \) and \( r = (\delta_1, \ldots, \delta_n) \in \mathbb{A}^n(\Delta_v) \) such that \( \sum_{i=1}^n c_i \delta_i = 1 \) with \( \delta_1, \ldots, \delta_n \) linearly independent over \( (kK^{p^e})_v \). We see that \( (X_1, \ldots, X_n) = (c_1 \delta_1, \ldots, c_n \delta_n) \) solves the system of linear equations

\[
\sum_{i=1}^n \frac{D^{(j)}_{K_v}(\delta_i)}{\delta_i} X_i = D^{(j)}_{\rho}(1) \quad 0 \leq j < p^e.
\]

(4)

For each \( i \), let \( d_i \in \Delta \) be a lift of \( \delta_i \) through the canonical isomorphism

\[
\Delta/\Delta \cap (kK^{p^e})^* \simeq \Delta_v/(\Delta \cap (kK^{p^e})^*)_v,
\]

(5)
which exists by Corollary 1. Then \(d_1, \ldots, d_n\) are linearly independent over \((kK^r)^e\). For any \(0 \leq j < p^e\), since \(D_{K_i}^{(j)}\) is \((kK^r)^e\)-linear, we have that \(D_{K_i}^{(j)}(\delta_i)/\delta_i = D_{K_i}^{(j)}(d_i)/\delta_i\) belongs to the subset \(C := \{D_{K_i}^{(j)}(g)/g : 0 \leq l < p^e, g \in R_e\}\) of \(K\), where \(R_e \subset \Delta\) is a complete set of representatives of \(\Delta/\Delta \cap (kK^r)^e\), which is finite by Proposition 4(a). By the \((kK^r)^e\)-linear independence of \(d_1, \ldots, d_n\), Garcia and Voloch [4, Theorem 1] asserts the existence of a subsystem of equations (4) in \((X_1, \ldots, X_n)\) which has a unique solution. Therefore, the finiteness of the subset \(C\) of \(K\) shows that \((c_1\delta_1, \ldots, c_n\delta_n)\) is contained in a finite subset of \(K\) depending only on \(n\) and \(C\), hence only on \((\Delta, n, e)\).  

Proposition 5. Suppose that \(M \geq 2\). Let \(e\) be a positive integer, and \(b \in A^M(K^*)\). Then there exists a finite union \(V'\) of proper \(K\)-subvarieties of \(H_b\) such that \(H_b'(\Gamma_v) = V'(\Gamma_v)\) for every \(v \in \Sigma_K\).

Proof. Fix \(v \in \Sigma_K\). Let \(b = (b_1, \ldots, b_M) \in A^M(K^*)\) and \((\gamma_1, \ldots, \gamma_M) \in H_b'(\Gamma_v)\), that is, \(\sum_{i=1}^M b_i\gamma_i = 1\) with \(\gamma_i \in \Gamma_v\) for all \(i\). Let \(d_v = \dim((kK^r)^e, \sum_{i=1}^M (kK^r)^e, b_i\gamma_i)\). We claim that \(d_v \geq 2\). Assume that this claim fails. Then \(b_i\gamma_i \in (kK^r)^e\) for all \(i\). The canonical isomorphism (5) in the proof of Lemma 13 yields \(r_i \in \Gamma\) such that \(\gamma_i r_i \in (kK^r)^e\). Then for each \(i\), we have \(b_i r_i^{-1} \in (kK^r)^e \cap K^* = (kK^r)^e\) by Lemma 12. This leads to the contradiction \(b \in A^M((kK^r)^e)\) and proves that \(d_v \geq 2\) as claimed.

Now we have \(d_v \in \{2, \ldots, M\}\). First, consider the case where \(d_v = M\). Let \(\Gamma' \subset K^*\) be the subgroup generated by \(\Gamma \cup \{b_1, \ldots, b_M\}\). Lemma 13 applied with \(c = (1, 1, \ldots, 1) \in A^M((kK^r)^e)\) and \(r = (b_1\gamma_1, \ldots, b_M\gamma_M) \in A^M(\Gamma_v)\) yields \(b_i\gamma_i \in Z(\Gamma', M, e)\) for all \(i\). For the remaining case, where \(d_v \in \{2, \ldots, M - 1\}\), after re-indexing we may assume that \(\{b_i\gamma_i\}_{1 \leq i \leq d_v}\) is linearly independent over \((kK^r)^e\) and that for each \(j \in \{d_v + 1, \ldots, M\}\) there exist \(c_{ji} \in (kK^r)^e\), \(1 \leq i \leq d_v\), such that \(b_j\gamma_j = \sum_{i=1}^{d_v} c_{ji} b_i\gamma_i\). For every \(j \in \{d_v + 1, \ldots, M\}\), Lemma 13 applied with \(c = (c_{j1}, \ldots, c_{jd_v}) \in A^M((kK^r)^e)\) and \(r = (\frac{b_{1\gamma_1}}{b_{j\gamma_j}}, \ldots\), \(\frac{b_{M\gamma_M}}{b_{j\gamma_j}}) \in A^d(\Gamma_v)\) implies \(c_{ji}b_{ji} := \ell_{ji} \in Z(\Gamma', d_v, e)\) for all \(1 \leq i \leq d_v\). Note that \(\sum_{i=1}^{d_v} (1 + \sum_{j=d_v+1}^M c_{ji}) b_i\gamma_i = 1\). Applying Lemma 13 with \(c = (1 + \sum_{j=d_v+1}^M c_{j1}, \ldots, 1 + \sum_{j=d_v+1}^M c_{jd_v}) \in A^d((kK^r)^e)\) and \(r = (b_1\gamma_1, \ldots, b_d\gamma_d) \in A^d(\Gamma_v)\), we obtain \(1 + \sum_{j=d_v+1}^M c_{ji}) b_i\gamma_i := f_i \in Z(\Gamma', d_v, e)\) for all \(1 \leq i \leq d_v\). Now one can check that \(f_i = b_i\gamma_i + \sum_{j=d_v+1}^M \ell_{ji} b_j\gamma_j\) for every \(1 \leq i \leq d_v\).

Taking the permutation of indices into consideration, we have just shown \(H_b'(\Gamma_v) \subset \bigcup_{(d, \Lambda, \Phi, \sigma) \in \mathcal{F}} V'(d, \Lambda, \Phi, \sigma)(\Gamma_v)\) for every \(v \in \Sigma_K\), where \(\mathcal{F}\) is the finite collection of tuples \((d, \Lambda, \Phi, \sigma)\) such that \(d \in \{2, \ldots, M\}\), \(\Lambda = (\lambda_{ji})_{1 \leq i \leq j \leq M} \in A^{d(M-d)}(Z(\Gamma', d, e))\), \(\Phi = (\phi_i)_{1 \leq i \leq d} \in A^d(Z(\Gamma', d, e))\), and \(\sigma \in S_M\), and \(V'(d, \Lambda, \Phi, \sigma)\) denotes the linear \(K\)-variety.
in $\mathbb{A}^M$ defined by

$$b_\sigma(i)X_\sigma(i) + \sum_{j=d+1}^M \lambda_{ji}b_\sigma(j)X_\sigma(j) = \phi_i, \quad 1 \leq i \leq d.$$ 

For each $(d, \Lambda, \Phi, \sigma) \in \mathcal{F}$, we have $d \geq 2$ and thus $V'(d, \Lambda, \Phi, \sigma) \cap H'_b \neq H'_b$. Therefore, we complete the proof by letting $V' = \bigcup_{(d, \Lambda, \Phi, \sigma) \in \mathcal{F}} (V'(d, \Lambda, \Phi, \sigma) \cap H'_b)$. 

\begin{proof}

Fix Proposition 6. Let $W$ be a finite union of proper linear $K$-subvarieties of $H'_b$ such that $H'_b(\Gamma_v) \subset \mathbb{A}^M((kK^p)_v) \subset V(\bar{\Gamma}_v)$ for every $v \in \Sigma_K$. 

\end{proof}

\begin{corollary}

Suppose that $M \geq 2$. Let $e$ be a positive integer, and $b \in \mathbb{A}^M(k^e)$. Then there exists a finite union $V'$ of proper linear $K$-subvarieties of $H'_b$ such that $H'_b(\Gamma_v) \subset \mathbb{A}^M((kK^p)_v) \subset V(\bar{\Gamma}_v)$ for every $v \in \Sigma_K$.

\end{corollary}

\begin{proof}

Fix $v \in \Sigma_K$. Let $b = (b_1, \ldots, b_M) \in \mathbb{A}^M(K^e)$ and $(\gamma_1, \ldots, \gamma_M) \in H'_b(\Gamma_v) \subset \mathbb{A}^M((kK^p)_v)$. Note that $\dim_{(kK^p)_v} (\sum_{i=1}^M (kK^p)_v b_i \gamma_i) = d_i > 1$, for otherwise since $b_i \in k^e$ for all $i$ we would obtain $(\gamma_1, \ldots, \gamma_M) \in \mathbb{A}^M((kK^p)_v)$, which is a contradiction. Now we have reached the conclusion at the end of the first paragraph of the proof of Proposition 5, and the remaining part of that proof concludes this corollary.

\end{proof}

\begin{proposition}

Let $d$ be the dimension of $W$. Suppose that $W$ is a union of homogeneous linear $K$-varieties, and that each $d$-dimensional irreducible component of $W$ is not $\rho(\Gamma)$-isotrivial. Then there exists a finite union $V$ of homogeneous linear $K$-subvarieties of $W$ with dimension smaller than $d$ such that $W(\Gamma_v) = V(\Gamma_v)$ for every $v \in \Sigma_K$; in particular, we have $W(\bar{\Gamma}) = V(\bar{\Gamma})$.

\end{proposition}

\begin{proof}

It suffices to show that there exists a finite union $V$ of homogeneous linear $K$-subvarieties of $W$ with dimension smaller than $d$ such that $W(\Gamma_v) = V(\Gamma_v)$ for every $v \in \Sigma_K$, for then $W(\bar{\Gamma}) \subset \prod_{v \in \Sigma} W(\Gamma_v) = \prod_{v \in \Sigma} V(\Gamma_v)$ and thus $W(\bar{\Gamma}) = V(\bar{\Gamma})$. To do this, fix $v \in \Sigma_K$. Since $\Gamma_v$ is contained in a field, the set of $\Gamma_v$-valued points of $W$ is the union of those sets of $\Gamma_v$-valued points of its irreducible components; hence, we may assume that $W$ is irreducible. After a permutation of coordinates in $\mathbb{A}^M$, we may suppose that $W$ is defined by

$$X_i = \sum_{j=1}^d b_{ij}X_j, \quad d + 1 \leq i \leq M,$$

where each $b_{ij}$ lies in $K$. For each $i \in \{d+1, \ldots, M\}$, we let $S_i = \{j : 1 \leq j \leq d, b_{ij} \neq 0\}$ and $\pi_i : \mathbb{A}^M \to \mathbb{A}^{|S_i|}$ be the map $(x_1, \ldots, x_M) \mapsto (x_j)_{j \in S_i}$; for each $\ell \in S_i$ we also define $\theta_{\ell} : \mathbb{A}^{|S_i|} \to \mathbb{A}^1$ by $(x_j)_{j \in S_i} \mapsto x_\ell$. Then we see that $\pi_i(W) = H'_b$, where $b_i = (b_{ij})_{j \in S_i} \in \mathbb{A}^{|S_i|}(K^e)$. 

\end{proof}
First, we treat the case where there is some $i \in \{d+1, \ldots, M\}$ such that $b_i \notin A_{|S|}(\rho(\Gamma))$, that is, there is a positive integer $e$ such that $b_i \notin A_{|S|}((kK^e)^{-1}\Gamma)$. Note that may assume that $W(\Gamma_{\nu}) \neq \emptyset$ and hence that $|S| \geq 2$ by Corollary 7. Thus Proposition 5 shows that there exists an union $V$ of finitely many proper linear $K$-subvarieties of $H'_\ell = \pi_i(W)$ such that $\pi_i(W)(\Gamma_{\nu}) \subset V(\Gamma_{\nu})$. Therefore, we have $W(\Gamma_{\nu}) \subset \pi_i^{-1}(\pi_i(W)(\Gamma_{\nu})) \subset \pi_i^{-1}(V(\Gamma_{\nu})) = \pi_i^{-1}(V(\Gamma_{\nu}))$. It is obvious that $\pi_i^{-1}(V) \cap W$ is an union of finitely many homogeneous linear $K$-subvarieties of $W$. Note that it is proper. Indeed, if $W = \pi_i^{-1}(V) \cap W \subset \pi_i^{-1}(V)$, then since $\pi_i(\pi_i^{-1}(V)) = V$, we would arrive the contradiction $\pi_i(W) = V$. This finishes the proof for the current case.

It remains to deal with the case where $b_i \in A_{|S|}(\rho(\Gamma))$ for all $i \in \{d+1, \ldots, M\}$. For each $i \in \{d+1, \ldots, M\}$, we let $\tau_i$ be the $\rho(\Gamma)$-automorphism on $A_{|S|}$ of $(x_j)_{j \in S_i} \mapsto (b_i j x_j)_{j \in S_i}$. Since (6) defines the homogeneous linear $\mathcal{W}$, which is not $\rho(\Gamma)$-isotrivial, while each equation in (6) defines a $\rho(\Gamma)$-isotrivial hyperplane, the same proof as Lemma 6.1 in [2] (which is stated in the case where $k$ is finite) shows that there exist $s \in \mathbb{N}$ and $i_1, \ldots, i_s \in \{d+1, \ldots, M\}$ and $j_1, \ldots, j_s \in \{1, \ldots, d\}$ with $|S_{i_j}| \geq 2$ and $j_\ell \in S_{i,J}$ for each $\ell \in \{1, \ldots, s\}$ such that

$$\begin{align*}
\frac{b_{i_1,j_1} b_{i_2,j_2} \cdots b_{i_{k_1-j_1},k_1-j_1}}{b_{i_1,j_1} b_{i_2,j_2} \cdots b_{i_{k_1-j_1},k_1-j_1}} \quad (7)
\end{align*}$$

does not lie in $k$, hence neither in $kK^e$ for some $e > 0$ by Lemma 2. Fix an arbitrary $r = (\gamma_1, \ldots, \gamma_M) \in W(\Gamma_{\nu})$. Since $\theta_j \circ \tau_i \circ \pi_i(r) = b_i j \frac{\gamma_j}{y_i}$ for each $i \in \{d+1, \ldots, M\}$ and $j \in S_{i,J}$, the quotient (7) is equal to

$$\begin{align*}
\frac{(\theta_{j_1} \circ \tau_{i_1} \circ \pi_{i_1}(r))(\theta_{j_2} \circ \tau_{i_2} \circ \pi_{i_2}(r)) \cdots (\theta_{j_{k_1-j_1}} \circ \tau_{i_{k_1-j_1}} \circ \pi_{i_{k_1-j_1}}(r))}{(\theta_{j_1} \circ \tau_{i_1} \circ \pi_{i_1}(r))(\theta_{j_2} \circ \tau_{i_2} \circ \pi_{i_2}(r)) \cdots (\theta_{j_{k_1-j_1}} \circ \tau_{i_{k_1-j_1}} \circ \pi_{i_{k_1-j_1}}(r))}
\end{align*}$$

By Lemma 12, we see that there exists some $\ell \in \{1, \ldots, s\}$ such that $\tau_{i_\ell} \circ \pi_{i_\ell}(r) \notin A_{|S|}(\rho(\Gamma))$. Note that $\tau_{i_\ell} \circ \pi_{i_\ell}(W)$ is the variety $H'_{\ell,1,\ldots,1}$ in $A_{|S|}$. Since $\tau_{i_\ell} \circ \pi_{i_\ell}(r) \in \tau_{i_\ell} \circ \pi_{i_\ell}(W)(\Gamma_{\nu}) \setminus A_{|S|}(K^e)$, Corollary 8 shows that there is some union $V(\Gamma_{\nu})$ of finitely many proper linear $K$-subvarieties of $\tau_{i_\ell} \circ \pi_{i_\ell}(W)$ such that $\tau_{i_\ell} \circ \pi_{i_\ell}(r) \in V(\Gamma_{\nu})$. Letting $V = \bigcup_{\ell=1}^s (W \cap (\tau_{i_\ell} \circ \pi_{i_\ell})^{-1}(V(\Gamma_{\nu})))$, we have just shown $W(\Gamma_{\nu}) \subset V(\Gamma_{\nu})$. An argument similar to the one given in the end of last paragraph shows that $W \cap (\tau_{i_\ell} \circ \pi_{i_\ell})^{-1}(V(\Gamma_{\nu})) \neq W$ for each $\ell$, and hence $V \neq W$. Since $W$ is irreducible, its dimension is greater than that of $V$, and this completes the proof.

Proof of Theorem 2. By repeated applications of Proposition 6, there exists a finite union $V$ of one-dimensional homogeneous linear $K$-subvarieties of $W$ such that $W(\Gamma) = V(\Gamma)$. Note that $\phi_i(V)$ has dimension zero for some $i \in \{1, \ldots, M\}$, hence
\( \phi_i(V) (\bar{\Gamma}) = \phi_i(V) (\Gamma) \) by Proposition 2, and therefore Lemma 1(c) yields that \( V(\bar{\Gamma}) = V(\Gamma) \). Now the first desired conclusion follows from Lemma 11. Under the additional hypothesis, each irreducible component of \( V \) is not \( \Gamma \)-isotrivial, thus \( \phi_i(V) (\bar{\Gamma}) = \emptyset \) and therefore \( \phi_i(V) (\bar{\Gamma}) = \phi_i(V) (\bar{\Gamma}) = \emptyset \). Hence, we conclude that \( W(\bar{\Gamma}) = V(\bar{\Gamma}) = \emptyset \) as desired. ■

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