國立清華大學七十七學年度
應數所博士班入學考試題

機率論：

1. Let Y be an integrable r.v. (random variable) on probability space $\{\Omega,\mathcal{F},P\}$ and $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$

   (i) Define the conditional expectation $E(Y|\mathcal{G})$ of Y relative to $\mathcal{G}$. (5%)

   (ii) Prove the existence and uniqueness (up to equivalence) of $E(Y|\mathcal{G})$ (15%)

   (iii) If X and Y are independent and $\mathcal{G} = \sigma(X)$, the σ-field generated by X. Prove that $E(Y|\mathcal{G}) = E(Y)$ a.s. (10%)

   (iv) Applying (ii) and (iii) prove that an integrable random variable is independent of itself if and only if it is constant a.s. (10%)

2. For a sequence $A_1,A_2,\ldots$ of events in a probability space $\{\Omega,\mathcal{F},P\}$ consider the σ-fields $\mathcal{F}_n = \sigma\{A_n,A_{n+1}\ldots\}$ and their intersection $\mathcal{T} = \cap_{n=1}^{\infty} \mathcal{F}_n$. $\mathcal{T}$ is called the tail σ-field associated with the sequence $\{A_n\}$, and its

elements are called tail events. If $A_1, A_2, \ldots$ is a sequence of independent events, prove that for each tail event A, P(A) is either 0 or 1. (10%)

3. (i) State Lindeberg's (central limit) theorem.(5%)

(ii) Suppose that $X_1, X_2, \ldots, X_n \ldots$ are independent r.v.'s with density

$$P\{X_j=\sqrt{j}\} = P\{X_j=-\sqrt{j}\} = \frac{1}{2}, \quad j = 1,2,\ldots\ .$$

Find two sequences of constants $\{a_n\}$ and $\{b_n\}$ such that $\frac{(X_1+\ldots+X_n)-a_n}{b_n}$ converges in distribution to N(0,1). Applying Lindeberg's theorem verify your answer. (10%)

4. Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. r.v.'s with density

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

(i) State the definition of stopping time (optional r.v.) relative to $\{X_n\}$. (5%)

(ii) For $c > 0$, define $T = \inf\{n \geq 1: S_n \geq c\}$, where $S_n = X_1+\ldots+X_n$. Verify that T is a stopping time relative to $\{X_n\}$. (10%)

(iii) Find the density function of T. In other words, compute $P\{T=k\}$ for $k = 1,2,\ldots$. (10%)

(iv) Find the distribution function of $S_T - c$. (Hint: For $y>0$, compute $P\{S_T-c>y\}$). (10%)

Recall that the density of S_n is

$$[\lambda^n/(n-1)!]x^{n-1}e^{-\lambda x} \quad \text{for } x \geq 0 \text{ and zero elsewhere.}$$

微分方程：

1. Explain the followig terms

15% (i) ω-limit set

(ii) orbital stability

(iii) unstable manifold.

2. Consider the following system

$$(*) \begin{cases} x' = x(1-\frac{x}{K}) - \frac{2xy}{1+x} \\ y' = (\frac{2x}{1+x} - 1)y \\ x(0) = x_0 > 0,\ y(0) = y_0 > 0, \end{cases} \qquad K>0 \text{ is a parumeter.}$$

25% (i) Show that (*) has a unique solution x(t), y(t) defined on $[0,\infty)$ which are positive and bounded

(ii) Do stability analysis for system (*) for various $K > 0$

(iii) Predict the asymptotic behavior of the solution x(t), y(t) as $t \to \infty$ for various $K > 0$.

3. Consider the equation

$$x'' + f(x)x' + h(x) = 0$$

10% where $xh(x) > 0,\ x \neq 0,\ f(x) > 0,\ x \neq 0$ and

$$H(x) = \int_0^x h(s)ds \to \infty \text{ as } |x| \to \infty$$

Show that the equilibrium solution $x \equiv 0$ is globally asymptotically stable.

(Hint: use Liapunov function $V(x,y) = \frac{y^2}{2} + H(x)$

where $y = x'$).

4. Solve $\quad xu_x + y\,u_y = \lambda u$

10% $\quad u(x,0) = g(x)$

by characteristc method.

5. Show that a solution $U \in C^2(\bar{\Omega})$ of

15%
$$\Delta U = 0 \quad \text{in } \Omega$$
$$U = f \quad \text{on } \partial\Omega$$

minimizes the Dirichlet integral $\int_\Omega |\nabla U|^2 dx$ among all functions in $C'(\bar{\Omega})$ with boundary values f.

6. Use energy method to show that the following initial-

15% boundary value problem

$$u_t = u_{xx} - u^3, \; 0 \le x \le 1$$
$$u(x,0) = u_0(x)$$
$$u(0,t) = a(t), \qquad u(1,t) = b(t)$$

has at most one smooth solution (Hint: Use the energy method on the difference of two solutions)

7. Let $\Omega = \{x \in \mathbb{R}^n : |x|>1\}$ and $u \in C^2(\bar{\Omega})$. Assume $\Delta u = 0$ in

10% Ω and $\lim_{x\to\infty} u(x) = 0$. Show that $\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$.

分析：

1. For a set $A \subseteq R^d$, let I_A denote its indicator function; i.e.

$$I_A(x) = \begin{cases} 1 & , \text{ if } x \in A \\ 0 & , \text{ if } x \notin A \end{cases}$$

Show that

10% (a) if $K \subseteq V \subseteq R^d$, K is compact, V is open, then there is a continuous function h(x) such that

$$I_K(x) \leq h(x) \leq I_V(x), \qquad x \in R^n.$$

15% (b) if $K \subseteq V_1 \cup V_2 \cup \ldots \cup V_n \subseteq R^d$, K is compact, $V_1, V_2, \ldots, V_n$ are open, then there exist continuous functions $h_1(x), h_2(x), \ldots, h_n(x)$ such that

$$h_i(x) \leq I_{V_i}(x), \; i = 1, 2, \ldots, n, \qquad x \in R^n$$

and

$$\sum_{i=1}^{n} h_i(x) = 1, \qquad x \in K.$$

2. Let μ be the Lebesgue measure on [0,1]. Let $\{f_n\}$ be a

15% sequence of real-valued measurable functions satisfying $\sup_n \int_0^1 f_n^2(x) d\mu(x) < \infty$. Show that for any $\epsilon > 0$, there is an $\delta > 0$ such that for any measurable set $E \subseteq [0,1]$ with $\mu(E) \leq \delta$ we have

$$\sup_n \int_E f_n(x) d\mu(x) \leq \epsilon.$$

3. Let $(X,\mathfrak{X},\mu)$ and $(Y,\mathcal{Y},\nu)$ be two σ-finite measure spaces.

15% (a) Define the product measure space

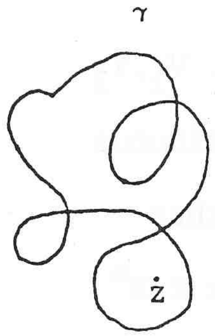
$$(X \times Y,\ \mathfrak{X} \times \mathcal{Y},\ \mu\times\nu).$$

10% (b) Show that if $f(x,y)$ is a nonnegative, product measurable function, then

$$\int_Y\int_X f(x,y)d\mu(x)d\nu(y) = \int_X\int_Y f(x,y)d\nu(y)d\mu(x).$$

4. Let $\mathbb{C}$ be the complex plane. Let γ be a closed

15% curve in $\mathbb{C}$; i.e. γ is a piecewise continuously differentiable map: $[0,1] \to \mathbb{C}$ with $\gamma(0) = \gamma(1)$.



Let

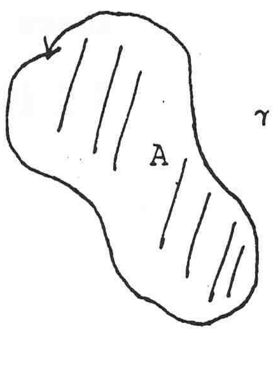
$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z},\quad \zeta \notin \{\gamma(t): 0\le t\le 1\}.$$

Show that $\mathrm{Ind}_\gamma(z)$ is an integer-valued function.

5. Let $\gamma = \{(x(t),y(t)): 0\le t\le 1\}$ be a positively oriented simple closed curve in R^2 (a curve is simple if it does not intersect with itself). Let ℓ be the length of γ, and A the area bounded by γ. Show that

10% (a) $A = \int_0^1 x(t)y'(t)dt = -\int_0^1 y(t)x'(t)dt$

10% (b) $A \le \dfrac{\ell^2}{4\pi}$



數理統計：

1.令X_1，X_2，…，爲獨立同態（iid）之二項隨機變數，$P(X_1=1)=1-P(X_1=0)=P$.令Y爲獨立於X數列之poisson變數，其平均值爲λ。證明

$$U=\sum_{i=1}^{Y} X_i \quad 和 \quad V=Y-\sum_{i=1}^{Y} X_i$$

爲獨立，並求它們的分布。20％

2.將1000 人依性別及是否色盲區分的結果如下表：

	男	女
正　常	442	514
色　盲	38	6

根據遺傳模式，上列數字宜有下列之相對頻率

	男	女
正　常	$p/2$	$p^2/2+pq$
色　盲	$q/2$	$q^2/2$

其中$q=1-p$ 爲基因集中色盲基因所佔之比例。請問上列數據及模式是否一致？20％

3.設$Y_i=\alpha+\beta X_i+e_i$，$i=1$，…，n，其中e_i爲獨立同態之$N(0,\sigma_e^2)$變數，X_i爲獨立同態之$N(\mu_x,\sigma_x^2)$變數，且e_i和X_i爲獨立。欲估計$\mu_y=\alpha+\beta\mu_x$，統計工作者常用

(i)　$\hat{\mu}_y(1)=\bar{Y}$，或

(ii)　$\hat{\mu}_y(2)=a+b\mu_x=\bar{Y}+b(\mu_x-\bar{X})$，其中$a$，$b$表示常用的$\alpha$，$\beta$之最小平方估計量，而$\mu_x$爲已知。

請問$\hat{\mu}_y(1)$和$\hat{\mu}_y(2)$何者較佳（具較小的MSE）？20％

4.假設在計錄n個獨立$N(0,\sigma^2)$變數時僅知其爲小於-1，介於-1與1之間或大於1.

(i)　請問σ之最小充分統計量爲何？

(ii)　證明σ之MLE各階動差（moment）皆爲∞。20％

5.令X_1，…，X_n爲獨立之常態變數、$EX_i=\xi_i\mu$，$Var(X_i)=\sigma^2$，此處之$n+2$個變數$\xi_i\in\{-1,+1\}$，$\mu\geq 0$和$\sigma^2>0$皆爲未知數。

(i)　求最大概似估計量$\hat{\xi}_i$，$\hat{\mu}$和$\hat{\sigma}^2$.

(ii)　證明$\hat{\mu}$和$\hat{\sigma}^2$不爲μ和σ^2之一致（consistent）估計量.

(iii)　求H_0：$\mu=0$對H_1：$\mu\neq 0$之概似比檢定量及其當$n\to\infty$時之漸近虛無分布. 20％

數學規劃：

1. Suppose that the following system has no solution

$$Ax = 0, \quad x \geq 0 \quad \text{and} \quad cx > 0$$

Devise another system that must have a solution. You have to prove that your answer is correct. (15%)

2. Given a linear (affine) function on R^n

$$h(x) = a^T x + b,$$

show that it is both convex and concave. Conversely, a real-valued, convex and concave function on R^n is linear. (15%)

3. Denote by (A,b,c) the standard maximum problem:

$$\max cx$$

$$Ax = b, \quad x \geq 0.$$

Assume that this problem has an optimal solution x^* and $v = cx^*$ is the optimal value of the objective function. Let $\{c^k\}_{k=1}^{\infty}$ be a sequence of vectors approaching c; and v_k the optimal value of the problem (A,b,c^k). Show that v_k approaches v. (15%)

4. Show that all basic feasible solutions of a assignment problem are integral vectors by first proving the total unimodularity of the constraint matrix. (15%)

5. Find all extreme rays of the convex cone

$$k = \{x \in R^n, \ Cx \geq 0\},$$

where C is an n×n non-singular matrix. (15%)

6. (i) The polar cone $\bar{S}$ of the cone S is defined as the cone $\{y: y \cdot x \geq 0, \text{ for all } x \in S\}$. Find a convex cone k for which $k \neq \bar{\bar{k}}$. (10%)

(ii) Show that if k_1 and k_2 are finite cones, then so are $k_1 + k_2$ and $k_1 \cap k_2$. (15%)